Part D

Higher Types

Chapter 7
Computations Over Two Types

In Chapter 4 we started our study of computation theories on domains of two types $\mathfrak{A} = \langle A, S, \mathbf{S} \rangle$ where $A = S \cup \mathrm{Tp}(S)$ and $\mathbf{S}$ is a coding scheme for S. Given a *normal* list $\mathbf{L}$ on $\mathfrak{A}$ we defined a recursion theory PR($\mathbf{L}$) generalizing Kleene recursion in higher types, and all of Chapter 4 was aimed at proving the following result (Theorem 4.4.1):

(a) PR($\mathbf{L}$) is p-normal, hence admits a selection operator over N,

(b) A is weakly but not strongly $\mathbf{L}$-finite, i.e. the $\mathbf{L}$-semicomputable relations are not closed under $\exists x \in \mathrm{Tp}(S)$.

(c) S is strongly $\mathbf{L}$-finite, i.e. the $\mathbf{L}$-semicomputable relations are closed under $\exists s \in S$.

The ultimate goal of this chapter is to see how far properties (a)–(c) determine normal recursion in higher types. In this study we meet a new and characteristic feature of computations on two types which was entirely absent in the general study of finite theories on one type (Chapter 3), viz. *reflection.*

7.1 Computations and Reflection

The setting is a computation theory Θ on $\mathfrak{A} = \langle A, S, \mathbf{S} \rangle$ satisfying properties (a)–(c), such theories are called *normal.* Since we are on two types, Θ also allows functional evaluation

$$f(x, y, \sigma) = x(y),$$

if $x \in \mathrm{Tp}(S)$ and $y \in S$. We also have extended our functional substitution

$$f(\sigma) = h(\lambda s \cdot g(s, \sigma), \sigma).$$

The code set C is assumed to be equal to N.

The reader should at this point recall Definitions 4.1.1, 4.1.2, 4.1.4 as well as Remark 4.1.5. Also recall the need of including the equality relation on S in the general case.

7.1.1 Definition. A computation theory Θ on $\mathfrak{A} = \langle A, S, \mathrm{S}\rangle$ is called *normal* if

(i) the equality relation on S is Θ-computable,
(ii) A is weakly and S is strongly Θ-finite,
(iii) Θ is p-normal.

As usual in the "finite" case we work with the norm or length function $|\cdot|_\Theta$ rather than with the subcomputation relation. Our computation theories are single-valued so we shall abbreviate $|a, \sigma, z|_\Theta$ to $|a, \sigma|_\Theta$ or in some cases to $|\{a\}_\Theta(\sigma)|_\Theta$.

Let $N \subseteq X \subseteq A$, and $\mathrm{Ord}(X) = \{|e, \sigma|_\Theta : \{e\}_\Theta(\sigma)\downarrow$ and σ a list from $X\}$. We introduce

$$\kappa^X = \sup \mathrm{Ord}(X)$$
$$\lambda^X = \text{order-type of } \mathrm{Ord}(X).$$

We shall be interested in the special cases $X = N$, $N \cup \{x_1, \ldots, x_m\}$, S, $S \cup \{x_1, \ldots, x_m\}$, A. Letting σ be the list $x_1, \ldots, x_m$, κ^X will be denoted by κ^0, κ^σ, κ^S, $\kappa^{S,\sigma}$, κ_Θ, respectively. And similarly for λ^X.

7.1.2 Remark. There is a well-known connection between prewellorderings and the ordinals κ^X. In particular, if $N \subseteq X \subseteq A$, then κ^X is the supremum of the lengths of the pwo's with domain $\subseteq A$ which are Θ-computable in elements from X. The supremum is not attained.

And if X admits a pairing scheme $\langle M_X, K_X, L_X\rangle$ Θ-computable in elements from X, then λ^X is the supremum of the lengths of the pwo's with domain $\subseteq X$ which are Θ-computable in elements from X. Again the supremum is not attained.

From this we may conclude that if $X = N$, $N \cup \{x_1, \ldots, x_m\}$, S, $S \cup \{x_1, \ldots, x_m\}$, then

$$\lambda^X < \kappa^X < \kappa_\Theta.$$

If $\sigma = (x_1, \ldots, x_m)$, then $\lambda^\sigma \leqslant \lambda^{S,\sigma} < \kappa^\sigma$.

Detailed proofs of these facts are given in lemmas 20 and 21 of Moldestad [105]. To give the flavor of such proofs we include a brief hint for the first case. Let P be a pwo which is Θ-computable in $\sigma = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_m \in X$ and $\operatorname{dom} P \subseteq A$. First find an index e_1 such that $\{e_1\}_\Theta(x, \sigma) \simeq 0$ if $x \in \operatorname{dom} P$, and if $x \in \operatorname{dom} P$ then $|e_1, y, \sigma|_\Theta < |e_1, x, \sigma|_\Theta$ for all y below x in the pwo P. Next, let e_2 be an index such that $\{e_2\}_\Theta(\sigma)\downarrow$ and $|e_1, x, \sigma|_\Theta < |e_2, \sigma|_\Theta$ for all $x \in \operatorname{dom} P$, e.g. let $\{e_2\}_\Theta(\sigma) \simeq E(f)$, where $f(x) \simeq \{e_1\}_\Theta(x, \sigma)$ if $x \in \operatorname{dom} P, f(x) \simeq 1$, otherwise. Obviously, $\kappa^X > |e_2, \sigma|_\Theta \geqslant |P|$. Conversely, given any $\nu < \kappa^X$, there is a computation in a list σ from X, $\{e\}_\Theta(\sigma)$, such that $|e, \sigma|_\Theta > \nu$. The subcomputation tree below $\{e\}_\Theta(\sigma)$ gives a pwo P such that $|P| = |e, \sigma|_\Theta > \nu$.

We now turn to a brief study of *reflection phenomena in higher types.* This notion was introduced in recursion theory by G. E. Sacks and further developed by L. Harrington in his thesis [53]. A Kechris in a set of unpublished notes from MIT [74] developed the general theory, see also his account in [76]. For the use of reflection in forcing arguments in higher types, see G. Sacks [143]. As we saw in Section 3.3, similar reflection properties are also of great importance for the general theory of inductive definability.

A computation-theoretic approach was developed by J. Moldestad [105]; we shall follow his account. As an introduction we present the following simple result.

7.1.3 Simple Reflection. *For all e, σ: If $\exists x \cdot |e, x, \sigma|_\Theta < \kappa^{S,\sigma}$, then $\exists x \cdot |e, x, \sigma| < \kappa^\sigma$.*

The premiss simply says:

$$\exists e' \in N \quad \exists s \in S[\{e'\}(s, \sigma)\downarrow \ \wedge\ \exists x \cdot |e, x, \sigma|_\Theta < |e', s, \sigma|_\Theta].$$

This is a Θ-semicomputable relation of σ: "$\exists x$" can be expressed by the E functional since A is weakly Θ-finite. "$\exists s \in S$" can be handled since S is strongly Θ-finite, and "$\exists e' \in N$" is no problem since Θ is p-normal, and we have selection over N.

7.1.4 Definition. Let μ be an ordinal $\leqslant \kappa$ $(= \kappa_\Theta)$. μ is called *σ-reflecting* if for all e:

$$\text{If } \exists x \cdot |e, x, \sigma| < \mu, \quad \text{then} \quad \exists x \cdot |e, x, \sigma| < \kappa^\sigma.$$

σ is here a list of elements from A. Note that the σ-reflecting ordinals are an initial segment. And κ is not σ-reflecting for all σ if A is not strongly Θ-finite.

7.1.5 Remark. As remarked above we shall not develop the general theory of σ-reflecting ordinals, but only that part of the theory which is needed for the characterization results of Section 7.2. But we cannot resist mentioning the following *characterization of strong Θ-finiteness*:

Let $B \subseteq A$ be Θ-computable. Then B is strongly Θ-finite iff for all e, σ: if $\exists x \in B \cdot \{e\}_\Theta(x, \sigma)\downarrow$, then $\exists x \in B \cdot |e, x, \sigma|_\Theta < \kappa^\sigma$. (For a proof and further refinements, see Lemma 25 in Moldestad [105].)

7.1.6 Further Reflection. *$\kappa^{S,P,\sigma}$ is σ-reflecting.*

Here $P = \{\langle e, \tau\rangle : \{e\}_\Theta(\tau, \sigma)\downarrow,\ \tau$ is a list from $S\}$. Note that P has a natural pwo of length $\lambda^{S,\sigma}$.

The proof of 7.1.6 will be a consequence of the following four propositions. We shall compare $\lambda^{S,\sigma}$ with the ordinal $|x|$ for any pwo x on S. Let $\text{pwo}_S(x)$ mean that x is a pwo with domain $\subseteq S$.

Proposition 1. "$\text{pwo}_S(x)\ \wedge\ |x| < \lambda^{S,\sigma}$" *is Θ-semicomputable as a relation of x, σ.*

The statement is equivalent to

$$\text{pwo}_S(x) \wedge \exists y \in S[y \in P \wedge \forall z \in \text{dom}(x)[|z|_x < |y|_P]],$$

where the notations $|\,|_x$ and $|\,|_P$ are self-explanatory. Note that the relation $\text{pwo}_S(x)$ is Θ-computable.

Proposition 2. *If* $\text{pwo}_S(x)$ *and* $|x| \geqslant \lambda^{S,\sigma}$, *then* P *is* Θ*-computable in* x, σ *and parameters from* S.

First compute an x' such that $|x'| > \lambda^{S,\sigma}$. Then there exists $r \in \text{dom}(x')$ such that $|r|_{x'} = \lambda^{S,\sigma}$. From x, σ, r we can decide P.

Proposition 3. *If* P *is* Θ*-computable in* σ, x *and parameters from* S, *then* $\kappa^{S,P,\sigma} \leqslant \kappa^{S,x,\sigma}$.

Proposition 4. "$\exists y \cdot |e, y, \sigma|_\Theta < \kappa^{S,x,\sigma}$" *is* Θ*-semicomputable as a relation of* e, σ, x. *There is an index* $\hat{e}$ *for this relation such that* $|\hat{e}, e, \sigma, x|_\Theta > \inf\{|e, y, \sigma|_\Theta : y \in A\}$.

The statement is equivalent to

$$\exists e' \in N \, \exists r \in S[\{e'\}_\Theta(r, x, \sigma)\downarrow \;\wedge\; \exists y \cdot |e, y, \sigma|_\Theta \leqslant |e', r, x, \sigma|_\Theta].$$

From these four propositions further reflection easily follows. So assume that $\exists y \cdot |e, y, \sigma| < \kappa^{S,P,\sigma}$. Define the following relation

$$R(\sigma, x) \quad \text{iff} \quad \text{pwo}_S(x) \wedge [|x| < \lambda^{S,\sigma} \cdot\vee\cdot \exists y \cdot |e, y, \sigma| < \kappa^{S,x,\sigma}].$$

(i) By Propositions 1 and 4 $R(\sigma, x)$ is Θ-semicomputable. There is an index f for R such that if $R(\sigma, x)$ and $|x| \geqslant \lambda^{S,\sigma}$, then

$$|f, \sigma, x|_\Theta > \inf\{|e, y, \sigma|_\Theta : y \in A\}.$$

(ii) From Propositions 1–3 and the assumption we conclude that if $\text{pwo}_S(x)$, then $R(\sigma, x)$ is true. Hence we can assume that $\lambda x \cdot \{f\}_\Theta(\sigma, x)$ is total. Let $\{g\}(\sigma) \simeq E(\lambda x \cdot \{f\}(\sigma, x))$, then $|g, \sigma|_\Theta\downarrow$ and $|g, \sigma|_\Theta > |f, \sigma, x|_\Theta$ for all x.

(iii) There is some x such that $\text{pwo}_S(x)$ and $|x| \geqslant \lambda^{S,\sigma}$. Combining (i) and (ii) we get

$$\kappa^\sigma > |g, \sigma|_\Theta > |f, \sigma, x|_\Theta > \inf\{|e, y, \sigma|_\Theta : y \in A\}.$$

Hence we conclude $\exists y \cdot |e, y, \sigma|_\Theta < \kappa^\sigma$, which completes the proof of 7.1.6.

Further reflection has the following compactness property as a corollary.

7.1.7 Compactness. *Assume that* B *is a set of subsets of* S *and that* B *is* Θ*-semicomputable in some list* σ. *Assume that* B *has as element a non-empty subset* α_0 *of*

S such that α_0 is Θ-semicomputable in σ. Then B contains a subset of S which is non-empty and Θ-computable in σ.

We indicate the proof. If $\alpha_0 \in B$ let α_μ be the approximation to α_0 up to length less than μ. Let g be an index such that if $|y| = \mu$, then $\alpha_\mu \in B$ iff $\{g\}(y, \sigma)\downarrow$.

If we can show that $\exists y \cdot |g, y, \sigma|_\Theta < \kappa^{P,\sigma}$, then by further reflection $\exists y \cdot |g, y, \sigma|_\Theta < \kappa^\sigma$. And from this we get a subset $\alpha_\tau \subseteq \alpha_0$ such that $\alpha_\tau \in B$ and α_τ is Θ-computable in σ.

But since $\kappa^{P,\sigma} > \kappa^{S,\sigma}$, let y be a computation $\{e\}(P, \sigma)$ such that $\kappa^{P,\sigma} > |y| > \kappa^{S,\sigma}$. Then $\alpha_{|y|} = \alpha_0$ because y is convergent and $|y| > \kappa^{S,\sigma}$, and $|g, y, \sigma| < \kappa^{P,\sigma}$ because $y = \langle e, P, \sigma\rangle$.

7.2 The General Plus-2 and Plus-1 Theorem

We start by fixing some notations. Let Θ be a normal theory on $\mathfrak{A} = \langle A, S, \mathrm{S}\rangle$:

$$\begin{aligned}
\mathrm{sc}(\Theta) &= \{X \subseteq A : X \text{ is } \Theta\text{-computable}\},\\
\mathrm{sc}(\Theta, \sigma) &= \{X \subseteq A : X \text{ is } \Theta\text{-computable in } \sigma\},\\
\mathrm{en}(\Theta) &= \{X \subseteq A : X \text{ is } \Theta\text{-semicomputable}\},\\
S\text{-en}(\Theta) &= \{X \subseteq S : X \text{ is } \Theta\text{-semicomputable}\}.
\end{aligned}$$

7.2.1 Theorem. *Let Θ be a normal theory on $\mathfrak{A}$. Then there exists a normal list* $\mathbf{L}$ *such that* S-en$(\Theta) = S$-en$(\mathbf{L})$ *and* sc$(\Theta, r) =$ sc$(\mathbf{L}, r)$ *for all $r \in S$.*

This is an abstract version of the plus-2 theorem of Harrington [53]. Harrington's original version was a reduction result: Starting out with a normal functional G of type $> n + 2$ he constructed a functional F of type $n + 2$ such that ${}_n\mathrm{en}(G) = {}_n\mathrm{en}(F)$. The proof used the fact that Tp(n) is *strongly* finite in G. Theorem 7.2.1 is an improvement, here we assume that A, which in the concrete setting of higher types corresponds to Tp(n), is *weakly* Θ-finite. And in general we should not assume more since Tp(n) is *not* strongly PR$({}^{n+2}E)$-finite. Thus Theorem 7.2.1 gives a kind of characterization result which we will supplement in Section 7.3.

We follow the detailed proof in Moldestad [105]. Moldestad's proof is patterned on the original Harrington proof in [53]. However, one refinement is necessary in order to go from strong to weak finiteness in the assumption; this refinement is the joint effort of Harrington and Moldestad.

We have also labelled the theorem a "plus-1" result. A plus-1 result was first proved by G. Sacks [143]. He generalized the notion of an abstract 1-section to the appropriate notion of abstract $k + 1$-section, and constructed by a forcing-type argument a functional F giving a concrete representation of the abstract $k + 1$-section. And Further Reflection 7.1.6 was an essential ingredient in his proof. In the present setting the starting point is different, viz. a normal theory Θ; thus we have both the section and the envelope. The section result is, as we shall see, a consequence of the result about envelopes, and the proof is an exercise in the use of reflection principles. We leave it to the reader to decide on how different the forcing construction of Sacks' is from the present construction. Note in this connection that the notion of abstract $k + 1$-section is not entirely "pure", there is something semicomputable involved.

We divide the proof of Theorem 7.2.1 into several parts:

7.2.2 Some Preliminary Material. Let $P = \{\langle e, \sigma\rangle : \{e\}_\Theta(\sigma)\downarrow$ and σ is a list from $S\}$. The ordinals $|e, \sigma|_\Theta$, where $\langle e, \sigma\rangle \in P$, are called Θ-*subconstructive*. Their order-type is $\lambda = \lambda^S$; let $\langle \eta_\nu : \nu < \lambda\rangle$ be an enumeration of the Θ-subconstructive ordinals.

We will construct a normal list **L** consisting of the equality relation on S, the quantifier functional E, and a functional G which shall code up information about S-en(Θ). G will be constructed in stages G_τ, each G_τ being a partial approximation to G containing sufficient information to generate the set $H_\tau = \{\langle e, \sigma\rangle : |e, \sigma|_{\mathbf{L}} < \tau, \sigma$ a list from $A\}$. If $\tau < \tau'$, then $G_{\tau'}$ will be an extension of G_τ.

G will be defined by

$$G(f) = \begin{cases} (\bigcup_\tau G_\tau)(f) & \text{if } f \in \bigcup_\tau \operatorname{dom} G_\tau \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let μ_ν be an enumeration of the **L**-*subconstructive* ordinals.

7.2.3 Further Preliminary Material. Let G_τ and H_τ be given. We note that

$$H_{\tau+1} = \{\langle e, \sigma\rangle : \text{all immediate subcomputations of } \{e\}_{\mathbf{L}}(\sigma) \text{ are in } H_\tau\}.$$

We observe that if $\lambda x \cdot \{e\}_{\mathbf{L}}(x, \sigma)$ is total and $\langle e, x, \sigma\rangle \in H_\tau$ for all x, then we must define G on this function at stage $\tau + 1$ if we have not previously done so. Thus we let $f = \lambda x \cdot \{e\}_{\mathbf{L}}(x, \sigma) \in \operatorname{dom} G_{\tau+1}$ and set $G_{\tau+1}(f) = 0$ or 1. $G_{\tau+1}$ is called a *trivial extension* of G_τ if $\operatorname{dom} G_{\tau+1} = \operatorname{dom} G_\tau \cup \{f : f$ as above$\}$ and $G_{\tau+1}(f) = 0$ if $f \in \operatorname{dom} G_{\tau+1} - G_\tau$.

7.2.4 On How G Shall Contain Information About the S-en(Θ). Not every extension should be trivial. Let τ be **L**-subconstructive, in fact, let $\tau = \mu_\nu$ for some $\nu < \lambda$. Information about the S-en(Θ) will be coded into G_τ at this stage.

If $x \in \Theta$ and $|x|_\Theta \leqslant \eta_\nu$, we shall take some function f_{xy} where $y \in \mathrm{PR}[\mathbf{L}]$ and $|y|_{\mathbf{L}} = \mu_\nu$ and such that $f_{xy} \notin \operatorname{dom} G_\mu$, $\mu < \tau$. We then let $f_{xy} \in \operatorname{dom} G_\tau$ and set $G_\tau(f_{xy}) = 1$.

From f_{xy} and G_τ we shall recover information about x inside **L** in order to get S-en(Θ) $\subseteq$ S-en(**L**). And f_{xy} should be **L**-computable in x, y when y is an **L**-computation of length τ.

Such functions f_{xy} exist:

Proposition 1. *Let* **L** *be any normal list. Let y be an* **L**-*computation of length* τ.

For each $x \in A$ there is a total function f_{xy} such that f_{xy} is **L**-*computable in x, y and if $x \neq x'$, then $f_{xy} \neq f_{x'y}$. If $f_{xy} = \lambda t \cdot \{e\}_{\mathbf{L}}(t, \sigma)$ for some e, σ then $\tau \leqslant |e, t, \sigma|_{\mathbf{L}}$ for some $t \in A$.*

We indicate briefly the proof. Let τ^+ be the least limit ordinal $\geqslant \tau$. The set of **L**-computations with length $< \tau^+$ is **L**-computable in y. Let f_y be defined by the following instructions: (i) $f_y(u) = 0$, if u is not of the form $\langle e, \sigma\rangle$. If $u = \langle e, \sigma\rangle$, then ask if $|e, t, \sigma|_{\mathbf{L}} < \tau^+$ for all t; (ii) If the answer is no, let $f_y(u) = 0$. (iii) If the answer is yes, let $f_y(u)$ be different from $\{e\}_{\mathbf{L}}(\langle e, \sigma\rangle, \sigma)$.

f_y is recursive in $\mathbf{L}, y$. And if $f = \lambda t \cdot \{e\}_{\mathbf{L}}(t, \sigma)$ is a total function such that $|e, t, \sigma|_{\mathbf{L}} < \tau^+$ for all t, then f and f_y differs at $t = \langle e, \sigma\rangle$.

We can now define f_{xy}:

$$f_{xy}(t) = \begin{cases} \langle f_y(t), x, 0\rangle & \text{if } x \in S \\ \langle f_y(t), x(t), 1\rangle & \text{if } x \in \mathrm{Tp}(S),\ t \in S \\ f_y(t) & \text{if } x, t \in \mathrm{Tp}(S). \end{cases}$$

Then f_{xy} is **L**-computable uniformly in x, y. And $f_{xy} \neq f_{x'y}$ if $x \neq x'$. Let $f_{xy} = \lambda t \cdot \{e\}_{\mathbf{L}}(t, \sigma)$ for some e, σ. It is not difficult to see that we can obtain f_y from f_{xy}. There is, in fact, an index e' such that $f_y = \lambda t \cdot \{e'\}_{\mathbf{L}}(t, \sigma)$ and $|e', t, \sigma|_{\mathbf{L}} < |e, t, \sigma|_{\mathbf{L}} + \omega$, for all t. If $|e, t, \sigma|_{\mathbf{L}} < \tau$ for all t, then $|e', t, \sigma|_{\mathbf{L}} < \tau^+$ for all t. This means that $f = \lambda t \cdot \{e'\}_{\mathbf{L}}(t, \sigma)$ would have to differ from f_y at $\langle e', \sigma\rangle$, but $f = f_y$.

7.2.5 Toward the Definition of G_τ. Actually, we have almost arrived. Suppose G_μ is defined for $\mu < \tau$. Then H_μ is constructed for $\mu < \tau$. Note that we can decide from $\bigcup_{\mu<\tau} H_\mu$ whether τ is **L**-subconstructive.

In the subconstructive case let $y \in \mathrm{PR}[\mathbf{L}]$ such that $|y|_{\mathbf{L}} = \tau$. If $\tau = \mu_\nu$, $\nu < \lambda$, let $x \in \Theta$ and $|x|_\Theta \leqslant \eta_\nu$. Define G^0 as the "zero-extension" of $\bigcup_{\mu<\tau} G_\mu$:

$$G^0(f) = \begin{cases} \left(\bigcup_{\mu<\tau} G_\mu\right)(f) & \text{if } f \in \bigcup_{\mu<\tau} \operatorname{dom} G_\mu \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{L}^0 = E, G^0, =_S$ is a normal list. Choose f_{xy} according to Proposition 1. Then f_{xy} is $\mathbf{L}^0$-computable in x, y, hence **L**-computable in x, y since G^0 is **L**-computable in y, and $f_{xy} \notin \operatorname{dom} G_\mu$ for any $\mu < \tau$.

7.2.6 Defining G_τ. Suppose G_μ and H_μ are defined for all $\mu < \tau$. There are two cases in the definition of G_τ.

Case 1. There exists an ordinal $\nu < \lambda$ such that ν is the order-type of all ordinals $< \tau$ which are **L**-subconstructive, i.e. $\{\mu_\rho : \mu_\rho < \tau\} = \{\mu_\rho : \rho < \nu < \lambda\}$.

I. *τ is **L**-subconstructive*, i.e. $\tau = \mu_\nu$. Let

$$G_\tau(f_{xy}) = 1$$

for all x, y such that $x \in \Theta$, $|x|_\Theta \leqslant \eta_\nu$ and $y \in \mathrm{PR}[\mathbf{L}]$ and $|y|_{\mathbf{L}} = \tau$, where f_{xy} is chosen according to 7.2.5 above. Note that we always include trivial extensions whenever relevant.

II. τ *is not* **L**-*subconstructive*, i.e. $\tau < \mu_\nu$. Let

$$\varepsilon = \eta_\nu - \sup\{\eta_\rho : \rho < \nu\}.$$

We ask the following *question*: Does there exist an ordinal π such that $\tau < \pi \leqslant \tau + \varepsilon$ and π is $\mathbf{L}^0$-subconstructive?

If the answer is *yes*: Let $G_\tau = \bigcup_{\mu<\tau} G_\mu$ if τ is a limit ordinal, and let G_τ be the trivial extension of G_μ if $\tau = \mu + 1$.

If the answer is *no*: Let G_τ be defined as in subcase I.

Case 2. Otherwise: Let $G_\tau = \bigcup_{\mu<\tau} G_\mu$ if τ is a limit ordinal, and let G_τ be the trivial extension of G_μ if $\tau = \mu + 1$.

This completes the definition of G_τ, and from the sequence G_τ we define G as in Section 7.2.2 above.

7.2.7 Proposition 2. *The order-type of the* **L**-*subconstructive ordinals is* $\geqslant \lambda$.

This should not come as a surprise. In fact, this is the way we have arranged the construction of G. Nevertheless, assume that ν = order-type of the **L**-subconstructives $< \lambda$.

Let $\tau = \sup\{\mu_\rho : \rho < \nu\}$; τ is not **L**-subconstructive. Back to the construction of G_τ: We must be in subcase II of case 1, and the answer to the question is no!

Let $x \in \Theta$, $|x|_\Theta = \eta_\nu$, and $x \in S$. Then $G(f_{xy}) = 1$ for all y such that $|y|_{\mathbf{L}} = \tau$. And

$$\tau = \text{least ordinal } \tau \text{ such that } \exists y[|y|_{\mathbf{L}} = \tau \ \wedge \ G(f_{xy}) = 1].$$

We should not forget to point out that there exists such $y \in \mathrm{PR}[\mathbf{L}]$.

We have now set the stage for an application of further reflection. Let Q code all **L**-computations with arguments from S. By Theorem 7.1.6 $\kappa^{Q,x}$ is x-reflecting and $\kappa^{Q,x} > \kappa^{S,x} = \kappa^S = \tau$, where the last equality follows from our assumption, and $\kappa^{S,x} = \kappa^S$ since $x \in S$.

Let m be an index such that $\{m\}_{\mathbf{L}}(Q, x)\downarrow$ and $|m, Q, x|_{\mathbf{L}} > \kappa^{S,x} = \tau$. Then

$$\exists y[|y|_{\mathbf{L}} < |m, Q, x|_{\mathbf{L}} \ \wedge \ G(f_{xy}) = 1].$$

By reflection, omitting a few pedantic details, we conclude

$$\exists y[|y|_{\mathbf{L}} < \tau \ \wedge \ G(f_{xy}) = 1].$$

But this is impossible by the definition of τ.

7.2.8 Proposition 3. S-en$(\Theta) \subseteq S$-en(**L**) *and for all* $r \in S$, sc$(\Theta, r) \subseteq$ sc$(\mathbf{L}, r)$.

Let $X \in S\text{-en}(\Theta)$; then $r \in X$ iff $\langle e, r\rangle \in P$ for some index e. Since we have enough subconstructives on the **L**-side we are in the "normal" case 1, I and conclude

$$r \in X \quad \text{iff} \quad \langle e, r\rangle \in P \quad \text{iff} \quad \exists y \in S[y \in \mathrm{PR}[\mathbf{L}] \wedge G(f_{\langle e,r\rangle y}) = 1].$$

Hence $X \in S\text{-en}(\mathbf{L})$.

For the section part, note that if $X \in \mathrm{sc}(\Theta, r)$ then we have indices e_1, e_2 such that

$$\begin{aligned} x \in X &\quad \text{iff} \quad \{e_1\}_\Theta(x, r)\downarrow. \\ x \notin X &\quad \text{iff} \quad \{e_2\}_\Theta(x, r)\downarrow. \end{aligned}$$

From this construct an index e such that $\lambda x \cdot \{e\}_\Theta(x, r)$ is total and $|e, x, r|_\Theta \geqslant \inf\{|e_1, x, r|_\Theta, |e_2, x, r|_\Theta\}$. Then we compute $E(\lambda x \cdot \{e\}_\Theta(x, r))$ to get a Θ-subconstructive level larger than the ordinals associated to e_1 and e_2 when they are defined. This can be matched by an **L**-subconstructive level. As above, this allows us to conclude that both X and $A - X$ are **L**-semicomputable in r.

7.2.9 Toward the Second Half of the Theorem. For the converse we need to analyze the construction of G and hence of $\mathrm{PR}[\mathbf{L}]$ inside Θ. We fix some notation

$$\begin{aligned} \eta &= \sup\{\eta_\nu : \nu < \lambda\} = \kappa_\Theta^S \\ \mu &= \sup\{\mu_\nu : \nu < \lambda\} \leqslant \kappa_\mathbf{L}^S. \end{aligned}$$

Proposition 4. (a) *There exists a total Θ-computable function f and a partial Θ-computable p such that:*

(i) *If $|e, \sigma|_\mathbf{L} < \mu$, then $\{e\}_\mathbf{L}(\sigma) \simeq \{f(e)\}_\Theta(\sigma)$.*

(ii) *If $|x|_\mathbf{L} < \mu$ or $|y|_\mathbf{L} < \mu$, then $p(x, y)\downarrow$, and $x \in \mathrm{PR}[\mathbf{L}] \wedge |y|_\mathbf{L} < \mu \wedge |x|_\mathbf{L} \leqslant |y|_\mathbf{L} \Rightarrow p(x, y) \simeq 0$. $|y|_\mathbf{L} < \mu \wedge |x|_\mathbf{L} > |y|_\mathbf{L} \Rightarrow p(x, y) \simeq 1$.*

(b) *There exists a total Θ-computable function f' and a partial Θ-computable function p' such that:*

(i) $\{e\}_\mathbf{L}(\sigma) \simeq \{f'(e)\}_\Theta(\sigma, P)$,

(*where P is the set defined in* 7.2.2).

(ii) *If $|x|_\mathbf{L} < \kappa_\mathbf{L}$ or $|y|_\mathbf{L} < \kappa_\mathbf{L}$, then $p'(x, y)\downarrow$, and $x \in \mathrm{PR}[\mathbf{L}] \wedge |x|_\mathbf{L} \leqslant |y|_\mathbf{L} \Rightarrow p'(x, y) \simeq 0$. $|x|_\mathbf{L} > |y|_\mathbf{L} \Rightarrow p'(x, y) \simeq 1$.*

The proof being an exercise in the use of the second recursion theorem is long and very computational. The overall strategy is as follows (we restrict ourselves to part (a) for the moment). Let $\rho < \mu$ and suppose that $\{e\}_\mathbf{L}(\sigma) \simeq \{f(e)\}_\Theta(\sigma)$ for all e, σ such that $|e, \sigma|_\mathbf{L} < \rho$ and that $p(x, y)$ is defined and has the right value when $\inf(|x|_\mathbf{L}, |y|_\mathbf{L}) < \rho$. When $|e, \sigma|_\mathbf{L} = \rho$ we shall describe $\{f(e)\}_\Theta(\sigma)$ in terms of $\{f(e')\}_\Theta(\sigma')$ and $p(x', y')$, where $\{e'\}_\mathbf{L}(\sigma')$ is an immediate subcomputation of $\{e\}_\mathbf{L}(\sigma)$ and $\inf(|x'|_\mathbf{L}, |y'|_\mathbf{L}) < \rho$. When $\inf(|x|_\mathbf{L}, |y|_\mathbf{L}) = \rho$ we shall describe $p(x, y)$ in terms of $p(x', y')$, $\{f(e')\}_\Theta(\sigma')$, where $\inf(|x'|_\mathbf{L}, |y'|_\mathbf{L}) < \rho$ and $|e', \sigma'|_\mathbf{L} < \rho$.

In the construction of f the case to worry about is an application of G, so suppose $|e, \sigma|_{\mathbf{L}} = \rho$ and

$$\{e\}_{\mathbf{L}}(\sigma) \simeq G(\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma)).$$

By the induction hypothesis $\{e'\}_{\mathbf{L}}(u, \sigma) \simeq \{f(e')\}_{\Theta}(u, \sigma)$ for all u. We must now be able to decide inside Θ if $\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma) = f_{xy}$ for some x, y, and if this is true calculate $G(f_{xy})$. We ask five questions (and note that by construction $\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma) \neq f_{xy}$ if $|y|_{\mathbf{L}} \geqslant \rho$).

Question 1: Are there x, y such that $|y|_{\mathbf{L}} < \rho$ and $\lambda u\{e'\}_{\mathbf{L}}(u, \sigma) = f_{xy}$?

NO: Set $G(\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma)) = 0$
YES: Go to question 2.

Question 2: Let $\tau < \rho$ be the ordinal such that for some x and y, $\tau = |y|_{\mathbf{L}}$ and $f_{xy} = \lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma)$. Is there an ordinal $\nu < \lambda$ such that $\mu_\xi < \tau$ when $\xi < \nu$ and $\mu_\nu \geqslant \tau$?

NO: Set $G(\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma)) = 0$
YES: Go to question 3.

Question 3: Let ν, τ be as above. Is there an x such that $|x|_\Theta \leqslant \eta_\nu$ and $\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma) = f_{xy}$, where $|y|_{\mathbf{L}} = \tau$?

NO: Set $G(\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma)) = 0$
YES: Go to question 4.

Question 4: Is τ **L**-subconstructive?

YES: Set $G(\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma)) = 1$
NO: Go to question 5.

Question 5: Let $\varepsilon = \eta_\nu - \sup\{\eta_\xi : \xi < x\}$. Is there an ordinal π such that $\tau < \pi \leqslant \tau + \varepsilon$ and π is $\mathbf{L}^0$-subconstructive?

YES: Set $G(\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma)) = 0$
NO: Set $G(\lambda u \cdot \{e'\}_{\mathbf{L}}(u, \sigma)) = 1$.

Each question must now be analyzed inside Θ. As an example we make some comments on the first question. We see that

$$\begin{aligned} |y|_{\mathbf{L}} < \rho \quad &\text{iff} \quad \exists u(|y|_{\mathbf{L}} \leqslant |e', u, \sigma|_{\mathbf{L}}) \\ &\text{iff} \quad \exists u \cdot p(y, \langle e', u, \sigma \rangle) \simeq 0. \end{aligned}$$

Observe that $\lambda u \cdot p(y, \langle e', u, \sigma \rangle)$ is total, hence $\exists u$ can be expressed by the E-functional, which means that the relation $|y|_{\mathbf{L}} < \rho$ is Θ-computable, uniformly in e, σ.

To describe f_{xy}, information about **L**-computations of length $< |y|_{\mathbf{L}}$ is needed. By the induction hypothesis this can be obtained from $\lambda e\sigma \cdot \{f(e)\}_{\Theta}(\sigma)$ and p when $|y|_{\mathbf{L}} < \rho$. In this case there is an index e_1 such that $f_{xy} = \lambda u\{e_1\}_{\Theta}(u, x, y, f(e), \sigma)$. We then have to decide the question

$$\exists x \exists y(|y|_{\mathbf{L}} < \rho \ \wedge \ f_{xy} = \lambda u\{f(e')\}_{\Theta}(u, \sigma)).$$

And this we argued that we can do, using E to express the quantifiers $\exists x \exists y$.

Question 2 is trivial in this case since $\rho < \mu$. It is when we come to part (b) of the proposition that we have to ask questions about P, hence the need to include P as argument. Also note that by bounding the search in question 5 we need not assume strong finiteness of the total domain A. With these hints we wish the reader the best of luck with the remaining details of the proof.

Remark. We need some more notation and a simple computational result (which actually is used in the proof of part (b) of the proposition).

We recall that P is the complete Θ-semicomputable set over S. Let Q be the corresponding **L**-set over S.

If $r \in Q$, then $|r|_{\mathbf{L}} = \mu_\nu$ for some ν, let $|r|_Q = \nu$, and set $|r|_Q =$ order-type of the **L**-subconstructives if $r \notin Q$.

If $r \in P$, then $|r|_\Theta = \eta_\nu$ for some ν, let $|r|_P = \nu$, and set $|r|_P =$ order-type of the Θ-subconstructives if $r \notin P$.

A simple computation from Proposition 4 shows that

$$\text{“}r \in P \ \wedge \ s \in Q \ \wedge \ |s|_{\mathbf{L}} < \mu \ \wedge \ |r|_P = |s|_Q\text{”},$$

is a Θ-semicomputable relation.

7.2.10 Proposition 5. *The order-type of the* **L**-*subconstructive ordinals is* λ.

We shall assume *not* and reflect down to a contradiction. So let there be an $s_0 \in Q$ such that $|s_0|_{\mathbf{L}} = \mu_\lambda$.

Some technical preliminaries are needed in order to reflect. Let $\lambda' \leqslant \lambda$ and set $P' = P_{\lambda'} = \{r \in P : |r|_P < \lambda'\}$ and $\eta' = \sup\{\eta_\rho + 1 : \rho < \lambda'\}$. Using P' in place of P we construct a functional G' and a list $\mathbf{L}'$. Note that G' and G agree up to μ', where $\mu' = \sup\{\mu_\rho + 1 : \rho < \lambda'\}$. And the functions f' and p' of Proposition 4 (b) act the same way with respect to P' as they do with respect to P.

We can now set the stage for the reflecting statement. Let e_1 be a Θ-index for the following statement which says that λ' is the order-type of the $\mathbf{L}'$-subconstructives below $|s_0|_{\mathbf{L}'}$.

(i) s_0 is an $\mathbf{L}'$-computation $\wedge$ $\forall r \in P' \exists s(|r|_P = |s|_Q \ \wedge \ |s|_{\mathbf{L}'} < |s_0|_{\mathbf{L}'}) \ \wedge$ $\forall s(|s|_{\mathbf{L}'} < |s_0|_{\mathbf{L}'} \rightarrow \exists r \in P'(|r|_{\mathbf{L}} = |s|_Q))$.

There further exist indices e_2, e_3, and e_4 such that

(ii) $\{e_2\}(P', s_0)\downarrow$ iff s_0 is an $\mathbf{L}'$-computation, in which case $|s_0|_{\mathbf{L}'} < |e_2, P', s_0|_\Theta$.

(iii) $\{e_3\}_\Theta(P')\downarrow$ and $|e_3, P'|_\Theta \geqslant \eta'$.

(iv) $\{e_4\}_\Theta(P', s_0)\downarrow$ iff s_0 is an $\mathbf{L}'$-computation, in which case $|e_4, P', s_0|_\Theta > \eta' + |s_0|_{\mathbf{L}'}$.

e_4 is constructed from e_2 and e_3.

The Reflecting Statement: The statement has three parts: There exists an $\lambda' \leqslant \lambda$ such that:

(a) $x = P_{\lambda'}$;
(b) s_0 is a convergent $\mathbf{L}'$-computation and λ' is the order-type of the $\mathbf{L}'$-subconstructives below $|s_0|_{\mathbf{L}'}$;
(c) if $r \notin x$, then $\eta' + |s_0|_{\mathbf{L}'} < |r|_\Theta$.

Part (a) of the statement can simply be expressed as "$x \subseteq P$ and $\forall r, r'(r \in x \wedge |r'|_\Theta \leqslant |r|_\Theta \rightarrow r' \in x)$". Part (b) is statement (i) above. Part (c) can be replaced by $\forall r(r \notin x \rightarrow |r|_\Theta > |e_4, x, s_0|_\Theta)$, which implies (c) when $x = P'$. Let e_5 be an Θ-index for these statements and define

$$\mathbb{B} = \{x : \{e_5\}_\Theta(x, s_0)\downarrow\}.$$

Then $P \in \mathbb{B}$ and by the compactness result 7.1.7 there exists a *proper* initial segment $P' = P_{\lambda'}$ of P, i.e. $\lambda' < \lambda$, such that $P' \in \mathbb{B}$.

Thus s_0 is a convergent $\mathbf{L}'$-computation, λ' is the order-type of the $\mathbf{L}'$-subconstructives below $|s_0|_{\mathbf{L}'}$ and if $r \notin P'$, then $\eta' + |s_0|_{\mathbf{L}'} < |r|_\Theta$.

We have reflected in order to show that s_0 is secured at an earlier stage than μ_λ. To do this we must go back to the construction of G and G'. We remarked above that G_ρ and G'_ρ are equal for all $\rho < \mu' = \sup\{\mu_\rho + 1 : \rho < \lambda'\}$. If we could extend this up to all $\rho < |s_0|_{\mathbf{L}'}$, then we would get $|s_0|_{\mathbf{L}} = |s_0|_{\mathbf{L}'} = \mu_{\lambda'} < \mu_\lambda = |s_0|_{\mathbf{L}}$. This will then be the desired contradiction which shows that the order-type of the $\mathbf{L}'$-subconstructives is at most, and hence equal to, λ.

It remains to fill in some of the details of this sketch.

Let $\tau = \mu'$. Note that the $\mathbf{L}$- and $\mathbf{L}'$-subconstructives agree below τ and have the same order-type λ'. And τ is $\mathbf{L}$-subconstructive iff it is $\mathbf{L}'$-subconstructive (see 7.2.5). And by part (b) of the reflecting statement $|s_0|_{\mathbf{L}'}$ is the λ'-th $\mathbf{L}'$-subconstructive.

Claim. *$G_\rho = G'_\rho$ for all $\rho < |s_0|_{\mathbf{L}'}$.*

If $\tau = |s_0|_{\mathbf{L}'}$, which is the case if τ is $\mathbf{L}$-subconstructive, then the claim follows from our preliminary remarks. So suppose $\tau < |s_0|_{\mathbf{L}'}$. τ is then *not* $\mathbf{L}$-subconstructive. We shall prove that $G_\tau = G'_\tau$.

In the construction of G'_τ we are in case 2 for the first time. In the construction of G_τ we are in case 1 since the order-type of the $\mathbf{L}$-subconstructive ordinals $< \tau$ is λ' and $\lambda' < \lambda$. Further, we are in subcase II because τ is not $\mathbf{L}$-subconstructive. The functional G^0 in the list $\mathbf{L}^0$ is the same as G', hence s_0 is a convergent $\mathbf{L}^0$-computation, $|s_0|_{\mathbf{L}'} = |s_0|_{\mathbf{L}^0}$, and $|s_0|_{\mathbf{L}^0}$ is the first ordinal $> \tau$ which is $\mathbf{L}^0$-subconstructive.

Let $\varepsilon = \eta_{\lambda'} - \sup\{\eta_\rho : \rho < \lambda'\}$. By (c) in the reflecting statement $\eta' + |s_0|_{\mathbf{L}^0} < |r|_\Theta$ if $r \notin P'$, i.e. if $|r|_\Theta \geqslant \eta_{\lambda'}$. Hence $\eta' + |s_0|_{\mathbf{L}^0} < \eta_{\lambda'}$. By definition $\eta' = \sup\{\eta_\rho + 1 : \rho < \lambda'\}$. Hence $\sup\{\eta_\rho : \rho < \lambda'\} + |s_0|_{\mathbf{L}^0} < \eta_{\lambda'}$. From this we conclude that

$$|s_0|_{\mathbf{L}^0} - \tau < \eta_{\lambda'} - \sup\{\eta_\rho : \rho < \lambda'\},$$

or

$$\tau < |s_0|_{\mathbf{L}^0} < \tau + \varepsilon.$$

But this means, since $|s_0|_{\mathbf{L}^0}$ is $\mathbf{L}^0$-subconstructive, that the answer to the question in II of case 1 is yes. We conclude that $G_\tau = G'_\tau$, and, in fact, that $G_\rho = G'_\rho$ for all ρ less than the next $\mathbf{L}^0$-subconstructive, which is $|s_0|_{\mathbf{L}'}$.

This proves the claim.

This means that the $\mathbf{L}'$-computations of length $< |s_0|_{\mathbf{L}'}$ are identical to the $\mathbf{L}$-computations of length $< |s_0|_{\mathbf{L}'}$, which means that $|s_0|_{\mathbf{L}'} = |s_0|_{\mathbf{L}}$. And this is impossible as explained above. The proof of Proposition 5 is complete.

7.2.11 Proposition 6. S-en(**L**) $\subseteq$ S-en(Θ) *and for all* $r \in S$, sc(**L**, r) $\subseteq$ sc(Θ, r).

Let $H = \{\langle e, \sigma\rangle : \{e\}_{\mathbf{L}}(\sigma)\downarrow \ \wedge\ |e, \sigma|_{\mathbf{L}} < \mu\}$. By Proposition 5 we know that $\mu = \sup\{\tau : \tau \text{ is } \mathbf{L}\text{-subconstructive}\} = \sup\{\mu_\nu : \nu < \lambda\}$. If $X \in S$-en(**L**), then there is an index e such that

$$r \in X \quad \text{iff} \quad \langle e, r\rangle \in H.$$

H, which is not necessarily a subset of S, is easily reduced to en(Θ), viz.

$$x \in H \quad \text{iff} \quad \exists r \in S\, \exists s \in S(r \in P\ \wedge\ s \in Q\ \wedge\ |s|_{\mathbf{L}} < \mu\ \wedge \\ |r|_P = |s|_Q\ \wedge\ |x|_{\mathbf{L}} < |s|_{\mathbf{L}}).$$

By the remark under 7.2.9 and part (a) of Proposition 4 we see that H is Θ-semicomputable.

For the final part observe that if $X \in$ sc(**L**, r), then there are indices e_1, e_2 such that

$$x \in X \quad \text{iff} \quad \langle e_1, x, r\rangle \in H$$
$$x \notin X \quad \text{iff} \quad \langle e_2, x, r\rangle \in H,$$

because the lengths can be dominated by an **L**-subconstructive level, see the similar proof of Proposition 3 in 7.2.8.

And, if the reader has not yet noticed, Propositions 3 and 6 prove Theorem 7.2.1!

7.3 Characterization in Higher Types

Theorem 7.2.1 is in a well-defined sense a lifting of the plus-1 Theorem 5.4.24 from one to higher types. Of course, something more is added. In one type, specifically over ω, there are no subindividuals S, hence there is no S-envelope part to Theorem 5.4.24. And proofs are quite different.

On the other hand, with minor changes the proof of Theorem 5.4.7 which characterizes Spector theories over ω, generalizes to higher types.

7.3.1 Theorem. *Let Θ be a normal theory on $\mathfrak{A}$. Then Θ is equivalent to* PR[**L**] *for a normal list* **L** *iff Θ is not Θ-Mahlo.*

Only minor changes are needed in the previous proof. First we need to modify Definition 5.4.1.

7.3.2 Definition. Let Θ and Ψ be normal computation theories on $\mathfrak{A}$. We define

$$\Psi <_1 \Theta \quad \text{iff} \quad \text{en}(\Psi) \subseteq \text{en}(\Theta) \wedge \exists x[\kappa_\Psi^x < \kappa_\Theta^x].$$

Over ω the ordinal κ_Ψ^x, $x \in \omega$, would just be the ordinal of the Spector theory.

Corresponding to the Fattening Lemma 5.4.4 we have the following result.

7.3.3 Fattening Lemma. *Let Θ be normal and* **L** *a Θ-computable list such that $\forall x[\kappa_{\mathbf{L}}^x = \kappa_\Theta^x]$. Then there is a normal list* **L**$'$ *such that $\Theta \sim$* PR[**L**$'$].

The proof is the same, we just have to relativize to a parameter from the domain A at certain places. For example, Definition 5.4.5 of Ord(f) by the following. Let f be a total unary function from $A \to S$. Then Ord(f) is the least ordinal τ such that for some $e, y, f = \lambda x\{e\}_\Theta(x, y)$ and $\tau = |e_1, e, y|_\Theta$. (Here e_1 is an index such that $\{e_1\}_\Theta(e, y)\downarrow$ iff $\lambda x \cdot \{e\}(x, y)$ is total, in which case $|e, x, y|_\Theta < |e_1, e, y|_\Theta$ for all x.)

We now construct G_0 exactly as in the proof of 5.4.4 and let **L**$' =$ **L**, $G, E, =_S$. We immediately conclude that en(**L**$'$) $\subseteq$ en(Θ).

To prove the converse we introduce a relativized version of κ, viz. for $y \in A$, let

$$\kappa_y = \sup\{\text{Ord}(f); f \text{ is } \mathbf{L}'\text{-computable in } y\}.$$

And corresponding to the claim we now have: $\kappa_y = \kappa_\Theta^y$ for all $y \in A$.

From this claim the fattening lemma immediately follows: Suppose $\{e\}_\Theta(\sigma)\downarrow$. By the claim there is an index m such that $f = \lambda t \cdot \{m\}_{\mathbf{L}'}(t, \langle e, \sigma\rangle)$ is total and $|e, \sigma|_\Theta \leqslant \text{Ord}(f)$.

By construction of G_0 we see that

$$G_0(\langle \lambda t \cdot \{m\}_{\mathbf{L}'}(t, \langle e, \sigma\rangle), e, \sigma\rangle) \simeq \{e\}_\Theta(\sigma) + 1.$$

Using selection over N, we pick an m as a function of e, σ, which gives us, exactly as before, the converse inclusion, viz. en(Θ) $\subseteq$ en(**L**$'$).

7.3.4 Definition. A normal theory Θ on $\mathfrak{A}$ is called Θ-*Mahlo* if for all normal and Θ-computable lists **L**, PR[**L**] $<_1 \Theta$.

Theorem 7.3.1 now follows as before. If $\Theta \sim \mathrm{PR}[\mathbf{L}]$ for a normal list **L**, then **L** is Θ-computable, and Θ, obviously, cannot be Θ-Mahlo, using **L** as a counterexample. Conversely, suppose that Θ is not Θ-Mahlo. Then there exists a Θ-computable list **L** such that PR[**L**] is not $<_1$ than Θ. Since by the Θ-computability of **L**, $\mathrm{en}(\mathbf{L}) \subseteq \mathrm{en}(\Theta)$, this means that $\forall x[\kappa_{\mathbf{L}}^x = \kappa_{\Theta}^x]$. The fattening lemma then gives us a normal list **L**′ such that $\Theta \sim \mathrm{PR}[\mathbf{L}']$.

7.3.5 Remark. We have been discussing computation theories on *one* and *two* types. In the one-type case we have a computation domain A with no extra structure assumed, in the two-type case A has the structure $A = S \cup \mathrm{Tp}(S)$, where $\mathrm{Tp}(S) = \omega^S$.

Various finiteness assumptions have been placed on the domains, the crucial distinction being *strong* versus *weak* finiteness. In the Spector theory case we have a theory on one type A which is assumed to be strongly finite in the sense of the theory. In the normal type-2 case we imposed the requirement that A is weakly finite, but S is strongly finite.

There are a number of comments to make. First recall that weak and strong finiteness coincides over ω.

In two types we could drop the requirement that S is strongly finite. In this case reflection phenomena disappear, and we are essentially back to the case of one weakly finite domain. This case is not without interest: The characterization Theorem 7.3.1 would still be true, and there is a non-trivial result in the theory of inductive definability here, see our discussion in Theorem 3.3.15 on $\mathrm{IND}(\Sigma_2^0)$ versus $\mathrm{IND}(\Pi_1^0)$.

But we could in the type-2 case make a move in the opposite direction, strengthening the axioms to the strong finiteness of A. This case has been studied by Hinman and Moschovakis [62] under the name of hyperprojective theory. Here again we can have some effect of the type structure, but the fascinating interplay of strong and weak finiteness in the normal case, is lost.