Part C

Infinite Theories

Chapter 5
Admissible Prewellorderings

The precomputation theories of the first chapter were to a large extent patterned on ordinary recursion theory, ORT. In part B we gave an analysis of "higher recursion theory" through the notion of a finite theory which generalizes ORT by moving up in types over the basic domain.

But there are different ways of extending ORT, e.g. from the integers to all or part of the ordinals. Or ORT can be rephrased as a recursion theory on HF, the hereditarily finite sets, and then be extended to other domains of sets, even to the total universe. Both approaches were followed, and we duly got various notions of ordinal and set recursion, leading to theories of primitive recursive functions, to the rudimentary functions, and to admissibility theory.

We shall not in this book retrace in any detail the line of development from ORT to recursion on ordinals and to the notion of an *admissible ordinal.* The literature on this topic is vast, but the reader would do well to consult one of the classics in the field, the 1967 paper of R. Jensen and C. Karp, *Primitive recursive set functions* [72]. From this one could go to the recent survey of R. Shore, *α-recursion theory* [152], and the *Short course on admissible recursion theory* [156] by S. Simpson, the latter being an advertisement—which we endorse—of a book-length exposition to come.

Pure α-recursion theory was soon transformed into a general theory of *admissible structures.* A thorough exposition of this field is given by J. Barwise in his book, *Admissible Sets and Structures* [11].

Remark. The references above are strictly pedagogical and do not imply a history of ordinal recursion theory and admissibility theory in any way—the names of S. Kripke and R. Platek have not even been mentioned.

Barwise and Shore give some historical remarks in their respective introductions. It remains to be seen what Simpson will do with the history of the subject in his book. We shall try to document the sources for those parts of the theory that we discuss. The reader should in particular see Section 5.3.

The line between set theory and recursion theory is sometimes difficult to draw. Admissibility theory on the ordinals is obviously recursion theory, but general admissible structures may be too "short and fat" to support a reasonable recursion theory. (For an example see F. Gregory [46].) We have decided to draw the line at *resolvable* structures, our reasons are as follows.

A basic result about admissible sets is

Gandy's Fixed-point Theorem. *Let A be an admissible set and Γ a Σ_1 positive inductive operator on A. Then the least fixed-point of Γ is Σ_1 on A.*

A discussion and proof can be found in Barwise [11].

For resolvable structures there is a converse. We recall that an admissible set A is *resolvable* if there is a total A-recursive function p such that

(i) $$A = \bigcup_{y \in A} p(y),$$

(ii) the relation $x \leqslant y$ iff $x \in p(y)$ defines a pwo of A.

Note that since a set is A-finite iff it is an element of A, the resolvability of A means that there is an A-computable pwo of A with A-finite initial segments.

The following result was proved by A. Nyberg [132].

Theorem. *Let $\mathfrak{A} = \langle A, \in, R_1, \ldots, R_k \rangle$ be a resolvable structure where A is transitive and closed under pairing. If $\mathfrak{A}$ satisfies Δ_0-separation then the following conditions are equivalent:*

(i) *$\mathfrak{A}$ is admissible.*

(ii) *Every Σ_1 positive inductive operator on $\mathfrak{A}$ has a Σ_1 definable least fixed-point.*

(iii) *The length of a Σ_1 positive induction does not exceed the ordinal of $\mathfrak{A}$.*

(iv) *There exists a positive first-order inductive definition on $\mathfrak{A}$ of length strictly greater than any Σ_1 positive induction on $\mathfrak{A}$.*

To a recursion theorist this result is an excellent conceptual justification for the following general notion of admissible prewellordering.

Definition (see 5.1.9). Let $(\mathfrak{A}, \preccurlyeq)$ be a computation domain with a pwo $\preccurlyeq$ and let $\mathbf{R}$ be a sequence of relations on the domain of $\mathfrak{A}$. The structure $(\mathfrak{A}, \preccurlyeq)$ is called an $\mathbf{R}$-*admissible prewellordering* if every $\Sigma_1(\preccurlyeq, \mathbf{R})$ positive inductive operator on $\mathfrak{A}$ has a $\Sigma_1(\preccurlyeq, \mathbf{R})$ definable least fixed-point.

Remarks. (1) The notion of $\mathbf{R}$-admissible pwo was introduced by Moschovakis [113] but without the conceptual analysis given by Nyberg's theorem. We like to think that Moschovakis had the recursion theorist's natural faith in the first recursion theorem. And it is nice to know that faith sometimes can be vindicated.

(2) Following Moschovakis we shall prove that admissible pwo's correspond to a certain class of infinite computation theories. Previous to this work C. E. Gordon gave a computation-theoretic analysis of admissibility in his thesis [45]. He showed that given an admissible structure $\mathfrak{A}$, Σ_1 definability on $\mathfrak{A}$ corresponds to multiple valued search computability (in the sense of Moschovakis [112]) in the $\in$-relation and the bounded quantifier

$$\mathrm{bE}(x, f) \to \begin{cases} 0 & \text{if } (\exists y \in x)[f(y) \to 0] \\ 1 & \text{if } (\forall y \in x)[f(y) \to 1]. \end{cases}$$

Search computability in $\in$ and bE is easily seen to take care of Σ_1 definability. For the converse one may use Gandy's fixed-point theorem.

We shall now proceed as follows. In Sections 5.1 and 5.2 we characterize admissible pwo's in computation-theoretic terms. The associated computation theory is the natural domain for degree theoretic arguments. In 5.3 we analyze the "next-admissible set" construction and venture to make a few historical remarks on how this idea developed. In the final section we apply the imbedding results of Section 5.3 to study the structure of finite theories over the integers.

5.1 Admissible Prewellorderings and Infinite Theories

Having given our motivation we make a detour via a class of infinite computation theories before developing the general theory of admissible prewellorderings. The class is not entirely arbitrary since we will show in 5.2 that it suffices to characterize the admissible prewellorderings.

So, let us start with a *p-normal* computation theory $\langle \Theta, < \rangle$ on a domain $(\mathfrak{A}, \preccurlyeq)$, where $\mathfrak{A}$ is a computation domain in which $A = C$ (thus equality is Θ-computable) and $\preccurlyeq$ is a prewellordering of A. As always, our theories are single-valued.

5.1.1 Assumption A. The structure $(\mathfrak{A}, \preccurlyeq)$ is Θ-*resolvable*, i.e.

(i) The domain A is not Θ-finite.

(ii) The pwo $\preccurlyeq$ is Θ-computable and the initial segments of $\preccurlyeq$ are uniformly Θ-finite.

The last requirement of 5.1.1 means that the functional $\mathbf{E}^{\preccurlyeq}$ is Θ-computable, where

$$\mathbf{E}^{\preccurlyeq}(f, x) \simeq \begin{cases} 0 & \text{if} \quad (\exists y \prec x)[f(y) \simeq 0] \\ 1 & \text{if} \quad (\forall y \prec x)[f(y) \simeq 1]. \end{cases}$$

For the definition and simple properties of finiteness we refer back to Section 2.5. See also Remark 5.1.4 below.

Our next assumption introduces the admissibility condition $\|\Theta\| = |\preccurlyeq|$, with a suitable effective addition. Recall that

$$\|\Theta\| = \sup\{|a, \sigma, z|_\Theta : (a, \sigma, z) \in \Theta\},$$

and $|\preccurlyeq|$ is the length of the pwo $\preccurlyeq$.

5.1.2 Assumption B. The assumption comes in two parts:

(i) $\|\Theta\| = |\preccurlyeq|$.

(ii) The set $\Theta^{|w|} = \{(a, \sigma, z) \in \Theta : |a, \sigma, z|_\Theta = |w|_{\preccurlyeq}\}$ is Θ-computable uniformly in $w \in A$.

Note that we can (due to the computability of $\mathbf{E}^{\preccurlyeq}$) replace $=$ by $<$ or $\leqslant$ in the definition of $\Theta^{|w|}$ and still obtain uniform Θ-computability.

We add one more assumption, the *Grilliot selection principle*, in order to close the Θ-semicomputable relations under existential quantification over the domain A. Since we insist on single-valued theories we forsake the option of adding "multiple-valued" search.

5.1.3 Assumption C. There exists a Θ-computable mapping $q(n)$ such that for all a, σ

$$\exists x \cdot \{a\}_\Theta(x, \sigma) \simeq 0 \quad \text{iff} \quad \{q(n)\}_\Theta(a, \sigma) \simeq 0.$$

And if $\exists x \cdot \{a\}_\Theta(x, \sigma) \simeq 0$, then

$$|q(n), a, \sigma, 0|_\Theta > \inf\{|a, x, \sigma, 0| : \{a\}(x, \sigma) \simeq 0\}.$$

This means that the functional

$$\mathbf{E}^{1/2}(f, \sigma) \simeq 0 \quad \text{iff} \quad \exists x \cdot f(x, \sigma) \simeq 0,$$

is Θ-computable, i.e. "half" of the usual $\mathbf{E}$ functional on A is Θ-computable.

5.1.4 Remark. Assumption C together with the p-normality of Θ gives a "nice" theory for en(Θ). From C follows the fact that the Θ-semicomputable relations are closed under $\exists$ and $\vee$. p-normality implies selection over N, which entails that $R \in \mathrm{sc}(\Theta)$ iff $R, \neg R \in \mathrm{en}(\Theta)$. Selection over N also ensures that the Θ-finite sets behave well.

5.1.5 Definition. Let $\mathfrak{A}$ be a computation domain with a pwo $\preccurlyeq$ and Θ a computation theory on $(\mathfrak{A}, \preccurlyeq)$. Θ is called an *infinite theory* on $(\mathfrak{A}, \preccurlyeq)$ iff Θ satisfies axioms A, B and C.

We give a few simple properties of infinite theories.

5.1.6 Proposition. *Let Θ be an infinite theory on $(\mathfrak{A}, \preccurlyeq)$. A set $B \subseteq A$ is Θ-finite iff it is Θ-computable and $\preccurlyeq$-bounded.*

We verify that Θ-finiteness implies $\preccurlyeq$-boundedness. Assume the contrary, then

$$A = \bigcup_{x \in B} \{y \in A : y \prec x\}.$$

But this means that A is a Θ-finite union of Θ-finite sets, hence A is Θ-finite, contradicting (i) of 5.1.1.

5.1.7 Proposition. *For each n there is a Θ-computable relation R_n such that*

$$\{a\}_\Theta(\sigma) \simeq z \quad \textit{iff} \quad \exists w R_n(a, \sigma, z, w).$$

We use the relation $\Theta^{|w|}$ of (ii) in 5.1.2.

Associated with the computation domain $\mathfrak{A}$, the prewellordering $\preccurlyeq$, and a sequence of relations $\mathbf{R}$ on $\mathfrak{A}$, we have a language

$$\mathbf{L} = \mathbf{L}(\mathfrak{A}, \preccurlyeq, \mathbf{R}).$$

As usual define classes $\Delta_0(\preccurlyeq, \mathbf{R})$, $\Sigma_1(\preccurlyeq, \mathbf{R})$, $\Pi_1(\preccurlyeq, \mathbf{R})$, and $\Delta_1(\preccurlyeq, \mathbf{R})$ over the domain A.

5.1.8 Proposition. *There exist Θ-computable relations $\mathbf{R}_\Theta = R_1, R_2$ on A such that* $\mathrm{en}(\Theta) = \Sigma_1(\preccurlyeq, \mathbf{R}_\Theta)$.

The axioms immediately entail that $\Sigma_1(\preccurlyeq, \mathbf{R}) \subseteq \mathrm{en}(\Theta)$ for any sequence $\mathbf{R}$ of Θ-computable relations on A. Conversely, let

$$\begin{aligned} &R_1(a, x, y, w) \quad \text{iff} \quad |a, x, y|_\Theta \leqslant |w|_\preccurlyeq \\ &R_2(a, b, c) \quad \text{iff} \quad a = S_1^1(b, c). \end{aligned}$$

A simple induction proof using Proposition 5.1.7 shows that $\mathrm{en}(\Theta) \subseteq \Sigma_1(\preccurlyeq, \mathbf{R}_\Theta)$.

Let us note a version of "Δ_0-separation": If S is $\Delta_0(\preccurlyeq, \mathbf{R})$, where $\mathbf{R}$ is Θ-computable, then the set

$$\{y \in A : y \prec x \wedge S(y)\},$$

is Θ-finite, uniformly in x.

We now return to the conceptually important notion of *admissible prewellordering*. First a notational convention, when we use $\Delta_0(\preccurlyeq, X, \mathbf{R})$ and $\Sigma_1(\preccurlyeq, X, \mathbf{R})$, we always require that X occurs *positively* in the formulas. Such formulas $\theta(\sigma, X)$ then define monotone operators

$$\Gamma_\theta(X) = \{\sigma : \theta(\sigma, X)\}.$$

As usual we let Γ_∞ be the least fixed point for Γ_θ and $|\Gamma_\theta|$ the ordinal of the inductive definition.

5.1.9 Definition. Let $(\mathfrak{A}, \preccurlyeq)$ be a computation domain with a pwo and $\mathbf{R}$ a sequence of relations on A. The structure $(\mathfrak{A}, \preccurlyeq)$ is called an $\mathbf{R}$-*admissible prewellordering* if for every $\Sigma_1(\preccurlyeq, X, \mathbf{R})$ formula θ with parameters from A, the fixed-point Γ_∞ of Γ_θ is a $\Sigma_1(\preccurlyeq, \mathbf{R})$ relation.

5.1.10 Proposition. *Let Θ be an infinite theory on $(\mathfrak{A}, \preccurlyeq)$ and* **R** *any sequence of relations extending* $\mathbf{R}_\Theta$. *Then the structure $(\mathfrak{A}, \preccurlyeq)$ is* **R**-*admissible.*

This is an immediate corollary of the first recursion theorem for Θ. Any formula $\theta(\sigma, X) \in \Sigma_1(\preccurlyeq, X, \mathbf{R})$ determines a Θ-computable functional

$$\varphi(f, \tau) \simeq 0 \quad \text{iff} \quad \theta(\tau, \{\sigma | f(\sigma) \simeq 0\}).$$

Let f^* be the fixed-point of φ. Then $\Gamma_\infty = \{\sigma : f^*(\sigma) \simeq 0\}$. Thus Γ_∞ is Θ-semi-computable, hence $\Sigma_1(\preccurlyeq, \mathbf{R})$ by Proposition 5.1.8.

We thus see that infinite theories Θ on a structure $(\mathfrak{A}, \preccurlyeq)$ give rise to admissible pwo's. We promise a converse. As a first step we shall, in a rather crude way, associate a recursion theory H with an **R**-admissible pwo $\preccurlyeq$.

5.1.11. Let $(\mathfrak{A}, \preccurlyeq)$ be **R**-admissible. Define

$$H = \mathrm{PR}[\mathbf{E}^{\preccurlyeq}, \preccurlyeq, \mathbf{R}, \mathbf{E}^{1/2}],$$

where $\mathbf{E}^{1/2}$ is the functional from Assumption C (5.1.3).

The following properties of H are immediate:

1. H is a computation theory on $(\mathfrak{A}, \preccurlyeq)$ satisfying axioms A and C.
2. $\mathrm{en}(H) = \Sigma_1(\preccurlyeq, \mathbf{R})$.

We make just one comment on 1 and 2. By the construction of H we see that $\Sigma_1(\preccurlyeq, \mathbf{R}) \subseteq \mathrm{en}(H)$. To prove the converse we give a $\Sigma_1(\preccurlyeq, X, \mathbf{R})$ inductive definition of $\mathrm{en}(H)$, and use the admissibility of the pwo to conclude that $\mathrm{en}(H)$ is $\Sigma_1(\preccurlyeq, \mathbf{R})$.

5.1.12 Proposition. *Let $(\mathfrak{A}, \preccurlyeq)$ be* **R**-*admissible and assume that* $\Sigma_1(\preccurlyeq, \mathbf{R}) - \Delta_1(\preccurlyeq, \mathbf{R}) \neq \varnothing$. *Let θ be a $\Delta_0(\preccurlyeq, \mathbf{R})$ formula. Then*

$$(\forall x \prec u)(\exists y)\theta(x, y) \Rightarrow (\exists w)(\forall x \prec u)(\exists y \prec w)\theta(x, y)$$

(Σ_1-*collection principle*).

The assumption that $\Sigma_1 - \Delta_1 \neq \varnothing$ is not serious. Indeed, the results of Section 5.2 can be used to show that it can be omitted.

For the proof we need a sublemma:

5.1.13 Sublemma. $\Sigma(\preccurlyeq, \mathbf{R}) = \Sigma_1(\preccurlyeq, \mathbf{R}) = \mathrm{en}(H)$.

We need to show that every Σ relation is in $\mathrm{en}(H)$. As a typical case take

$$\forall x \prec u \exists y R(x, y),$$

where R is Δ_0. By assumption there exists an H-computable function f such that

$$R(x, y) \quad \text{iff} \quad f(x, y) \simeq 0.$$

Hence $\exists y R(x, y)$ iff $\{\exists_f\}(y) \simeq 0$, where $\exists_f$ is computable from f and the given index for $\mathbf{E}^{1/2}$ in H. Let g be introduced by $g(y) \simeq 1$ iff $\{\exists_f\}(y) \simeq 0$. Then

$$\forall x \prec u \exists y R(x, y) \quad \text{iff} \quad \mathbf{E}^{\preccurlyeq}(g, u) \simeq 1.$$

The construction is uniform in the parameters, and thus we can proceed by induction.

Back to the proof of 5.1.12: Choose a $\Sigma_1(\preccurlyeq, \mathbf{R})$ relation $U(\sigma)$ such that $\neg U \notin \Sigma(\preccurlyeq, \mathbf{R})$. We note that $|\preccurlyeq|$ is a limit ordinal, otherwise A would be H-finite.

Assume now:

1. $(\forall x \prec u)(\exists y)\theta(x, y)$.
2. $(\forall w)(\exists x \prec u)(\forall y \prec w)\neg\theta(x, y)$.

This means that for all w we must have some $x \prec u$ and some $y \succcurlyeq w$ such that $\theta(x, y)$. This is used in proving the equivalence in 4 below:

3. $U(\sigma)$ iff $\exists z U_0(\sigma, z)$,

where $U_0 \in \Delta_0(\preccurlyeq, \mathbf{R})$.

4. $$\begin{aligned} \neg U(\sigma) \quad &\text{iff} \quad (\forall z)\neg U_0(\sigma, z) \\ &\text{iff} \quad (\forall x \prec u)(\exists y)[\theta(x, y) \wedge (\forall t \prec y)\neg U_0(\sigma, t)]. \end{aligned}$$

From 4 it follows that $\neg U$ is $\Sigma(\preccurlyeq, \mathbf{R})$—a contradiction that proves Proposition 5.1.12.

Above we showed that en(H) is $\Sigma_1(\preccurlyeq, \mathbf{R})$ by constructing a Σ_1-inductive definition for en(H). If we are going to prove axiom B, in particular, the admissibility condition $\|H\| = |\preccurlyeq|$, we need to have an estimate of the ordinals of inductive definitions on $(\mathfrak{A}, \preccurlyeq)$ in terms of the ordinal of the pwo $\preccurlyeq$. We should expect that $|\Gamma_\theta| \leqslant |\preccurlyeq|$ for all $\Sigma_1(\preccurlyeq, X, \mathbf{R})$ inductive operators, which we indeed will prove in 5.1.15 below. In order to prove the equality $\|H\| = |\preccurlyeq|$, we must be able to carry out the construction of $H = \mathrm{PR}[\ldots]$ in sufficiently many steps. And then we must verify that the equality is "effective" in the sense of axiom B, 5.1.2. But that is the topic of the next section. Here we start by proving an auxiliary lemma.

5.1.14 Lemma. *Let $(\mathfrak{A}, \preccurlyeq)$ be $\mathbf{R}$-admissible and let $\theta(\sigma, X)$ define a monotone $\Sigma_1(\preccurlyeq, X, \mathbf{R})$ inductive operator. The relation*

$$P(\sigma, x) \quad \textit{iff} \quad \sigma \in \Gamma_{|x|},$$

is $\Sigma_1(\preccurlyeq, \mathbf{R})$. (*Note that* $\Gamma_{|x|}$ *is the x'th stage of the inductively defined set* Γ_∞.) *If* θ *is* $\Delta_0(\preccurlyeq, X, \mathbf{R})$, *the relation P is* $\Delta_1(\preccurlyeq, \mathbf{R})$.

The proof is carried out inside the associated theory $H = \mathrm{PR}[\mathbf{E}^{\preccurlyeq}, \preccurlyeq, \mathbf{R}, \mathbf{E}^{1/2}]$. First we construct a code $\hat{p}$ such that

$$\{\hat{p}\}(\sigma, x) \simeq 0 \quad \text{iff} \quad P(\sigma, x).$$

If θ is Δ_0, we also get a code $\hat{q}$ such that

$$\{\hat{q}\}(\sigma, x) \simeq 0 \quad \text{iff} \quad \neg P(\sigma, x).$$

We start the construction of $\hat{p}$ (and $\hat{q}$) by rewriting:

$$\begin{aligned} P(\sigma, x) \quad & \text{iff} \quad \sigma \in \Gamma_{|x|} \\ & \text{iff} \quad \sigma \in \Gamma_\theta\Big(\bigcup_{y \prec x} \Gamma_{|y|}\Big) \\ & \text{iff} \quad \theta(\sigma, \{\sigma' : (\exists y \prec x) P(\sigma', y)\}). \end{aligned}$$

Let f be the function defined by

$$f(p, \sigma, x) \simeq 0 \quad \text{iff} \quad \theta(\sigma, \{\sigma' : (\exists y \prec x)\cdot\{p\}(\sigma', y) \simeq 0\}).$$

We see that f is H-computable, since $\{p\}(\sigma', y) \simeq 0$ is H-semicomputable and $\theta(\sigma, X)$ is positive in X.

By the second recursion theorem there exists a code $\hat{p}$ such that $\{\hat{p}\}(\sigma, x) = (\hat{p}, \sigma, x)$. By induction we now verify that

$$\{\hat{p}\}(\sigma, x) \simeq 0 \quad \text{iff} \quad \sigma \in \Gamma_{|x|}.$$

Assume this true for all $y \prec x$:

$$\begin{aligned} \sigma \in \Gamma_{|x|} \quad & \text{iff} \quad \theta(\sigma, \{\sigma' : (\exists y \prec x)\cdot \sigma' \in \Gamma_{|y|}\}) \\ & \text{iff} \quad \theta(\sigma, \{\sigma' : (\exists y \prec x)\cdot\{\hat{p}\}(\sigma', y) \simeq 0\}) \\ & \text{iff} \quad f(\hat{p}, \sigma, x) \simeq 0 \\ & \text{iff} \quad \{\hat{p}\}(\sigma, x) \simeq 0. \end{aligned}$$

If θ is Δ_0, then

$$\neg P(\sigma, X) \quad \text{iff} \quad \neg\theta(\sigma, \{\sigma' : \neg(\forall y \prec x)\neg P(\sigma', y)\}),$$

is by Proposition 5.1.12 a $\Sigma_1(\preccurlyeq, \mathbf{R})$ relation. Hence there exists an H-computable g such that

$$g(q, \sigma, x) \simeq 0 \quad \text{iff} \quad \neg\theta(\sigma, \{\sigma' : \neg(\forall y \prec x)\cdot\{q\}(\sigma, y) \simeq 0\}).$$

We may then proceed as above to produce a code $\hat{q}$ such that $\{\hat{q}\}(\sigma, x) \simeq 0$ iff $\neg P(\sigma, x)$.

5.1.15 Proposition. *Let $(\mathfrak{A}, \preccurlyeq)$ be* **R**-*admissible and let $\theta(\sigma, X)$ define a monotone $\Sigma_1(\preccurlyeq, X, \mathbf{R})$ inductive operator. Then*

$$|\Gamma_\theta| \leqslant |\preccurlyeq|,$$

where Γ_θ is the inductive operator associated with θ.

Since $|\preccurlyeq|$ is a limit number, we must be able to show that $\sigma \in \Gamma_\theta(\Gamma_{|\preccurlyeq|})$ implies that $\exists x[\sigma \in \Gamma_\theta(\Gamma_{|x|})]$, i.e.

$$\theta(\sigma, \{\sigma' : \exists x P(\sigma', x)\}) \Rightarrow \exists x \cdot \theta(\sigma, \{\sigma' : P(\sigma', x)\}).$$

Or, more generally, we show that if $\theta'(X)$ is $\Sigma_1(\preccurlyeq, X, \mathbf{R})$ (in parameters from A), then

$$\theta'(\{\sigma : \exists x P(\sigma, x)\}) \Rightarrow \exists x \cdot \theta'(\{\sigma : P(\sigma, x)\}).$$

But this follows since θ' is positive in X, hence the $\exists$-quantifier can be advanced, e.g.

$$\begin{aligned}(\forall y \preccurlyeq z)(\exists x)P(\sigma, x) &\Rightarrow (\exists w)(\forall y \preccurlyeq z)(\exists x \preccurlyeq w)P(\sigma, x)\\ &\Rightarrow (\exists w)(\forall y \preccurlyeq z)P(\sigma, w),\end{aligned}$$

because of the monotone character of P.

5.2 The Characterization Theorem

We saw in Proposition 5.1.10 that if Θ is an infinite theory on a domain $(\mathfrak{A}, \preccurlyeq)$ and $\mathbf{R}$ is any sequence of relations extending the relations $\mathbf{R}_\theta$ of Proposition 5.1.8, then $(\mathfrak{A}, \preccurlyeq)$ is $\mathbf{R}$-admissible. In 5.1.11 we made a few steps toward proving a converse. We constructed a theory $H = \mathrm{PR}[\mathrm{E}^{\preccurlyeq}, \leqslant, \mathbf{R}, \mathsf{E}^{1/2}]$ which by the very construction satisfied Assumption A of 5.1.1 and Assumption C of 5.1.3. We further noted that $\mathrm{en}(H) = \Sigma_1(\preccurlyeq, \mathbf{R})$. And it is a consequence of Proposition 5.1.15 that $\|H\| \leqslant |\preccurlyeq|$, which goes some way toward verifying Assumption B of 5.1.2. Our program is now to make a more refined construction of H, in fact, slowing up the construction of H, so as to obtain the converse inequality $|\preccurlyeq| \leqslant \|H\|$. And by a careful analysis of the construction we shall be able to get the "effective" content of the equality $\|H\| = |\preccurlyeq|$, i.e. Assumption B.

Let $(\mathfrak{A}, \preccurlyeq)$ be a computation domain with a pwo $\preccurlyeq$. We assume that the code set of $\mathfrak{A}$ is equal to the whole domain and that $\mathfrak{A}$ includes a pairing structure. We start from the following basic assumption:

5.2.1 Assumption. Let $\mathbf{R} = R_1, \ldots, R_k$ be relations on A such that $(\mathfrak{A}, \preccurlyeq)$ is $\mathbf{R}$-admissible.

We shall now construct a theory $\Theta = \mathrm{PR}[[\preccurlyeq, \mathbf{R}]]$ which in many respects equals the prime recursion theory $H = \mathrm{PR}[\mathbf{E}^{\preccurlyeq}, \preccurlyeq, \mathbf{R}, \exists^{1/2}]$.

Our main problem in constructing $\mathrm{PR}[[\preccurlyeq, \mathbf{R}]]$ is to "delay" the definition of the inductive operator Γ such as to get in the end the inequality $\|\Theta\| = |\Gamma| \leqslant |\preccurlyeq|$. We use the following trick.

5.2.2 Definition. For every set Δ of tuples on A of length $\geqslant 2$ we set

$$\Delta^{+} = \{x : (\forall u \prec x)[(\langle 12, 0\rangle, u, u) \in \Delta]\}.$$

The intention is that $\langle 12, 0\rangle$ will be a special index for the identity function. Δ^{+} will always be an initial segment of A, and, in particular, $\emptyset^{+} = \{x \in A : |x| = 0\}$, where $|x|$ is the ordinal of x in the pwo $\preccurlyeq$.

5.2.3 Construction of the Inductive Operator Γ. We give a few but typical cases:

1. Successor function: If $\langle 1, 0\rangle, x, s(x) \in \Theta^{+}$, then

$$(\langle 1, 0\rangle, x, s(x)) \in \Gamma(\Theta).$$

2. Substitution: If $\langle 6, 0\rangle, \hat{f}, \hat{g}, \sigma, z \in \Theta^{+}$ and $\exists u \in \Theta^{+}[(\hat{f}, u, \sigma, z) \in \Theta \wedge (\hat{g}, \sigma, u) \in \Theta]$, then

$$(\langle 6, 0\rangle, \hat{f}, \hat{g}, \sigma, z) \in \Gamma(\Theta).$$

(The indices $\langle 2, 0\rangle, \ldots, \langle 5, 0\rangle$ are used for introducing the pairing structure and definition by cases. In the same way $\langle 7, 0\rangle$ is used for $\mathbf{P}$ and $\langle 8, \ldots\rangle$ for the *s-m-n* function.)

3. Introduction of $\preccurlyeq$ and $\mathbf{R}$*:* We let

$\preccurlyeq$ have the code $\langle 9, 1\rangle$
R_i have the code $\langle 9, 1 + i\rangle$,

and add the obvious inductive clauses.

4. Closure under $\exists$-quantification: If $\langle 10, 0\rangle, \hat{f}, \sigma, 0 \in \Theta^{+}$ and if $(\exists x \in \Theta^{+})\cdot[(\hat{f}, x, \sigma, 0) \in \Theta]$, then

$$(\langle 10, 0\rangle, \hat{f}, \sigma, 0) \in \Gamma(\Theta).$$

5. Introduction of the functional $\mathbf{E}^{\preccurlyeq}$*:* If $\langle 11, 0\rangle, \hat{f}, x, 0 \in \Theta^{+}$ and $(\exists y \prec x)\cdot[(\hat{f}, y, 0) \in \Theta]$, then

$$(\langle 11, 0\rangle, \hat{f}, x, 0) \in \Gamma(\Theta).$$

If $\langle 11, 0\rangle, \hat{f}, x, 1 \in \Theta^+$ and $(\forall y \prec x)[(\hat{f}, y, 1) \in \Theta]$, then

$$(\langle 11, 0\rangle, \hat{f}, x, 1) \in \Gamma(\Theta).$$

(Recall that Θ^+ is always an initial segment.)

6. *The identity function:* If $y \in \Theta^+$, then

$$(\langle 12, 0\rangle, y, y) \in \Gamma(\Theta).$$

Note how this clause differs slightly from the previous ones but, as subsequent lemmas will show, the difference is important.

The operator Γ is monotone, we define as usual

$$\Theta^\xi = \Gamma\left(\bigcup_{\eta<\xi} \Theta^\eta\right),$$

and can now introduce the associated theory.

5.2.4 Definition. Let **R** be introduced by Assumption 5.2.1 and Γ constructed by 5.2.3. We define

$$\Theta = \mathrm{PR}[[\preccurlyeq, \mathbf{R}]] = \Theta^\infty = \bigcup_\xi \Theta^\xi.$$

And for $(a, \sigma, z) \in \Theta$ we set

$$|a, \sigma, z|_\Theta = \text{least } \xi \text{ such that } (a, \sigma, z) \in \Theta^\xi.$$

5.2.5 Lemma. $(\langle 12, 0\rangle, y, y) \in \Theta^\xi$ *iff* $|y| \leqslant \xi$.

The proof is by induction, so assume that the lemma is true for all $\eta < \xi$. Let $(\langle 12, 0\rangle, y, y) \in \Theta^\xi - \bigcup_{\eta<\xi} \Theta^\eta$. By 6 of 5.2.3 this means that

$$y \in \left(\bigcup_{\eta<\xi} \Theta^\eta\right)^+,$$

i.e.$(\forall u \prec y)[(\langle 12, 0\rangle, u, u) \in \bigcup_{\eta<\xi} \Theta^\eta$. But this in turn means that $(\forall u \prec y)$. $(\exists n < \xi)[|u| \leqslant \eta]$, i.e. $|y| \leqslant \xi$. The argument also works in reverse which proves the lemma.

5.2.6 Lemma. *If* $a \neq \langle 12, 0\rangle$ *and* $(a, x_1, \ldots, x_n, z) \in \Theta^\xi$, *then* $|a|, |x_1|, \ldots, |x_n|, |z| \leqslant \xi$.

From the assumption that $(a, x_1, \dots, x_n, z) \in \Gamma(\bigcup_{\eta<\xi} \Theta^\eta)$ and $a \neq \langle 12, 0\rangle$, it follows that $a, x_1, \dots, x_n, z \in (\bigcup_{\eta<\xi} \Theta^\eta)^+$: But for any $y \in (\bigcup_{\eta<\xi} \Theta^\eta)^+$ we have the truth of

$$(\forall u \prec y)(\exists \eta < \xi)[(\langle 12, 0\rangle, u, u) \in \Theta^\eta].$$

By Lemma 5.2.5 this means that $(\forall u \prec y)(\exists \eta < \xi)[|u| \leqslant \eta]$, i.e. $|y| \leqslant \xi$.

5.2.7 Lemma. PR[[$\preccurlyeq$, **R**]] *with the given length function is a p-normal computation theory on* $(\mathfrak{A}, \preccurlyeq)$ *in which* $\preccurlyeq$, **R**, *and* $\mathbf{E}^{\preccurlyeq}$ *are computable.* en(PR[[$\preccurlyeq$, **R**]]) *is closed under* $\exists$*-quantification.*

This lemma is a direct consequence of the construction. For p-normality we can use the proof of Proposition 3.1.12 of Section 3.1, replacing the functional $\mathbf{E}_A$ of that proof by the functionals $\mathbf{E}^{1/2}$ and $\mathbf{E}^{\preccurlyeq}$ introduced in 4 and 5 of Construction 5.2.3.

5.2.8 Lemma. en(PR[[$\preccurlyeq$, **R**]]) $= \Sigma_1(\preccurlyeq, \mathbf{R})$.

There are two things to verify:

(i) The inductive operator of 5.2.3 is of $\Sigma_1(\preccurlyeq, X, \mathbf{R})$ form, hence by **R**-admissibility of $(\mathfrak{A}, \preccurlyeq)$, has a $\Sigma_1(\preccurlyeq, \mathbf{R})$ least fixed-point; it follows that en(PR[[$\preccurlyeq$, **R**]]) $\subseteq \Sigma_1(\preccurlyeq, \mathbf{R})$.

(ii) A simple analysis of $\Delta_0(\preccurlyeq, R)$ relations shows that they are computable in PR[[$\preccurlyeq$, **R**]]. Closure under $\exists$-quantifier shows that $\Sigma_1(\preccurlyeq, \mathbf{R}) \subseteq$ en(PR[[$\preccurlyeq$, **R**]]).

We also note that sc(PR[[$\preccurlyeq$, **R**]]) $= \Delta_1(\preccurlyeq, \mathbf{R})$, this being a consequence of p-normality (see Remark 5.1.4).

From what we have proved so far we see that PR[[$\preccurlyeq$, **R**]] satisfies Assumptions A and C. And it follows from Lemmas 5.1.15 and 5.2.5 that $\|$PR[[$\preccurlyeq$, **R**]]$\| = |\preccurlyeq|$, which is the first part of Assumption B. We proceed to a more detailed analysis of the construction.

5.2.9 Definition. To the operator $\Gamma(\Theta)$ we associate an operator $\Gamma(w, \Theta)$ obtained by relativizing the quantifiers in $\Gamma(\Theta)$ to "$\preccurlyeq w$" and replacing "$x \in \Theta^+$" by "$x \in \Theta^+ \wedge x \preccurlyeq w$". (Note that $\langle 12, 0\rangle$ is exempted from this restriction.) Θ_w^ξ is defined as usual.

5.2.10 Lemma. *If* $\xi \leqslant |w|$, *then* $\Theta^\xi = \Theta_w^\xi$.

The proof is by induction on subcomputations using Lemma 5.2.6 in an essential way.

5.2.11 Definition. Let Θ be a computation theory on $(\mathfrak{A}, \preccurlyeq)$. Set

$$X(\Theta) = \{(m, x) : m \in N \wedge m \geqslant 2 \wedge ((x)_1, \dots, (x)_m) \in \Theta\}$$
$$\mathbf{\Theta}(X) = \{((x)_1, \dots, (x)_m) : (m, x) \in X\}.$$

Introduce the following operators:

$$\bar{\Gamma}(X) = X(\Gamma(\boldsymbol{\Theta}(X))).$$
$$\bar{\Gamma}(w, X) = X(\Gamma(w, \boldsymbol{\Theta}(X))).$$

The corresponding stages X^{ξ} and X_w^{ξ} are defined as usual.

Note that we have the following equalities:

$$X_w^{\xi} = X(\Theta_w^{\xi}) \qquad X^{\xi} = X(\Theta^{\xi})$$
$$\Theta_w^{\xi} = \boldsymbol{\Theta}(X_w^{\xi}) \qquad \Theta^{\xi} = \boldsymbol{\Theta}(X^{\xi}),$$

i.e. coding and decoding are carried along the stages.

5.2.12 Lemma. PR[[$\preccurlyeq$, **R**]] *satisfies Assumption* B *of* 5.1.2.

We first note that the operator $\bar{\Gamma}(x, X)$ is defined by a $\Delta_0(\preccurlyeq, X, R)$ positive formula, hence by Lemma 5.1.14 the relation

$$(m, x) \in X_w^{|y|},$$

is $\Delta_1(\preccurlyeq, \mathbf{R})$.

By Lemma 5.2.10 we have

$$\begin{aligned}(m, x) \in X^{|y|} \quad &\text{iff} \quad (\exists w)[y \preccurlyeq w \ \wedge \ (m, x) \in X_w^{|y|}] \\ &\text{iff} \quad (\forall w)[y \preccurlyeq w \ \rightarrow \ (m, x) \in X_w^{|y|}],\end{aligned}$$

so the relation $(m, x) \in X^{|y|}$ is $\Delta_1(\preccurlyeq, \mathbf{R})$.

Since $\bar{\Gamma}$ is defined by a $\Sigma_1(\preccurlyeq, X, \mathbf{R})$ formula, we get $|\bar{\Gamma}| \leqslant |\preccurlyeq|$ by Proposition 5.1.15. Hence if $(m, x) \in X(\Theta)$, there is a w such that $[m, x] = |w|$, where $[m, x]$ is the ordinal of the coded computation tuple (m, x). But

$$[m, x] = |w| \quad \text{iff} \quad (m, x) \in X^{|w|} \wedge (\forall u \prec w)[(m, x) \notin X^{|u|}],$$

so the relation $[m, x] = |w|$ is $\Delta_1(\preccurlyeq, \mathbf{R})$, i.e. PR[[$\preccurlyeq$, **R**]]-computable, which is the substance of (ii) in Assumption B.

5.2.13 Theorem. *Let* $(\mathfrak{A}, \preccurlyeq)$ *be an* **R**-*admissible prewellordering. There exists a p-normal computation theory* $\Theta =$ PR[[$\preccurlyeq$, **R**]] *on* $\mathfrak{A}$ *satisfying A, B, and C such that* $\mathrm{en}(\Theta) = \Sigma_1(\preccurlyeq, \mathbf{R})$ *and* $\mathrm{sc}(\Theta) = \Delta_1(\preccurlyeq, \mathbf{R})$.

The development of Sections 5.1 and 5.2 is patterned on Section 10 of Moschovakis [113]. There are many differences due to the fact that he uses multiple-valued theories whereas we have insisted on single-valued ones. But many of the key technical points are taken from his paper.

We stop short with Theorem 5.2.13. One could go on to investigate to what extent an infinite theory is characterized by its associated pwo: If we start with an infinite theory Θ and construct the associated pwo (5.1.10) with respect to the relations $\mathbf{R}_\Theta$ and then pass to the theory PR[[$\preccurlyeq$, $\mathbf{R}_\Theta$]], are then the two theories equivalent? The answer is "almost", we need to be more careful in the construction of PR[[$\preccurlyeq$, $\mathbf{R}_\Theta$]]; the interested reader may consult theorem (xx) of Section 10 in Moschovakis [113] for technical inspiration and then prove a similar result in the present framework.

5.2.14 Remark. If a multiple-valued theory Θ has a pmv selection operator $q(a, \sigma)$, we get a "nice" theory for en(Θ) in a straightforward way, see Section 1.3.

It could have been the case that multiple-valued selection was necessary for theories associated with an admissible pwo. In a set-theoretic context we assume computability of the union operator, i.e. if $x \in A$, then $f(x) = \bigcup x$ is also an element of A (f is a *rudimentary operation*). And if x is a unit set, then $f(x) = \bigcup x$ is the unique element of x, i.e. we have a special kind of selection operator. This is precisely what we need in proving that $R(\sigma)$ is computable if both $R(\sigma)$ and $\neg R(\sigma)$ are semicomputable. We are led to the construction of a set B_σ (uniformly in σ) which has 0 as its only member if $R(\sigma)$ and 1 as its only member if $\neg R(\sigma)$. $f(B_\sigma) = \bigcup B_\sigma$ would then be the characteristic function of $R(\sigma)$, proving that it is computable.

Over an arbitrary **R**-admissible pwo $(\mathfrak{A}, \preccurlyeq)$ the operation $\bigcup$ makes no sense. So we use multiple valued search: $q(a, \sigma)$ selects a set of elements satisfying a condition $\{a\}(x, \sigma) \simeq 0$. And if for all σ there is a unique x satisfying the condition, then $q(a, \sigma)$ defines a single-valued mapping. (The reader should at this point look back at the proof of Theorem 1.3.4.)

The union operator leads to a new element of the domain, multiple-valued selection leads to a subset. It is an interesting and important technical point that one can extend the formalism of recursion theory to include pmv functions. They can even be made to work in the context of priority arguments, see Stoltenberg-Hansen [163]. But we may have *conceptual* doubts, strong enough to resist a technical point, however ingenious and elegant.

5.3 The Imbedding Theorem

The interplay of recursion theory and set theory has been a rich source of ideas for the general theory. In this section we shall analyze the "next-admissible" set/ordinal construction. We start by tracing some of the history.

"Higher" recursion theory started in the mid 1950's with S. C. Kleene's work on *the analytic hierarchy*, *constructive ordinals*, and *the hyperarithmetic sets* [79–81].

This work was followed by a number of basic contributions by *Clifford Spector* (1930–1961).

(i) In his paper of 1955, *Recursive wellorderings* [158], he introduced and used as a basic tool the boundedness theorem for hyperarithmetic theory (i.e. every Σ_1^1 subset of $\mathcal{O}$, the set of ordinal notations, is bounded by some recursive ordinal).

(ii) In his 1959 paper *Hyperarithmetic quantifiers* [160] he proved that Π_1^1 equals Σ_1^1 on HYP, i.e. every Π_1^1 predicate $P(a)$ can be written in the form

$$P(a) \quad \text{iff} \quad \exists\alpha(\alpha \in \text{HYP} \wedge R(\alpha, a)),$$

where HYP is the set of hyperarithmetic functions and R is an arithmetic relation.

(iii) In his study of *Ordinals of inductive definitions* in [161] he established a number of basic results, in particular, that

$$|\Pi_1^1\text{-mon}| = |\Pi_1^0| = \omega_1,$$

ω_1 being the first non-recursive ordinal.

(iv) Less known, and published only as an abstract, is a paper from the 1957 Cornell Summer Institute for Symbolic Logic, *Recursive ordinals and predicative set theory* [159]. Restated in modern terminology he proves that L_{ω_1} is the collection of hereditarily hyperarithmetic sets. (See also Hao Wang's contribution [171] to the same meeting where, in particular, the connection with Godel's notion of *constructibility* is emphasized.)

Writing history is not easy. Too often one restates and interprets the past from our present and "correct" point of view. As a result one is often amazed at the lack of insight sometimes shown by our predecessors.

Thus from our "correct" point of view it is remarkable that Spector did not put the pieces together and arrive at our way of looking at general recursion theory. From (ii) and (iv) he "knew" that Π_1^1 corresponds to Σ_1 on L_{ω_1}. From (iii) he "had" the connection with inductive definability. And a basic tool in the proofs was the boundedness theorem of (i), which is the clue to the set-theoretic description of L_{ω_1}, i.e. to admissibility theory. It was all in his hands!

Perhaps, not. The paper referred to in (iv) was related to Hao Wang's program for building a constructive or predicative foundation for mathematics. At the time it could be described as an example of "applied" recursion theory. Today it can be viewed as a central piece of the "pure" theory.

But Spector, who died 31 years old, had some of the basic technical results which we have exploited over and over again in arriving at our present "correct" version of the theory.

Remark. In his paper of 1959, *Quantification of number-theoretic functions* [84], Kleene proves that the ramified analytic hierarchy up to ω_1, R_{ω_1}, equals Δ_1^1. There is no reference to Spector's Cornell paper [159]. Did one not see any connection?

We have briefly focused on the work of Spector. It soon flowered into a rich general theory. Kreisel's and Sacks' 1965 paper, *Metarecursive sets* [91], marked an important stage. It gave both a conceptual analysis of the fundamental notions, in particular, of the notion of *finiteness* (see the introduction to Chapter 3), and contributed significantly to the techniques of ordinal recursion theory. At the same time, S. Kripke [92] developed the more general theory of recursion on an arbitrary *admissible ordinal.* A topic developed independently by R. Platek in his 1966 Stanford thesis [133], where also the theory of *admissible sets* is studied as an important part of the general theory.

In all this work the connection between hyperarithmetic theory and L_{ω_1} was of central importance. The metarecursion theory of Kreisel and Sacks was in a sense "constructed" out of hyperarithmetic theory and notations for recursive ordinals. And what had been a guiding principle was soon made into a precise theory: the "next-admissible" set/ordinal construction.

The basic reference is the 1971 paper of Barwise, Gandy, and Moschovakis, *The next-admissible set* [14]. Let A be a transitive set closed under the formation of unordered pairs. Define

$$A^+ = \bigcap \{M;\ M \text{ is admissible and } A \in M\}.$$

The first basic result in their study is:

(1) A^+ is admissible, in fact,
$A^+ = L_\kappa(A)$,
where $\kappa = \sup\{|\Gamma| : \Gamma \text{ is first-order positive inductive operator on } A\}$.

This result was proved using the theory of hyperprojective sets of Moschovakis [112]. A further result is

(2) A subset S of A is hyperprojective iff $S \in A^+$.

In the context of hyperarithmetic on ω hyperprojective theory is the same as Δ^1_1. In this case (1) constructs L_{ω_1} and (2) asserts that a subset of ω is Δ^1_1 iff it is an element of L_{ω_1}.

Hyperprojective theories are a special class of Spector theories. The above results immediately call for a generalization. This was provided in the context of Spector classes over transitive sets by Moschovakis in Chapter 9 of his book [115] on elementary induction. We discussed the problem in the context of Spector theories over general computation domains in our *On axiomatizing recursion theory* [26]. A proof of these results would use the theory of admissible sets with urelements, which was developed by Barwise [10] at the same time.

The "next-admissible" ordinal construction was carried out by P. Aczel [5] at the same time, formalizing the construction of metarecursion theory from hyperarithmetic theory.

It is time to be technical. Let Θ be a Spector theory on a computation domain $\mathfrak{A}$. There are two basic objects associated with Θ. First, an ordinal $\|\Theta\| = \sup\{|a, \sigma, z|;\ (a, \sigma, z) \in \Theta\}$. And next a relation R defined by

$$R(x, \alpha) \quad \text{iff} \quad x \in \Theta \wedge |x|_\Theta = \alpha.$$

These two objects are the essential ingredients in the "next-admissible" construction.

If we want to construct the next-admissible set, we should look at $L_{\|\Theta\|}[A; R]$, where A, the underlying set of $\mathfrak{A}$, is a set of urelements. If we want to construct the next-admissible ordinal we should look for a two-sorted theory $(A, \|\Theta\|)$, i.e. an admissible ordinal with urelements.

Both approaches are possible. But we shall follow a third way, and look for an R-admissible prewellordering above the given Spector theory Θ on $\mathfrak{A}$. There is an immediate candidate.

Let Θ be a Spector theory on $\mathfrak{A} = \langle A, A, \ldots \rangle$. Define the ordinal $\|\Theta\|$ as above. Let us be a bit more careful with the relation R:

$$R(x, \alpha) \text{ iff } x = \langle n, \langle x_1, \ldots, x_n \rangle\rangle \wedge (x_1, \ldots, x_n) \in \Theta \wedge |x_1, \ldots, x_n|_\Theta = \alpha.$$

5.3.1 Definition. We define a prewellordering $(\mathfrak{A}^*, \preccurlyeq)$ by setting

$$A^* = A \times \|\Theta\|,$$

and

$$(x, \alpha) \preccurlyeq (y, \beta) \text{ iff } \alpha \leqslant \beta.$$

For $(\mathfrak{A}^*, \preccurlyeq)$ we construct an appropriate language $L^*(\preccurlyeq, R)$ and introduce as usual classes $\Delta_0(\preccurlyeq, R)$ and $\Sigma_1(\preccurlyeq, R)$.

5.3.2 Lemma. *Let $B \subseteq A$. Then $B \in \mathrm{en}(\Theta)$ iff B is $\Sigma_1(\preccurlyeq, R)$.*

(We use the simple imbedding $x \to (x, 0)$ of A into A^*.) For the proof assume first that B is Θ-semicomputable, i.e. for some code a,

$$\begin{aligned} x \in B \text{ iff } & (a, x, 0) \in \Theta \\ \text{iff } & \exists y \in A^*[(y)_0 = \langle 3, \langle a, x, 0\rangle\rangle \wedge R(y)]. \end{aligned}$$

Remember that we have coding-decoding in a pwo.

Conversely, assume that B is $\Sigma_1(\preccurlyeq, R)$. Then for some $\Delta_0(\preccurlyeq, R)$ formula Φ,

$$\begin{aligned} x \in B \text{ iff } & \exists y \in A \times \|\Theta\| \cdot \Phi((x, 0), y) \\ \text{iff } & \exists a, z, w \in A[(a, z, 0) \in \Theta \wedge \Phi((x, 0), (w, |a, z, 0|))]. \end{aligned}$$

Here the matrix is Θ-semicomputable. Note that bounded quantification in Φ can be handled by *finiteness* and *prewellordering*, the characteristic properties of a Spector theory.

It remains to show that the pwo $(\mathfrak{A}^*, \preccurlyeq)$ is R-admissible. So let $\theta(x, X)$ be a $\Sigma_1(\preccurlyeq, X, R)$ formula in which X occurs positively. We must show that the least fixed-point of the associated inductive operator Γ_θ is $\Sigma_1(\preccurlyeq, R)$. In view of Lemma 5.3.2 all we need to do is to use the first recursion theorem for the underlying Spector theory.

With Θ we associate a Θ-computable functional φ in the following way (here we think of a as (a_1, a_2)):

$$\varphi(f, a) \simeq 0 \quad \text{iff} \quad a_2 \in \Theta \wedge \theta((a_1, |a_2|), \{(b_1, |b_2|) : \exists b[R((b, |b_2|) \wedge f((b_1, b)) \simeq 0]\}).$$

By necessity there is some coding involved!

Let f^* be the least—and Θ-computable—fixed-point for φ. Define X^* by

$$x \in X^* \quad \text{iff} \quad \exists b[R(b, x_2) \wedge f^*((x_1, b)) \simeq 0].$$

By Lemma 5.3.2 X^* is $\Sigma_1(\preccurlyeq, R)$. Assume now that $\theta(x, X^*)$ is true, we must verify that $x \in X^*$. Now x will be of the form (x_1, α) where $x_1 \in A$ and $\alpha < \|\Theta\|$. Let x' be (x_1, b) where $|b|_\Theta = \alpha$. We then get $\varphi(f^*, (x_1, b)) \simeq 0$, from which we conclude $f^*((x_1, b)) \simeq 0$. But then it follows that $\exists b[R(b, \alpha) \wedge f^*((x_1, b)) \simeq 0]$, i.e. $x \in X^*$, which was what we had to prove. And minimality of X^* follows from the minimality of f^*. QED.

5.3.3 Theorem. *Let Θ be a Spector theory on a computation domain $\mathfrak{A}$. Let $(\mathfrak{A}^*, \preccurlyeq)$ be the* pwo *introduced in Definition* 5.3.1 *and let $R(x, \alpha)$ be the relation: $x \in \Theta \wedge |x|_\Theta = \alpha$. Then*

(i) $(\mathfrak{A}^*, \preccurlyeq)$ *is R-admissible:*

(ii) *A subset $B \subseteq A$ is Θ-semicomputable iff it is $\Sigma_1(\preccurlyeq, R)$ under the imbedding* $x \to (x, 0)$.

This is our version of the "next-admissible" construction. And we feel that our analysis has isolated the crucial recursion-theoretic content: *the first recursion theorem.*

Adding Theorem 5.2.13 to the above construction yields a good infinite theory Θ^* over Θ. And, as we will show in the next chapter, it is this infinite theory Θ^* which will be the setting for "post-Fridberg" recursion theory, i.e. priority arguments and fine structure theory, which by "pull-back" should yield information about the given theory Θ.

5.3.4 Remark. A similar point of view was taken in Chapter 6 of Barwise, *Admissible Sets and Structures* [11]. He used his construction $\mathrm{HYP}_{\mathfrak{M}}$, the "next-admissible" over $\mathfrak{M}$, to develop the theory of inductive definability over $\mathfrak{M}$. (See a remark on this in connection with Example 3.3.7.)

5.4 Spector Theories Over ω

Two important examples of Spector theories over ω are: (i) prime recursion in a total, normal type-2 functional, and (ii) prime recursion in ${}^2E$ and a consistent partial functional $F_{\mathbf{Q}}^{\#}$, derived from a monotone quantifier $\mathbf{Q}$ (see Example 3.1.3). From the representation Theorem 3.2.9 we know that these examples are exhaustive. One question remains: *Can we characterize those Spector theories which are equivalent to prime recursion in some total and normal type-2 functional?*

In discussing this question we shall have more to say about the interplay between set theory and recursion theory, thus continuing the discussion of the last section.

Let Θ and Ψ be Spector theories on ω. Theorem 3.2.8 tells us that $\Theta \sim \Psi$ iff $\mathrm{en}(\Theta) = \mathrm{en}(\Psi)$ and that $\Psi \leqslant \Theta$ iff $\mathrm{en}(\Psi) \subseteq \mathrm{en}(\Theta)$. We shall introduce a special notion for "strictly less than". But first a piece of notation: For any theory Θ we shall use α_Θ for the ordinal $\|\Theta\|$ of the theory. If F is a normal type-2 functional over ω we use α_F to denote the ordinal of the theory $\mathrm{PR}[F]$.

5.4.1 Definition. Let Θ and Ψ be Spector theories over ω. We define

$$\Psi <_1 \Theta \quad \text{iff} \quad \mathrm{en}(\Psi) \subseteq \mathrm{en}(\Theta) \wedge \alpha_\Psi < \alpha_\Theta.$$

5.4.2 Remark. We note the obvious consequences of the definition: If $\Psi \leqslant \Psi'$, $\Psi' <_1 \Theta'$, $\Theta' \leqslant \Theta$, then $\Psi <_1 \Theta$. And $\Psi <_1 \Theta$ implies that $\mathrm{en}(\Psi) \subsetneqq \mathrm{en}(\Theta)$.

We remind the reader that a functional F is Θ-computable if it is weakly Θ-computable, i.e. $F(g, \sigma)$ is Θ-computable if for some primitive recursive $f\colon \omega \to \omega$

$$F(\lambda\tau\cdot\{e\}_\Theta(\tau, \sigma_1), \sigma) \simeq \{f(n)\}_\Theta(e, \sigma_1, \sigma).$$

5.4.3 Remark. The following facts are immediate

(i) If F is Θ-computable, then $\mathrm{PR}[F] \subseteq \Theta$.
(ii) If Θ is Spector and $\Theta \sim \mathrm{PR}[F]$, then F is Θ-computable.

If Θ is a Spector theory and F is a Θ-computable total functional such that $\alpha_F = \alpha_\Theta$ it could still happen that $\mathrm{en}(F)$ (i.e. $\mathrm{en}(\mathrm{PR}[F])$) was strictly contained in $\mathrm{en}(\Theta)$. F could be too "thin" to code up all computations in Θ. What the next result shows is that we can use F to construct another normal and total G such that $\mathrm{PR}[G] \sim \Theta$.

5.4.4 Fattening Lemma. *Let Θ be a Spector theory and F a total Θ-computable functional such that $\alpha_F = \alpha_\Theta$. Then there is a total normal G such that $\Theta \sim$* $\mathrm{PR}[G]$.

For the proof we first pick an index e_1 such that $\{e_1\}_\Theta(e)\downarrow$ iff $\lambda x\{e\}_\Theta(x)$ is total, in which case $|\{e\}_\Theta(x)| < |\{e_1\}_\Theta(e)|$, for all $x \in \omega$. We further define for f a total function from ω to ω:

5.4.5 Definition. $\mathrm{Ord}(f) = \inf\{|\{e_1\}_\Theta(e)|_\Theta : f = \lambda x\cdot\{e\}_\Theta(x)\}$.

Using $\mathrm{Ord}(f)$ as a "cut-off" we construct a functional G_0 as follows:
If $\mathrm{Ord}(f)$ is defined, then

$$G_0(\langle f, e, \sigma\rangle) = \begin{cases} \{e\}_\Theta(\sigma) + 1 & \text{if } |\{e\}_\Theta(\sigma)| \leqslant \mathrm{Ord}(f) \\ 0 & \text{otherwise,} \end{cases}$$

where $\langle f, e, \sigma\rangle = \lambda x \cdot \langle f(x), e, \sigma\rangle$. To make G_0 total we set $G_0(g) = 0$ if g is not of the form $\langle f, e, \sigma\rangle$, or if $g = \langle f, e, \sigma\rangle$ and $\mathrm{Ord}(f)$ is undefined.

We note first that G_0 is Θ-computable. We have the following instructions for computing G_0 on a function φ with index e': First use ${}^2E$ to check if φ is total. If φ is total, check if $\varphi = \langle f, e, \sigma\rangle$ for some f, e, σ. If not, let $G_0(\varphi) = 0$. If $\varphi = \langle f, e, \sigma\rangle$, then $\mathrm{Ord}(f)\downarrow$ since φ is Θ-computable. It remains to check if $|\{e\}_\Theta(\sigma)| \leqslant \mathrm{Ord}(f)$. But since Θ is Spector, the relation

$$\forall h(f = \lambda x\{h\}_\Theta(x) \Rightarrow |\{e\}_\Theta(\sigma)|_\Theta \leqslant |\{e_1\}_\Theta(h)|_\Theta),$$

is Θ-computable. Hence we have Θ-computability of G_0. To ensure normality, take the join of G_0 and ${}^2E$, which we denote by G_0'. To ensure that G_0' does not close off too soon, take further the join of G_0' and the F we started with, denote the result by G.

One half of the lemma is now immediate, $\mathrm{en}(G) \subseteq \mathrm{en}(\Theta)$. The converse needs more work. First define

$$\kappa = \sup\{\mathrm{Ord}(f) : f \text{ is computable in } G\}.$$

Claim. $\kappa = \alpha_\Theta$.

Granted this claim the "fattening lemma" follows immediately. Suppose $\{e\}_\Theta(\sigma)\downarrow$. By the claim there is an index m such that $f = \lambda t \cdot \{m\}_G(t)$ is total and $|\{e\}_\Theta(\sigma)| \leqslant \mathrm{Ord}(f)$. Hence $G_0(\langle \lambda t \cdot \{m\}_G(t), e, \sigma\rangle) \simeq \{e\}_\Theta(\sigma) + 1$. Since $\mathrm{PR}[G]$ is Spector we have a selection operator $\nu(e, \sigma)$, and we can define

$$\{e\}_\Theta(\sigma) \simeq G(\langle \lambda t \cdot \{\nu(e, \sigma)\}_G(t), e, \sigma\rangle) - 1.$$

Thus $\mathrm{en}(\Theta) \subseteq \mathrm{en}(G)$.

To prove the claim we first note that if f is G-recursive, then f is Θ-computable, hence $\mathrm{Ord}(f)\downarrow$ and $< \alpha_\Theta$. Therefore, $\kappa \leqslant \alpha_\Theta$.

To prove the converse we use the fact that $\alpha_F = \alpha_\Theta$. Hence $\alpha_G = \alpha_\Theta$, since obviously $\alpha_F \leqslant \alpha_G \leqslant \alpha_\Theta$. Thus we have to prove that $\alpha_G \leqslant \kappa$. Since α_G is the supremum of the lengths of G-recursive prewellorderings on ω, it suffices to prove that for each such prewellordering there is a G-recursive f such that $\mathrm{Ord}(f) \geqslant$ the length of the prewellordering. This is an exercise in the use of the second recursion theorem, which we omit. However, we shall in Chapter 7 return to this point in the context of computation theories on two types.

5.4.6 Definition. A Spector theory Θ on ω is called Θ-*Mahlo* if for all normal Θ-computable F there exists a Spector theory Θ' such that

(i) $\Theta' <_1 \Theta$,
(ii) F is Θ'computable.

Below we shall comment on the relationship of this notion of Mahloness to the notion of Mahloness in ordinal recursion theory. Here we have the characterization theorem.

5.4.7 Theorem. *Let Θ be a Spector theory on ω. Then Θ is equivalent to* PR[G] *for a normal, total type-2 G iff Θ is not Θ-Mahlo.*

We have here restricted ourselves to ω as domain. A more general result is valid. We return to this in Chapter 7 in the context of recursion theories on two types.

Half of the theorem is immediate: Let $\Theta \sim \mathrm{PR}[G]$, then, using G in Definition 5.4.6, we see that there is no Θ' satisfying (i) and (ii). Conversely, assume that Θ is not Θ-Mahlo. Hence there exists a normal Θ-computable F such that, in particular, PR[F] is *not* $<_1$ than Θ. Since by Remark 5.4.3 $\mathrm{en}(F) \subseteq \mathrm{en}(\Theta)$, this means that $\alpha_F = \alpha_\Theta$. Hence, by the "Fattening Lemma" 5.4.4 there exists a total, normal G such that $\Theta \sim \mathrm{PR}[G]$.

"Mahlo is Mahlo", we shall prove that the definition of Θ-Mahlo in 5.4.6 is the same as the ordinal-theoretic notion of Mahloness. And this will give us an opportunity to elaborate further on the relationship between Spector theories on ω and admissibility theory.

From a Spector theory Θ we derive an ordinal $\alpha_\Theta = \|\Theta\| = \sup\{|a, \sigma, z|_\Theta : (a, \sigma, z) \in \Theta\}$ and a relation $R_\Theta(x, \alpha)$ iff $x \in \Theta \wedge |x|_\Theta = \alpha$. From α_Θ and R_Θ we can construct the admissible set $L_{\alpha_\Theta}[R_\Theta]$. And we know from the imbedding theorem that if Θ is Spector, then α_Θ is R-admissible. (We shall further discuss this in connection with the one-section Theorem 5.4.24.)

As a variant of the standard procedure we shall now look at admissible ordinals from the standpoint of Spector theories. Θ is given. For $\tau \leqslant \alpha_\Theta$ introduce

$$\Theta_\tau = \{(a, \sigma, z) : \{a\}_\Theta(\sigma) \simeq z \wedge |\{a\}_\Theta(\sigma)|_\Theta < \tau\}.$$

5.4.8 Definition. τ is called Θ-*admissible* if Θ_τ is a Spector theory.

5.4.9 Definition. (i) A relation R on ω is called Θ_τ-*semicomputable* if there exists an index e such that

$$R(\sigma) \quad \text{iff} \quad |\{e\}_\Theta(\sigma)| < \tau.$$

(ii) Let $\pi\colon \tau^n \to \tau$ be a (partial) function. π is called Θ_τ-*computable* if the relation

$$\{(x, \ldots, y) : |x|_\Theta, \ldots, |y|_\Theta < \tau \wedge \pi(|x|_\Theta, \ldots) \simeq |y|_\Theta\},$$

is Θ_τ-semicomputable.

(iii) π is called Θ-*computable* if it is Θ_{α_Θ}-computable.

5.4.10 Definition. An ordinal $\tau \leqslant \alpha_\Theta$ is called Θ-*Mahlo* iff

(i) τ is Θ-admissible.

(ii) Every normal Θ_τ-computable function π has a Θ-admissible fixed-point less than τ.

We use standard set-theoretic terminology: π as a function from ordinals to ordinals is *normal* if it is strictly increasing and continuous at limit ordinals.

We give two simple results to show that the above definitions are the standard ones.

5.4.11 Proposition. *Let Θ be a Spector theory on ω. Let $\nu < \alpha_\Theta$ and π a partial Θ-computable functional. If $\pi(\xi)\downarrow$ for all $\xi < \nu$, then there exists a $\nu' < \alpha_\Theta$ such that $\pi(\xi) < \nu'$, for all $\xi < \nu$.*

For the proof note that the set

$$\{(x, y) : |x|_\Theta, |y|_\Theta < \alpha_\Theta \quad \text{and} \quad \pi(|x|) \simeq |y|\},$$

is Θ-semicomputable. Being in the Spector case, we have a selection function $\nu(x)$ such that if $\exists y[\pi(|x|) \simeq |y|]$, then $\nu(x)\downarrow$ and $\pi(|x|) \simeq |\nu(x)|$. It is not difficult to construct a computation x_0 such that $|\nu(x)|_\Theta < |x_0|_\Theta$ for all x such that $|x|_\Theta < \nu$. Let $\nu' = |x_0|_\Theta$.

5.4.12 Proposition. *Let F be a total, normal type-2 functional on ω. Then α_F is the least F-admissible ordinal.*

F-admissible is, of course, the same as PR[F]-admissible. For the proof let τ be a limit ordinal such that $\omega < \tau < \alpha_F$. We must show that τ is not F-admissible. Since $\tau < \alpha_F$ there must be F-computations of length τ, and since τ is a limit ordinal this computation must be an application of F to some function $\lambda x\{e\}_F(x, \sigma)$, where the function is total and $\tau = \sup\{|\{e\}_F(x, \sigma)|_F + 1 : x \in \omega\}$. Define π as $\pi(n) = |\{e\}_F(n, \sigma)|_F$. Then $\tau = \sup\{\pi(n) : n \in \omega\}$. If τ were F-admissible, then π would be F_τ-computable. 5.4.11 would then tell us that $\sup\{\pi(n) : n \in \omega\} < \tau$, a contradiction.

5.4.13 Remarks. These results should convince the reader that we are just looking at admissibility theory from a different point of view. We can even prove more: If α is admissible and projectible to ω (i.e. there is a one-one mapping π from α into ω which is α-recursive in constants less than α) then there is a Spector theory Θ on ω such that $\alpha_\Theta = \alpha$. We shall return to this point below in connection with the one-section result.

We claimed above that "Mahlo is Mahlo":

5.4.14 Theorem. *Let Θ be a Spector theory on ω. Then Θ is Θ-Mahlo (in the sense of Definition 5.4.6) iff α_Θ is Θ-Mahlo (in the sense of Definition 5.4.10).*

The proof is split into two lemmas.

5.4.15 Lemma. *Let Θ be a Spector theory on ω and F a normal Θ-computable type-2 functional. There exists a Θ-computable normal function π with no Θ-admissible fixed-points $< \alpha_F$.*

5.4.16 Lemma. *Let Θ be a Spector theory on ω and π a normal Θ-computable function. There exists a Θ-computable normal F such that α_F is Θ-admissible and a fixed-point for π.*

The theorem is a simple combination of the lemmas. Let first α_Θ be Θ-Mahlo and F a normal Θ-computable functional. By 5.4.15 there is a π with no Θ-admissible fixed-points $< \alpha_F$. But α_Θ is Θ-Mahlo, so π has Θ-admissible fixed points $< \alpha_\Theta$. Thus $\alpha_F < \alpha_\Theta$, and Θ is easily seen to be Θ-Mahlo. Conversely, let Θ be Θ-Mahlo and π a normal Θ-computable function. By 5.4.16 we have a Θ-computable F such that α_F is Θ-admissible and a fixed-point for π. Since Θ is Θ-Mahlo, $\alpha_F < \alpha_\Theta$. Thus α_Θ is Θ-Mahlo.

It remains to prove the lemmas; the reader not interested in the technical details may move on to Remark 5.4.17.

For Lemma 5.4.15 we first note that since F is Θ-computable there is an index t such that

$$x \in \mathbf{C}_F \quad \text{iff} \quad \langle t, x \rangle \in \mathbf{C}_\Theta,$$

where $\mathbf{C}_\Theta$ is, as before, the coded set of convergent computations. Use now Proposition 5.4.11 to prove the following two facts:

If $\nu < \alpha_\Theta$, there exists $\mu < \alpha_\Theta$ such that for all x,

$$|x|_F < \nu \Rightarrow |\langle t, x \rangle|_\Theta < \mu.$$

If $\nu < \alpha_\Theta$, there exists $\mu < \alpha_\Theta$ such that for all x,

$$|\langle t, x \rangle|_\Theta < \nu \Rightarrow |x|_F < \mu.$$

This done, define π as follows: $\pi(0) = 0$ and π is continuous at limit ordinals. $\pi(\nu + 1)$ is the least ordinal μ such that

(i) $\pi(\nu) < \mu$.
(ii) For all x, $|x|_F < \nu \Rightarrow |\langle t, x \rangle|_\Theta < \mu$,
(iii) For all x, $|\langle t, x \rangle|_\Theta < \nu \Rightarrow |x|_F < \mu$.

Using the second-recursion theorem we see that π is Θ-computable. π is normal by construction, and $\pi(\nu) < \alpha_\Theta$ whenever $\nu < \alpha_\Theta$. It remains to verify that π has no Θ-admissible fixed-points $< \alpha_F$.

Assume that $\tau < \alpha_F$, τ a limit ordinal, and $\pi(\tau) = \tau$. As before we have an index e such that the function $\lambda x \cdot \{e\}_F(x, \sigma)$ is total, $|\{e\}_F(x, \sigma)|_F < \tau$ for all x, and

(iv) $\quad \sup\{|\{e\}_F(x, \sigma)|_F : x \in \omega\} = \tau.$

From (ii) we conclude that $|\langle t, \langle e, x, \sigma\rangle\rangle|_\Theta < \pi(\tau) = \tau$, all x. Whence,

(v) $\quad \sup\{|\langle t, \langle e, x, \sigma\rangle\rangle|_\Theta : x \in \omega\} = \tau,$

viz. if the sup in (v) was $\tau' < \tau$, then by (iii) $|\{e\}_F(x, \sigma)|_F < \pi(\tau' + 1) < \pi(\tau) = \tau$, contradicting (iv) above.

If τ was Θ-admissible, then $\rho(x) = |\langle t, \langle \rho, x, \sigma\rangle\rangle|_\Theta$, $x \in \omega$, would be Θ_τ-computable. By 5.4.11 this would give $\sup\{\rho(x) : x \in \omega\} < \tau$, contradicting (v) above. Hence π has no Θ-admissible fixed-points $< \alpha_F$.

The proof of Lemma 5.4.16 necessitates a few preparatory remarks. First of all we need to keep track of how ordinals of computations in Θ grow. If $\nu < \alpha_\Theta$ by virtue of 5.4.11 there exists an ordinal $\mu < \alpha_\Theta$ such that:

(i) (substitution) If there exists an u such that $\{e\}_\Theta(\sigma) \simeq u$ and $\{f\}_\Theta(u, \sigma) \simeq x$ and $|\{e\}_\Theta(\sigma)|_\Theta$, $|\{f\}_\Theta(u, \sigma)|_\Theta < \nu$, then $|\{g_1(e, f, n)\}_\Theta(\sigma)| < \mu$, where $n = \mathrm{lh}(\sigma)$ and g_1 is an index for substitution.

(ii) (prewellordering) If $|x|_\Theta < \nu$ or $|y|_\Theta < \nu$, then $|\{\hat{p}\}_\Theta(x, y)| < \mu$.

(iii) (application of $\mathbf{E}_\omega$) If for some x, $\{e\}_\Theta(x, \sigma) \simeq 0$ and $|\{e\}_\Theta(x, \sigma)|_\Theta < \nu$, then $|\{g_2(n)\}_\Theta(e, \sigma)|_\Theta < \mu$, and if for all x there is some $y \neq 0$ such that $\{e\}_\Theta(x, \sigma) \simeq y$ and $|\{e\}_\Theta(x, \sigma)| < \nu$, then $|\{g_2(n)\}_\Theta(e, \sigma)|_\Theta < \mu$, where $n = \mathrm{lh}(\sigma)$ and $g_2(n)$ is an index for $\mathbf{E}_\omega$.

We have similar clauses for other functions and functionals entering into the axiomatic description of Θ. Let $\rho(\nu)$ be the least μ satisfying the conditions above. ρ is seen to be Θ-computable.

We now start the proof of Lemma 5.4.16. Recall from 5.4.5 the notion $\mathrm{Ord}(f)$, defined whenever f is a total Θ-computable function.

Let f be total and Θ-computable. Let $\nu = \mathrm{Ord}(f)$ and set $\mu = \sup(\pi(\nu), \rho(\nu))$. Then

$$F_0(\langle f, n, 0\rangle) = \begin{cases} 0 & \text{if } \ n = \langle e, \sigma, y\rangle, \{e\}_\Theta(\sigma) \simeq y, \quad |\{e\}_\Theta(\sigma)|_\Theta < \nu, \\ 1 & \text{otherwise.} \end{cases}$$

$$F_0(\langle f, n, 1\rangle) = \begin{cases} 0 & \text{if } \ n = \langle e, \sigma, y\rangle, \{e\}_\Theta(\sigma) \simeq y, \quad |\{e\}_\Theta(\sigma)|_\Theta < \mu, \\ 1 & \text{otherwise.} \end{cases}$$

Let $F_0(g) = 1$ if g is not Θ-computable, or g is not of the forms $\langle f, n, 0\rangle$, $\langle f, n, 1\rangle$. F_0 is easily seen to be Θ-computable. Let F be the join of F_0 and E.

We note that if f is F-recursive and total, then f is also Θ-computable, hence $\mathrm{Ord}(f)\downarrow$. Let

(a) $\lambda = \sup\{\mathrm{Ord}(f) : f \text{ is total, } F\text{-recursive}\}.$

Let $f = \lambda x\cdot\{e\}_F$ be total and $\nu = \mathrm{Ord}(f)$. We see that $\lambda n\cdot F_0(\langle f, n, 0\rangle)$ is the characteristic function of the set

$$B_e = \{\langle e', \sigma, y\rangle : \{e'\}_\Theta(\sigma) \simeq y \wedge |\{e'\}_\Theta(\sigma)|_\Theta < \nu\}.$$

In the same way $\lambda n\cdot F_0(\langle f, n, 1\rangle)$ is the characteristic function of

$$C_e = \{\langle e', \sigma, y\rangle : \{e'\}_\Theta(\sigma) \simeq y \wedge |\{e'\}_\Theta(\sigma)|_\Theta < \mu\},$$

where we remember that $\mu = \sup(\pi(\nu), \rho(\nu))$. Thus B_e, C_e are F-recursive, uniformly in e.

Also remember that if $\inf(|x|_\Theta, |y|_\Theta) < \nu$ then $|\{\hat{p}\}_\Theta(x, y)|_\Theta < \rho(\nu) \leqslant \mu$. The set

$$\{(x, y) : |x|_\Theta < \nu \text{ or } |y|_\Theta < \nu, \text{ and } \langle \hat{p}, x, y, 0\rangle \in C_e\},$$

is a prewellordering of length ν and is F-recursive, since B_e and C_e are F-recursive. Any F-recursive pwo has length $< \alpha_F$. Hence $\nu < \alpha_F$.

Since ν above is of the form $\mathrm{Ord}(f)$, where f is total and F-recursive, it follows that the ordinal λ introduced in (a) satisfies $\lambda \leqslant \alpha_F$. If we had equality, $\lambda = \alpha_F$, Lemma 5.4.16 would immediately follow:

(1) To prove that α_F is a fixed-point for π, it suffices to prove that $\nu < \alpha_F$ implies $\pi(\nu) < \alpha_F$. So let $\nu < \alpha_F$ and choose an e such that $f = \lambda x\{e\}_F(x)$ is total and $\mathrm{Ord}(f) = \nu' \geqslant \nu$. Such an e exists since $\lambda \geqslant \alpha_F$. Let $\mu = \sup(\pi(\nu'), \rho(\nu'))$. C_e is F-recursive and from it we can construct a total and F-recursive f' different from all f'' with $\mathrm{Ord}(f'') \leqslant \mu$. Thus $\mu < \mathrm{Ord}(f') < \lambda \leqslant \alpha_F$. Since $\nu \leqslant \nu'$ the desired inequalities follow.

(2) To prove that α_F is Θ-admissible it suffices to prove that if $|\{e\}_\Theta(x, \sigma)|_\Theta < \alpha_F$ for all x, then there is an ordinal $\mu < \alpha_F$ such that $|\{e\}_\Theta(x, \sigma)|_\Theta < \mu$ for all x. But up to length α_F enough information about Θ-computations is coded into F_0. In fact if $|\{e\}_\Theta(x, \sigma)| < \alpha_F$ for all x then it is possible to construct an index e' such that $\{e\}_\Theta(x, \sigma) \simeq \{e'\}_F(x, \sigma)$ and $|\{e\}_\Theta(x, \sigma)|_\Theta < |\{e'\}_F(x, \sigma)|_F$ for all x. Then let μ be the length of the computation $E(\lambda x\cdot\{e'\}_F(x, \sigma))$.

We know that $\lambda \leqslant \alpha_F$. It remains to prove equality. (And, note that equality was used in (1) above.)

We do this by constructing a function $\sigma: \lambda \to \alpha_F$ such that σ is F-recursive and cofinal in α_F. Admissibility, i.e. 5.4.11, then implies that $\lambda = \alpha_F$.

Replacing Θ by F we introduce a notion $\mathrm{Ord}_F(f)$, for f total and F-recursive. A simple diagonal construction will tell us that

(b) $\alpha_F = \sup\{\mathrm{Ord}_F(f) : f \text{ is total, } F\text{-recursive}\}$.

From (a) and (b) there is a short step to a suitable function σ, viz. for $\nu < \lambda$ set

(c) $\sigma(\nu) = \inf\{\mathrm{Ord}_F(f) : \mathrm{Ord}(f) > \nu\}$.

We must prove cofinality and F-recursiveness.

The F-recursiveness of σ is obtained by a painstaking analysis of the definition, writing out each part in its ultimate recursion-theoretic details (Moldestad [105], pp. 106–107). For cofinality, assume to the contrary that

$$\sup\{\sigma(\nu) : \nu < \lambda\} = \mu < \alpha_F,$$

which means that $\sup\{\mathrm{Ord}(f) : \mathrm{Ord}_F(f) < \mu\} = \lambda$. This, however, contradicts the following fact: If $\mu < \alpha_F$, then

(d) $\nu = \sup\{\mathrm{Ord}(f) : \mathrm{Ord}_F(f) < \mu\} < \lambda$.

For the proof of (d) let μ and ν be fixed. The set $D = \{e : |\{e_1\}_F(e)|_F < \mu\}$ is F-recursive, where e_1 is the index used in the definition of $\mathrm{Ord}_F(f)$. Note that $\mathrm{Ord}_F(f) < \mu$ iff $\exists e \in D \cdot [f = \lambda x \cdot \{e\}_F(x)]$. Let

$$B = \bigcup_{e \in D} B_e.$$

This set is F-recursive since B_e is F-recursive uniformly in e. And we observe that

$$\langle e', \sigma, y\rangle \in B \quad \text{iff} \quad \{e'\}_\Theta(\sigma) \simeq y \wedge |\{e'\}_\Theta(\sigma)|_\Theta < \nu,$$

where ν is the ordinal introduced in (d). Now, if f is a total Θ-computable function such that $\mathrm{Ord}(f) \leqslant \nu$, then there is an index e such that $f = \lambda x \cdot \{e\}_\Theta(x)$ and $\forall x \exists y \langle e, x, y\rangle \in B$. Hence the set

$$E = \{e : \forall x \exists y \langle e, x, y\rangle \in B\},$$

which is F-recursive, contains Θ-indexes for all total Θ-computable functions f with $\mathrm{Ord}(f) \leqslant \nu$, in particular, E contains Θ-indexes for all f such that $\mathrm{Ord}_F(f) < \mu$. Once more, a diagonal construction will yield a function f' which is F-recursive and different from all total Θ-computable functions with Θ-index in E. Hence, $\nu < \mathrm{Ord}(f') < \lambda$, which concludes the proof of (d).

5.4.17 Remark. We add a brief remark on the sources for the theory of Section 5.4 up to this point. The fact that Θ is equivalent to a theory PR[G] for a normal type-2 G iff the ordinal α_Θ of Θ is not Θ-Mahlo is due independently to S. Simpson and to L. Harrington and A. Kechris, see [58] and [56]. The notion of Θ-Mahlo (5.4.6) can be found in the seminar report of A. Kechris [74], where Theorem 5.4.7 for both one and two domains are proved. Implicit in this work are both the Fattening Lemma 5.4.4 and the fact that "Mahlo is Mahlo". We have followed

Moldestad [105] in our exposition. In particular the detailed and explicit constructions in Lemmas 5.4.15 and 5.4.16 are taken from his study.

We have now successfully characterized those Spector theories which are equivalent to prime computability in a normal type-2 functional over ω. Restricting ourselves to sections we can go further and show that for *every* Spector theory Θ on ω there is a normal type-2 functional F such that $\mathrm{sc}(\Theta) = \mathrm{sc}(F)$, but in general the envelopes will be different. This is the "plus-one" theorem of G. E. Sacks [142].

5.4.18 Definition. Let M be a non-empty transitive set. M is called an *abstract 1-section* if it is closed under pairing and union and satisfies the following axioms:

(1) *Local countability:* $\forall x$ [x is countable].
(2) Δ_0*-separation:* $\exists x \forall y[y \in x \leftrightarrow y \in a \wedge \varphi(y)]$.
(3) Δ_0*-dependent choice:* $\forall x \exists y \varphi(x, y) \to \exists h \forall n \varphi(h(n), h(n+1))$, where $\varphi(x)$ and $\varphi(x, y)$ are Δ_0-formulas (with parameters) and h is a function from ω to M.

The reader will note that if M is an abstract 1-section, then M is an admissible set and each element of M is hereditarily countable.

This leads to the topic of codings: Each set $x \in \mathrm{HC}$ (the hereditarily countable sets) can be encoded by an $\alpha \in \omega^\omega$. If α is a code, let $m(\alpha) \in \mathrm{HC}$ be the set encoded by α. By induction on the set theoretic rank of x we can define a relation:

(i) α is a code and $m(\alpha) = x$,

α is a code for the set $\{m(\alpha_n) : n \in \omega\}$, α_n being the usual projection of α. The set of codes is Π_1^1, i.e. semicomputable in ${}^2E$.

5.4.19. Proposition. *Let α be a code and $\varphi(x)$ a Δ_0-formula. The relation $P(\beta)$ iff $\exists n[\beta = \alpha_n \wedge \varphi(m(\beta))]$ is recursive in α, ${}^2E$.*

By now this is familiar: bounded quantification corresponds to number quantification over ω.

So we come to our main construction. Let Θ be a Spector theory on ω. Let $m(\mathrm{sc}(\Theta))$ be the set of all sets in HC with code in $\mathrm{sc}(\Theta)$.

5.4.20 Proposition. *Let Θ be a Spector theory on ω. Then $m(\mathrm{sc}(\Theta))$ is an abstract 1-section.*

We verify Δ_0-separation and Δ_0-dependent choice. For Δ_0-separation let $\varphi(y)$ be Δ_0 and $\alpha \in \mathrm{sc}(\Theta)$ be a code. We must find a code $\alpha_0 \in \mathrm{sc}(\Theta)$ such that

$$\forall y[y \in m(\alpha_0) \text{ iff } y \in m(\alpha) \wedge \varphi(y)].$$

By Proposition 5.4.19 the relation $P(\beta)$ is Θ-computable. Let α_0 be an enumeration of all β's that satisfy $P(\beta)$.

Turning to Δ_0-DC let $\varphi(x, y)$ be a Δ_0-formula such that

(ii) $\forall x \exists y \varphi(x, y)$,

is true in $m(\mathrm{sc}(\Theta))$.

Let $\{a\}_\Theta$ be a code in $\mathrm{sc}(\Theta)$. Define a set

$$Q_a = \{n \in \omega : \{n\}_\Theta \text{ is a code } \wedge \ \varphi(m(\{a\}_\Theta), m(\{n\}_\Theta))\}.$$

Q_a is Θ-semicomputable uniformly in a. We now use a selection operator $\nu(a)$ such that whenever $\{a\}_\Theta$ is a code then $\nu(a) \in Q_a$. We then define a function h by recursion

$$h(0) = n_0, \quad \{n_0\}_\Theta \text{ a code for } \varnothing.$$
$$h(n + 1) = \nu(h(n)).$$

Then, clearly, $\forall n \varphi(m(\{h(n)\}_\Theta), m(\{h(n + 1)\}_\Theta))$.

5.4.21 Remark. In the proof of 5.4.20 we asserted that the relation $P(\beta)$ is Θ-computable. Strictly speaking this makes no sense: Our Spector theories are the "light-faced" version of hyperarithmetic theory and $P(\beta)$ is a "bold-faced" relation. But using essentially Proposition 3.1.12, we can pass from the "light-faced" to the "bold-faced" version as in hyperarithmetic theory.

$m(\mathrm{sc}(\Theta))$ can be given a more precise description. It is in fact equal to $L_{\alpha_\Theta}[R_\Theta]$, where α_Θ and R_Θ have their usual meaning. And $\alpha_\Theta = m(\mathrm{sc}(\Theta)) \cap \mathrm{On}$. In this setting the imbedding theorem of Section 5.3 asserts that Θ-semicomputability corresponds to $\Sigma_1(\langle L_{\alpha_\Theta}[R_\Theta], \in, R_\Theta\rangle)$. We shall not prove this in detail, for it is not needed in the actual proof of Sack's "plus-one" theorem.

This theorem asks if it is possible to define a normal type-2 F such that $\mathrm{sc}(\Theta) = L_{\alpha_\Theta}[R_\Theta] \cap 2^\omega = \mathrm{sc}(F)$ for any Spector theory Θ on ω.

The answer is yes, but there are several stumbling-blocks in the proof. Theorems 5.4.7 and 5.4.14 should warn us that there is no trivial way of pulling R_Θ back to a functional over ω. One point is that the notion of code involves the notion of well-foundedness, and well-foundedness is not computable in every Spector theory. This is the first obstacle to get around.

A second obstacle comes from Proposition 5.4.12. It is not at all obvious (in fact, it may be false) that α_Θ is the least R_Θ-admissible ordinal, which it should be.

But here is a lead: use forcing. Given any abstract one-section M we can generically construct an R such that $M = L_\alpha[R]$, where $\alpha = M \cap \mathrm{On}$, and α is the least R-admissible ordinal. In particular, α will not be R-Mahlo. And this is a result we can apply. But let us first digress and make some historical remarks.

Forcing was rather soon applied to problems in arithmetic and recursion theory, some of the early and influential papers are Feferman [24], Gandy-Sacks [43], and Sacks [141]. In the context of admissible sets Jensen included a section on forcing in his lecture notes [70], proving, characteristically, some very deep theorems. Unfortunately Jensen's lecture notes have remained unpublished. Our next result is, in fact, a simple version of a result of Jensen. We follow the exposition in Normann [121], who saw how to apply this result to the "plus-one" theorem, avoiding the somewhat complicated hierarchy for recursion in higher types introduced by Sacks.

5.4.22 Proposition. *Let M be an abstract 1-section and $\alpha = M \cap \mathrm{On}$. There exists an $R \subseteq \alpha$ such that $M = L_\alpha[R]$ and α is the least R-admissible ordinal.*

Introduce the conditions $\mathbf{P}$ as follows:

(1) $p \in \mathbf{P}$ iff $p \subseteq \alpha$ and no ordinal $\leqslant \sup p$ is p-admissible.
(2) $p \leqslant q$ iff $q = p \cap \mathrm{rnk}(q)$.

The forcing relation will be defined directly for Δ_0-formulas and then extended to all formulas in the usual way.

(3) $p \Vdash \varphi(x_1, \ldots, x_n, p)$ iff $x_1, \ldots, x_n \in L_{\mathrm{rnk}(p)}[p]$ and $\langle L_{\mathrm{rnk}(p)}[p], \in, p\rangle \models \varphi(x_1, \ldots, x_n \dot{p})$.

Here $\varphi(x_1, \ldots, x_n, \dot{p})$ is a Δ_0-formula containing the symbol $\dot{p}$ to be interpreted by the set p. To be really careful (or pedantic?) we should also have distinguished between the variable x_i in $\varphi(x_1, \ldots, x_i, \ldots, x_n, \dot{p})$ and the set $x_i \in L_{\mathrm{rnk}(p)}[p]$. We also remind the reader that $\mathrm{rnk}(x)$ is the usual set-theoretic notion of rank.

From (3) we see immediately that $\Vdash^{\Delta_0}$ is Δ_1-definable. Let R be generic with respect to $\langle \mathbf{P}, \leqslant\rangle$. We want to show that

(4) $M = L_\alpha[R]$.

The part of (4) that requires some work follows from the following lemma.

5.4.23 Lemma. *For all conditions p and all $x \in M$ there is an extension $q \leqslant p$ such that $x \in L_{\mathrm{rnk}(q)}[q]$.*

Actually it is sufficient to prove the simpler result that there exists a $q \leqslant p$ such that the *code* for x belongs to $L_{\mathrm{rnk}(q)}[q]$, since a set belongs to an abstract 1-section iff its code, which is a subset of ω, belongs to the 1-section. And it is easy to extend p to a q encoding the code of x, viz. put $q = p \cup \{\mathrm{rnk}(p) + n;$ n belongs to the code of $x\}$.

The main thing to verify is $\Delta_0(R)$-collection. And as usual assume the contrary. Then there is some Δ_0-formula φ and a set $u \in M$ such that for some $p \subseteq R$

(5) $p \Vdash (\forall x)(\exists y)\varphi(x, y, \dot{R})$.
(6) $p \Vdash (\forall v)(\exists x \in u)(\forall y \in v)\neg\varphi(x, y, \dot{R})$.

We can rewrite (5) and (6) in the following forms

(7) $(\forall q \leqslant p)(\forall x)(\exists r \leqslant q)(\exists y)\cdot r \Vdash \varphi(x, y, \dot{R})$.

(8) $(\forall q \leqslant p)(\forall v)(\exists r \leqslant q)\cdot r \Vdash \exists x \in u\, \forall y \in v \neg\varphi(x, y, \dot{R})$.

From (7) we can derive

(9) $(\forall q \leqslant p)(\exists r \leqslant q)(\forall x \in L_{\mathrm{rnk}(q)}[q])(\exists y \in L_{\mathrm{rnk}(r)}[r])\cdot r \Vdash \varphi(x, y, \dot{R})$.

This is obtained in the following way. Let $r_0 \leqslant q$ take a wellordering of type ω of $L_{\mathrm{rnk}(q)}[q]$ inside the model. By Σ_1-DC we may choose a sequence $\langle r_i \rangle_{i \in \omega}$ (here is a point where the definability of the forcing-relation enters) such that if $r = \bigcup_{i \in \omega} r_i$, then r_{i+1} is of minimal rank such that $r_{i+1} \leqslant r_i$ and if x_i is element number $i + 1$ in $L_{\mathrm{rnk}(q)}[q]$ then $r_{i+1} \Vdash \varphi(x_i, y_i, \dot{R})$, for some element y_i. rnk(r) is not r-admissible, hence $r \in \mathbf{P}$ and (9) is verified.

Wellorder $u = \{x_i : i \in \omega\}$ inside M and use (9) to get a sequence $\langle q_i, y_i \rangle$ such that

(10)
(i) $q_i \leqslant p$ and $u \in L_{\mathrm{rnk}(q_0)}[q_0]$.
(ii) $q_{i+1} \leqslant q_i$.
(iii) $(\forall x \in L_{\mathrm{rnk}(q_i)}[q_i])(\exists y \in L_{\mathrm{rnk}(q_i+1)}[q_{i+1}])\cdot q_{i+1} \Vdash \varphi(x, y, \dot{R})$.
(iv) $(\forall i \in \omega)\cdot q_{i+1} \Vdash \varphi(x_i, y_i, \dot{R})$.

Let $q = \bigcup_{i \in \omega} q_i$. We must first verify that q is a condition. First observe from (iii) that $L_{\mathrm{rnk}(q)}[q] \Vdash \forall x \exists y \varphi(x, y)$. Suppose that $\exists v \in L_{\mathrm{rnk}(q)}[q]$ such that for $(\forall i \in \omega)(\exists y \in v)\varphi(x_i, y)$. Since $q = \bigcup q_k$, there must be some q_k such that $v \in L_{\mathrm{rnk}(q_k)}[q_k]$. Then $q_k \Vdash (\forall x \in u)(\exists y \in v)\varphi(x, y, \dot{R})$ (recall the definition (3) of the forcing relation). But this contradicts (8), hence q is a condition.

The same type of argument applied once again will finish off the proof of $\Delta_0(R)$-collection. Let $s \leqslant q$ be such that $\langle y_i \rangle_{i \in \omega}$, $v = \{y_i : i \in \omega\} \in L_{\mathrm{rnk}(s)}[s]$. Then

$$L_{\mathrm{rnk}(s)}[s] \models (\forall x \in u)(\exists y \in v)\varphi(x, y, \dot{R}).$$

But this contradicts (8). Putting things together we now have a full proof of Proposition 5.4.22.

We are now ready for the main result supplementing the characterization Theorem 5.4.7.

5.4.24 Plus-One Theorem. *Let Θ be a Spector theory on ω. There exists a normal type-2 functional F on ω such that* sc(Θ) = sc(F).

Let Θ be given and construct its associated one-section $m(\mathrm{sc}(\Theta)) = L_\alpha[R]$, where $\alpha = \|\Theta\|$ and R obtained as in Proposition 5.4.22. α is the least R-admissible, hence not R-Mahlo. $\Sigma_1(\langle L_\alpha[R], \in, R \rangle)$ defines a Spector class, hence a Spector theory on ω, call the theory Θ^*. This theory is not Θ^*-Mahlo, hence by Theorem 5.4.7, it is equivalent to a Spector theory PR[F], for some total, normal type-2 F over ω. Since it is easy to see that sc(Θ) is determined by $m(\mathrm{sc}(\Theta))$, it follows that

$\mathrm{sc}(\Theta) = \mathrm{sc}(F)$. But note that $\mathrm{en}(\Theta)$ may differ from $\mathrm{en}(\Theta^*)$. The latter corresponds to $\Sigma_1(\langle L_\alpha[R], \in, R\rangle)$ restricted to ω, but since R is obtained by a forcing argument, which does not preserve Δ_1-definability, this may differ from $\Sigma_1(\langle L_\alpha[R_\Theta], \in, R_\Theta\rangle)$.

5.4.25 Example. (This is a simple version of a result in J. Bergstra [15].) There exist two normal type-2 functionals F_1, F_2 such that

(1) $\mathrm{en}(F_1) \neq \mathrm{en}(F_2)$
(2) $\mathrm{sc}(F_1, \alpha) = \mathrm{sc}(F_2, \alpha)$, for all $\alpha \in \omega^\omega$.

The envelope cannot be reconstructed from its section.

We let $F_1 = {}^2E$ and F_2 the recursive join of ${}^2E$ and a functional F_{α_0}, where $\alpha_0 \notin 1\text{-sc}({}^3E)$ and

$$F_{\alpha_0}(\beta) = \begin{cases} 1 & \text{if } \alpha_0 = \beta \\ 0 & \text{otherwise.} \end{cases}$$

To verify (1) assume that $F_{\alpha_0} \leqslant {}^2E$. Then $1\text{-sc}(F_2, {}^3E) \subseteq 1\text{-sc}({}^2E, {}^3E) = 1\text{-sc}({}^3E)$. But this is a contradiction since $\alpha_0 \in 1\text{-sc}(F_2, {}^3E)$ but does not lie in $1\text{-sc}({}^3E)$.

Let α be given. If α_0 *is* $\Delta_1^{1,\alpha}$, then F_{α_0} is recursive in ${}^2E, \alpha$; hence $1\text{-sc}(F_2, \alpha) \subseteq 1\text{-sc}({}^2E, \alpha)$. If α_0 *is not* $\Delta_1^{1,\alpha}$, we will show that F_{α_0} has no effect on the 1-sections, and (2) will again follow.

So suppose that we are making a calculation

$$\{e\}(t) = F_{\alpha_0}(\lambda v \cdot \{e'\}(v, t)),$$

where for some t_0, $\{e\}(t_0) = 1$ (otherwise $\lambda t \cdot \{e\}(t)$ would just be the characteristic function of ω) and $\lambda t \cdot \{e\}(t)$ is total. And suppose that this is the "first" (in length of computations) where we are breaking out of $1\text{-sc}({}^2E, \alpha)$. This means that $\lambda v \cdot \{e'\}(v, t_0) \in 1\text{-sc}({}^2E, \alpha)$. But this is impossible since $\{e\}(t_0) = 1$ implies that

$$\alpha_0 = \lambda v \cdot \{e'\}(v, t_0),$$

and we had assumed that α_0 is not in $\Delta_1^{1,\alpha}$.