Chapter XII
Initial Segments of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$

Having embedded minimal degrees below $\mathbf{0}'$, it is natural to try the embed other uppersemilattices as initial segments of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. We prove such embedding theorems in this chapter. In the first four sections, we present a detailed proof of the embeddability of an arbitrary finite lattice as an initial segment of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. Extensions of this result to other usls or to embeddings below degrees other than $\mathbf{0}'$ are discussed in Sec. 5. These results are applied to prove theorems about $\mathscr{D}$ and $\mathscr{D}[\mathbf{0}, \mathbf{0}']$.

1. Weakly Uniform Trees

Let $\mathscr{L}$ be a fixed finite lattice, with elements $0 = u_0, u_1, \ldots, u_n = 1$. Fix a weakly homogeneous sequential table Θ for $\mathscr{L}$ as in Appendix B.2. Θ is then the union of an increasing sequence $\Theta_0 \subseteq \Theta_1 \subseteq \cdots$ of finite sets of $n + 1$-tuples. Θ gives rise to a recursive function f defined by $f(k) = |\Theta_k|$ for all $k \in N$, and hence to the set of strings $\mathscr{S}_f = \{\sigma \in \mathscr{S}: \forall i \in N(\sigma(i) < f(i))\}$.

It is tempting to try to embed $\mathscr{L}$ as an initial segment of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ by combining the proofs of Theorems VII.4.1 and IX.2.1, and so, to use partial uniform trees to construct the desired initial segment. There are severe problems, however, in carrying out such a program. For suppose that an attempt is being made to construct a uniform (binary) e-splitting partial subtree of Id_2. Let us suppose that $T(\emptyset)$, $T(0)$ and $T(1)$ have been defined. The partial trees of Chap. IX now require

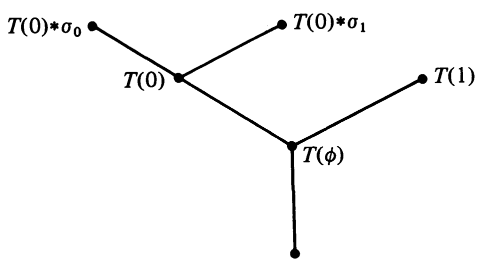

Fig. 1.1

that an appropriate e-splitting of $T(0)$, if found, be erected on T, independently of what is happening above $T(1)$ (see Fig. 1.1). Uniformity, however, ties the branches above $T(0)$ and above $T(1)$ together. We cannot wait to place $T(0) * \sigma_0$ and $T(0) * \sigma_1$ on T while we search for e-splittings which are compatible both above $T(0)$ and above $T(1)$, (i.e., τ_0 and τ_1 such that $\langle T(i) * \tau_0, T(i) * \tau_1 \rangle$ is an e-splitting for $i = 0, 1$) since we may never find an e-splitting of $T(1)$ on Id_2. Yet if we erect $T(0) * \sigma_0$ and $T(0) * \sigma_1$ on T and $T(1)$ must later be extended on T (due to a diagonalization requirement), then $T(1) * \sigma_0$ and $T(1) * \sigma_1$ must be erected on T to preserve the uniformity of T, even though they may not e-split $T(1)$ and even if there are other e-splittings of $T(1)$ on Id_2.

There are several ways to circumvent this problem. The one which we adopt here is to use weakly uniform trees in place of uniform trees. Such trees were discussed in the exercises for Chap. VII, where it was shown that they could be used to embed finite lattices as initial segments of $\mathscr{D}$. Weakly uniform trees differ from uniform trees in that the condition which requires a local isomorphism of extensions of any two strings of the same length is dropped. This enables us to build modified e-splitting trees; the definition of e-splitting trees is also weakened to require that there be infinitely many levels at which all strings which reach that level and do not satisfy a certain congruence relationship, form an e-splitting. (The previous definition required that all levels have this property.) Weak uniformity allows for proofs of interpolation lemmas without causing too much damage to the ability to prove a computation lemma. Total weakly uniform trees are defined below.

1.1 Definition. For all $\sigma, \tau \in \mathscr{S}_f$ and all $i \leqslant n$, define $\sigma \equiv_i \tau$ if $\sigma^{[i]}(x) = \tau^{[i]}(x)$ for all $x < \min(\{\mathrm{lh}(\sigma), \mathrm{lh}(\tau)\})$.

1.2 Definition. A *weakly uniform f-tree* is a function $T\colon \mathscr{S}_f \to \mathscr{S}_f$ which has the following properties:

(i) (Well-defined levels): $\forall \sigma, \tau \in \mathscr{S}_f(\mathrm{lh}(\sigma) = \mathrm{lh}(\tau) \to \mathrm{lh}(T(\sigma)) = \mathrm{lh}(T(\tau)))$.

(ii) (Congruence preserving): $\forall \sigma, \tau \in \mathscr{S}_f\, \forall i \leqslant n(\sigma \equiv_i \tau \leftrightarrow T(\sigma) \equiv_i T(\tau))$.

In order to embed initial segments below $\mathbf{0}'$, we will work with partial weakly uniform f-trees. Such trees are obtained by weakening Definition 1.2 so that T need not be a total function; but we must be careful to make the domains of the resulting trees relatively nice. We will require that the *levels* of the trees be nicely organized into *plateaus*. Levels and plateaus are best defined in terms of interval notation.

1.3 Definition. Let T be a partial f-tree satisfying 1.2(i) on its domain. *Level i* of T is the interval $I_{T(\xi), T(\eta)}$ where $I_{\sigma,\tau} = [\mathrm{lh}(\sigma), \mathrm{lh}(\tau))$ and ξ and η are strings in the domain of T such that $\mathrm{lh}(\xi) = i = \mathrm{lh}(\eta) - 1$. If $T(\emptyset) \neq \emptyset$, then $I_{\emptyset, T(\emptyset)}$ is *level -1* of T.

Certain levels of T are special in that only one string which is not terminal on T ends at the top of the level. These levels give rise to the *focal lengths* of the tree, which, in turn, determine the *plateaus* of the tree.

1.4 Definition. Let T be a tree and let $\sigma \subset T$ be given. σ is a *potential focal point* of T if there is no $\tau \subset T$ such that $\mathrm{lh}(\sigma) = \mathrm{lh}(\tau)$, $\tau \neq \sigma$, and τ is not terminal on T. σ is a *focal point* of T if σ is a potential focal point of T and σ is not terminal on T. r is a *focal length* of T if $r = \mathrm{lh}(\sigma)$ for some potential focal point σ of T.

It follows easily from Definition 1.4 that no two focal points of a tree have the same length, and that the set of focal points of a tree is linearly ordered by $\subseteq$. Also, if the tree T is finite, then all strings on T of greatest length are potential focal points of T, and each has, as its length the greatest focal length of T. A tree with focal points is pictured below.

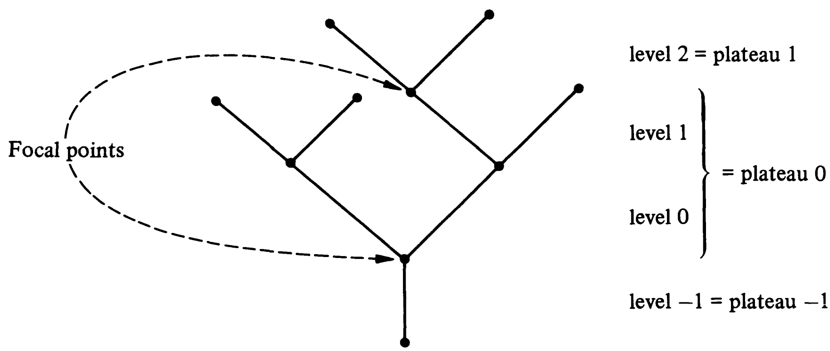

Fig. 1.2

Focal lengths determine plateaus of a tree as follows.

1.5 Definition. Let T be a partial f-tree satisfying 1.2(i) on its domain. Let $0 \leqslant r_0 < r_1 < \cdots$ be the focal lengths of T. *Plateau i* of T is the interval $[r_i, r_{i+1})$ (if r_{i+1} is defined). If $r_0 \neq 0$ then $[0, r_0)$ is *plateau* -1 of T. The *height of plateau i* is r_{i+1}. The *height of T* is the height of the last plateau of T if there is such a plateau, and is undefined otherwise. We write $\mathrm{ht}(T)$ for the height of T.

The ability to prove interpolation lemmas will come from the *fullness* of plateaus in weakly uniform trees.

1.6 Definition. The interval $[s, t)$ is *full on T* if there is no terminal σ on T such that $s < \mathrm{lh}(\sigma) < t$, but there is a string $\tau \subset T$ such that $\mathrm{lh}(\tau) \geqslant t$.

Figure 1.2 pictures a tree whose plateaus are full. Figure 1.3 below pictures a tree in which plateau 0 is not full.

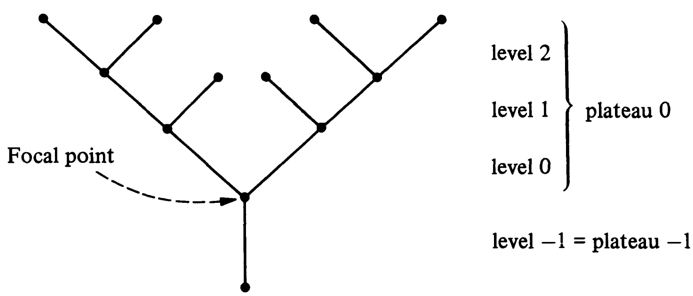

Fig. 1.3

The partial trees with which we will be working will have only full plateaus. They are now defined.

1.7 Definition. A partial f-tree T: $\mathscr{S}_f \to \mathscr{S}_f$ is *weakly uniform* if it satisfies:

(i) (Well-defined Levels): $\forall \sigma, \tau \in \mathscr{S}_f(T(\sigma)\downarrow \,\&\, T(\tau)\downarrow \,\&\, \mathrm{lh}(\sigma) = \mathrm{lh}(\tau) \to \mathrm{lh}(T(\sigma)) = \mathrm{lh}(T(\tau)))$.

(ii) (Congruence Preserving): $\forall \sigma, \tau \in \mathscr{S}_f \forall i \leqslant n(T(\sigma)\downarrow \,\&\, T(\tau)\downarrow \,\&\, \sigma \equiv_i \tau \leftrightarrow T(\sigma) \equiv_i T(\tau))$.

(iii) (Fullness): Every plateau of T is full.

Henceforth, we will use *tree* to denote a weakly uniform partial f-tree.

The function g such that $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$ is chosen to lie on infinitely many trees, each of which forces the satisfaction of a requirement. Two properties which these trees may possess are now defined. Recall the definition of $\sigma^{\langle i \rangle}$ from VI.2.10.

1.8 Definition. A tree T is *$\langle e, i, j\rangle$-differentiating* if there is an $x < \mathrm{lh}(T(\emptyset))$ such that $\Phi_e^{T(\emptyset)^{\langle j \rangle}}(x)\downarrow \neq T(\emptyset)^{\langle i \rangle}(x)\downarrow$.

1.9 Remark. If T is $\langle e, i, j\rangle$-differentiating and g is an infinite branch of T, then $\Phi_e^{g^{\langle j \rangle}} \neq g^{\langle i \rangle}$.

1.10 Definition. A tree T is *$\langle e, k\rangle$-divergent* for $k \leqslant n$, if there is an $x \in N$ such that for all $\sigma \subset T$, $\Phi_e^{\sigma^{\langle k \rangle}}(x)\uparrow$.

1.11 Remark. If T is $\langle e, k\rangle$-divergent and g is an infinite branch of T, then $\Phi_e^{g^{\langle k \rangle}}$ is not total.

The most important trees for the construction of any initial segment of $\mathscr{D}$ are the e-splitting trees. These trees are always the most difficult ones to construct. We will, in fact, weaken the notion of e-splitting tree and use this weaker notion for the constructions of this chapter. The weaker notion will require that certain levels of the tree be designated as *e-splitting levels*. The existence of infinitely many e-splitting levels will allow us to prove a computation lemma. But we will be able to extend many strings on T without having to make pairs of extensions form an e-splitting.

1.12 Definition. Let T be a tree and let $k \leqslant n$ and $e \in N$ be given. Level i of T is an *e-splitting level of T for k* if for all ξ, $\eta \in \mathscr{S}_f$ satisfying $\mathrm{lh}(\xi) = \mathrm{lh}(\eta) = i + 1$, $T(\xi)\downarrow$, $T(\eta)\downarrow$, and $\xi \not\equiv_k \eta$, it is the case that $\langle T(\xi), T(\eta)\rangle$ e-splits $T(\emptyset)$.

1.13 Definition. Let $k \leqslant n$ and e, $s^* \in N$ be given, and let $\{T_s\colon s \geqslant s^*\}$ be a recursive sequence of weakly uniform finite trees which approximates to the partial recursive tree $T = \bigcup\{T_s\colon s \geqslant s^*\}$. Then T is a *weak e-splitting tree for k generated by* $\{T_s\colon s \geqslant s^*\}$ if the following conditions hold:

(i) There are no e-splittings mod k on T.

(ii) The last level of every plateau of each T_s is an e-splitting level of T_s for k.

Whenever we refer to a tree T as a weak e-splitting tree for k, there will be an implicit underlying approximation $\{T_s\colon s \geqslant s^*\}$ to T which generates T as a weak e-splitting tree for k. We now prove a computation lemma for trees which are weak e-splitting for some $k \leqslant n$.

1.14 Computation Lemma. *Let $k \leqslant n$ and $e \in N$ be given, and let T be a partial recursive tree which is weak e-splitting for k. Let g be an infinite branch of T such that Φ_e^g is total. Then $\Phi_e^g \equiv_T g^{\langle k \rangle}$.*

Proof. We first show how to compute $\Phi_e^g(x)$ recursively from $g^{\langle k \rangle}$. Search for $\sigma \subset T$ such that $\sigma \equiv_k g$ and $\Phi_e^\sigma(x)\downarrow$. Let $\tau \subset g$ be given such that $\tau \subset T$ and $\Phi_e^\tau(x)\downarrow$. Such σ and τ must exist since Φ_e^g is total. Since $\sigma \equiv_k \tau$ and there are no e-splittings mod k on T, $\Phi_e^\sigma(x) = \Phi_e^\tau(x) = \Phi_e^g(x)$. Since $\Phi_e^\sigma(x)$ was computed following a procedure which is uniformly recursive in $g^{\langle k \rangle}$, $\Phi_e^g \leqslant_T g^{\langle k \rangle}$.

We now show how to recover $g^{\langle k \rangle}$ recursively from Φ_e^g. The reader may find Fig. 1.4 helpful for following the proof. In that figure, T_s appears in solid lines, and its extension to T is denoted by a dotted line. We proceed by induction on j, finding, at step j, $\sigma_j \subset T$ such that $\sigma_j = T(\xi_j)$, $\mathrm{lh}(\xi_j) = j$, and $\sigma_j \equiv_k g$. When $j = 0$, we choose $\sigma_0 = T(\emptyset)$.

At step $j+1$, expressing T as $\cup\{T_s : s \geqslant s^*\}$ as in Definition 1.13, find the least $s \geqslant s^*$ and the smallest level r of T_s such that:

(1) Level $r = [u, v)$ is an e-splitting level of T_s for k and $v > \mathrm{lh}(\sigma_j)$.

(2) $\exists \tau \subset T_s(\mathrm{lh}(\tau) = v \,\&\, \tau \equiv_k \sigma_j)$.

Such r and s can be found because of Definition 1.13(ii). Note that since each T_s is weakly uniform, each plateau of each T_s is full, so by Definition 1.13(ii), the interval $[\mathrm{lh}(\sigma_j), v)$ is full on T_s. Fix $T(\eta) = \rho \subset \tau$ such that $\mathrm{lh}(\rho) = \mathrm{lh}(\sigma_j)$. If $\mu, \nu \subset T_s$ are such that $\rho \subseteq \mu$, $\rho \subseteq \nu$, $\mathrm{lh}(\mu) = \mathrm{lh}(\nu) = v$ and $\mu \not\equiv_k \nu$, then by (1), $\langle \mu, \nu \rangle$ e-splits on some x. Hence Φ_e^g can be used to eliminate at least one of μ and ν as a potential candidate for a string σ such that $\mathrm{lh}(\sigma) = v$ and $\sigma \equiv_k g$. Complete this elimination process, ending with μ. (If no string remains at the end, choose μ arbitrarily.) Let $\mu = T(\eta * \delta)$. Choose $\sigma_{j+1} = T(\xi_{j+1})$ such that $\sigma_{j+1} \subseteq \mu$ and $\mathrm{lh}(\xi_{j+1}) = j+1$. Since $\sigma_{j+1} \equiv_k \mu$, it suffices to show that $\mu \equiv_k g$. Fix $\alpha, \beta \in \mathscr{S}_f$ such that $\mathrm{lh}(\alpha) = \mathrm{lh}(\eta)$, $\mathrm{lh}(\beta) = \mathrm{lh}(\eta * \delta)$, and $T(\alpha) \subset T(\beta) \subset g$. Let $\beta = \alpha * \gamma$. Since $[\mathrm{lh}(\sigma_j), v)$ is full on T_s, $T_s(\eta * \gamma)\downarrow$. By the choice of s satisfying (1) and (2), $T(\eta * \gamma)$ cannot be eliminated during the above process since, by 1.7(ii), $\eta \equiv_k \sigma_j$ and there are no e-splittings mod k on T, hence on T_s. Thus $\mu \equiv_k T(\eta * \gamma) \equiv_k T(\beta) \equiv_k g$. ▯

The types of tree mentioned in this section can be used to embed $\mathscr{L}$ as an initial segment of $\mathscr{D}$ as follows.

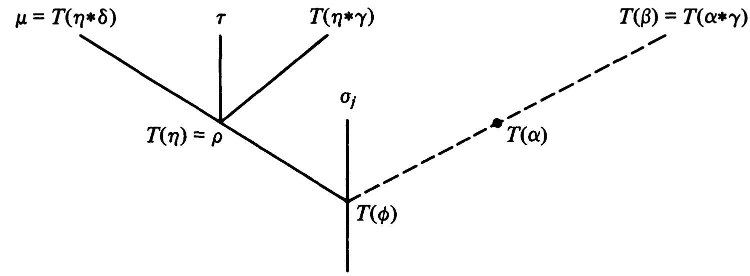

Fig. 1.4

1.15 Proposition. *Let $g\colon N \to N$ be given. Assume that:*

(i) *For all $e \in N$ and $i, j \leqslant n$ such that $u_i \nleqslant u_j$, there is a partial recursive tree T such that $g \subset T$ and either T is $\langle e, i, j\rangle$-differentiating or $\langle e, j\rangle$-divergent.*

(ii) *For all $e \in N$, there is a partial recursive tree T such that $g \subset T$ and either T is $\langle e, n\rangle$-divergent or there is a $k \leqslant n$ for which T is a weak e-splitting tree for k. Then $\mathscr{L} \simeq \mathscr{D}[\mathbf{0}, \mathbf{g}]$.*

Proof. By the properties of tables, if $u_i \leqslant u_j$ then $g^{\langle i\rangle} \leqslant_T g^{\langle j\rangle}$. Suppose that $u_i \nleqslant u_j$. Then by (i), Remark 1.9 and Remark 1.11, $g^{\langle i\rangle} \nleqslant_T g^{\langle j\rangle}$. Hence the map $u_i \mapsto g^{\langle i\rangle}$ is a poset isomorphism. It therefore suffices to show that $\mathscr{D}[\mathbf{0}, \mathbf{g}] = \{\mathbf{g}^{\langle i\rangle}\colon i \leqslant n\}$. Assume that $h \leqslant_T g$. Then there is an $e \in N$ such that $\Phi_e^g = h$. Since Φ_e^g is total g cannot be on an $\langle e, n\rangle$-divergent tree. Hence by (ii) and the Computation Lemma, $h \equiv_T g^{\langle k\rangle}$ for some $k \leqslant n$, so $h \in \{g^{\langle i\rangle}\colon i \leqslant n\}$. ▯

1.16 Remarks. Our first proof embedding $\mathscr{L}$ as an initial segment of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ used *quasi-uniform* trees in place of weakly uniform trees. These trees were sparse rather than full, and really consisted of a pair of trees, a tree T^* whose range was the difference of recursively enumerable sets sitting inside a partial recursive tree T. T^* was uniform, allowing proofs of interpolation lemmas, and T carried just enough strings to permit the proof of a complicated computation lemma. The situation was much more complex than with weakly uniform trees.

The proof using weakly uniform trees occurred to us after a discussion with S. Simpson in which Simpson commented that weakly uniform trees could be used in place of uniform trees to prove the results of Chap. VI. Proofs in Chap. VI would then become slightly more complicated, but these trees could be used to construct minimal degrees which were bi-immune free. It later occurred to us that the use of weakly uniform trees would simplify the proofs of this chapter.

2. Subtree Constructions

The proof that $\mathscr{L} \simeq \mathscr{D}[\mathbf{0}, \mathbf{g}]$ for some $\mathbf{g} < \mathbf{0}'$ will make use of a full approximation construction. All trees used will be constructed simultaneously with g. However, the construction of the trees can be isolated from the construction of g if the full approximation construction satisfies certain conditions. The subtrees needed are constructed in this section under such an assumption.

The construction of subtrees in a full approximation construction is a dynamic process which we view as follows. Let $T = \bigcup\{T_t\colon t \geqslant s\}$ be given, where $\{T_t\colon t \geqslant s\}$ is an increasing recursive sequence of finite trees. A subtree $T^* = \bigcup\{T_t^*\colon t \geqslant s^*\}$ is constructed, where $s^* \geqslant s$ and $\{T_t^*\colon t \geqslant s^*\}$ is an increasing recursive sequence of finite trees, and for each $t \geqslant s^*$, T_t^* is a subtree of T_t.

At each stage $t \geqslant s^*$, T_{t-1}^* will *receive* a set of strings S_t^* as input. The way in which T_t^* extends T_{t-1}^* depends on information conveyed by S_t^*. T_t^* may be prevented from extending T_{t-1}^* because an appropriate string cannot be found on T_t. This information is conveyed to T by having T^* transmit a string α_t which will be received by T and used to define T_{t+1}. T may also receive strings from trees other than T^*. However, if T^* *controls* T, then T and T^* will be related in a way which will guarantee the success of the construction.

Strings received by trees will carry instructions with them. The information received by a tree will be coded as a set of ordered pairs of the form $\langle \alpha, i \rangle$, with $\alpha \in \mathscr{S}_f$ and $i \leqslant 3$. If $\langle \alpha, i \rangle$ is received by the tree T_s at stage $s + 1$, then $\langle \alpha, i \rangle$ will instruct T_{s+1} to extend T_s in a specified way. The following are the types of extensions which may be specified. They are pictured in the next two figures.

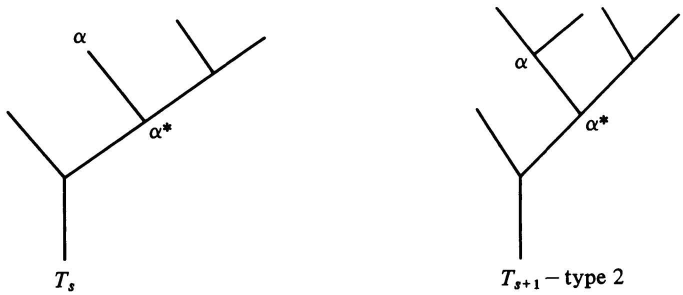

Fig. 2.1

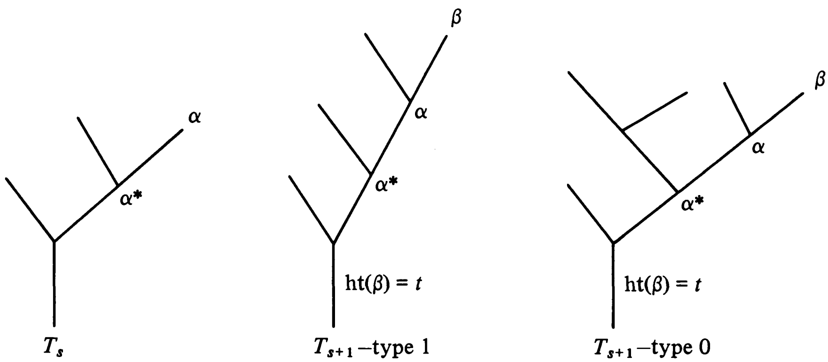

Fig. 2.2

2.1 Definition. Let T_s be a tree, and let $\alpha \subset T_s$ be given. Let α^* be the longest focal point of T_s such that $\alpha^* \subseteq \alpha$, and fix $\eta, \eta^* \in \mathscr{S}_f$ such that $T(\eta) = \alpha$ and $T(\eta^*) = \alpha^*$. Fix $t \in N$ such that $t \geqslant \mathrm{ht}(T_s)$ and the greatest $m \in N$ such that $T_s(\delta)\downarrow$ for some $\delta \in \mathscr{S}_f$ for which $\mathrm{lh}(\delta) = m$. T_{s+1} is a *type 0 extension of T_s for α of height t* if:

(i) T_{s+1} extends T_s.

(ii) $\mathrm{ht}(T_{s+1}) = t$.

(iii) $\mathrm{dom}(T_{s+1}) = \mathrm{dom}(T_s) \cup \{\lambda \colon \eta^* \subseteq \lambda \,\&\, \mathrm{lh}(\lambda) \leqslant m + 1\}$.

T_{s+1} is a *type 1 extension of T_s for α of height t* if (i) and (ii) hold as well as:

(iv) $\mathrm{lh}(\alpha) = \mathrm{ht}(T_s)$.

(v) $\mathrm{dom}(T_{s+1}) = \mathrm{dom}(T_s) \cup \{\lambda \colon \eta \subseteq \lambda \,\&\, \mathrm{lh}(\lambda) = m + 1\}$.

T_{s+1} is a *type 2 extension of T_s for* α if (i) holds as well as:

(vi) $\quad \mathrm{ht}(T_{s+1}) = \mathrm{ht}(T_s)$.

(vii) $\quad \mathrm{dom}(T_{s+1}) = \mathrm{dom}(T_s) \cup \{\lambda\colon \eta^* \subseteq \lambda \,\&\, \mathrm{lh}(\lambda) \leqslant m\}$.

2.2 Remark. The reception of $\langle \alpha, i\rangle$ by the tree T_s at stage $s+1$ will convey the instruction to carry out Objective i whenever possible. The objectives are listed below.

Objective 0. Combine plateaus and create a new (splitting) level. This objective will be met when T_{s+1} is a type 0 extension of T_s for α of height t, where $t > \mathrm{ht}(T_s)$ is specified. There is a type 2 extension T^* of T_s for α within T_{s+1}, the existence of which is crucial to the proofs of the interpolation lemmas. T_{s+1} adds one level to T^* (which will be an e-splitting level if a weak e-splitting tree is being constructed).

Objective 1. Designate a new focal point. This objective will be met when T_{s+1} is a type 1 extension of T_s for α of height t for some $t > \mathrm{ht}(T_s)$. It is used to force $T = \bigcup\{T_s\colon s \geqslant s^*\}$ to be infinite, with infinitely many focal points.

Objective 2. Specify an $\langle e, k\rangle$-divergent extension tree. When T_s receives $\langle \alpha, 2\rangle$, α is a potential focal point of T_s, and a tree T_s^* is specified such that $T_s^* \subseteq T_s$ with $\alpha = T_s^*(\xi)$. T_{t+1} is instructed to preserve α as a potential focal point, while a search for suitable strings in $\mathrm{PExt}_f(T^*, \xi)$ proceeds. If this search is unsuccessful, then $\mathrm{PExt}_f(T^*, \xi)$ will be $\langle e, k\rangle$-divergent for some $k \leqslant n$.

Objective 3. Specify a tree with no e-splittings mod k. The process is the same as in Objective 2, except that if the search is unsuccessful, then $\mathrm{PExt}_f(T^*, \xi)$ will have no e-splittings mod k for some $k < n$.

Since an oracle of degree $\mathbf{0}^{(2)}$ is not available, we will not be able to determine in advance whether or not searches as in Objective 2 and Objective 3 will succeed. This differs from the situation in Chap. VII.

Reception of strings will be subject to the following constraints.

2.3 Remark. Let the tree T_s receive $\langle \alpha, i\rangle$ and $\langle \beta, j\rangle$ at stage $s+1$. Then the following conditions will hold:

(i) $\alpha \subset T_s$.

(ii) If $i = 0$, then $\mathrm{lh}(\alpha) < \mathrm{ht}(T_s)$.

(iii) If $i = 1$, then α is a potential focal point of T_s which is not a focal point of T_s.

(iv) If $i \in \{2, 3\}$ and T_{s-1} does not receive $\langle \alpha, i\rangle$, then α is a potential focal point of T_s.

(v) $\alpha \subseteq \beta$ or $\beta \subset \alpha$.

If $T \subseteq T^*$ and T^* is sufficiently large, then there will be an extension $T^\#$ of T of type i such that $T^\# \subseteq T^*$. The proof of the existence of $T^\#$ under the following hypotheses is left to the reader.

2.4 Lemma. *Let T and T^* be trees such that $T \subseteq T^*$ and T is finite. Let $\alpha \in \mathscr{S}_f$ and $i \leqslant 2$ be given such that $\alpha \subset T$. Then there is a type i extension $T^\#$ of T for α of height t such that $T^\# \subseteq T^*$ under the following circumstances:*

(i) $i = 0$, $\text{lh}(\alpha) < \text{ht}(T) < \text{ht}(T^*) = t$, *and if* α^* *is the longest focal point of* T *such that* $\alpha^* \subseteq \alpha$ *then* α^* *is in the last plateau of* T^*.

(ii) $i = 1$, $\text{ht}(T^*) = t$, 2.3(iii) *holds for* T *in place of* T_s, *and* α *is not terminal on* T^* *but is in the last plateau of* T^*.

(iii) $i = 2$, $\text{ht}(T) = t$, *and* T^* *contains a type* 0 *extension of* T *for* α.

When a tree T_s receives information at stage $s + 1$, it will decide to process at most one bit of information $\langle \alpha, i \rangle$. This choice is made as follows.

2.5 Definition. Let T_s be a tree which receives the set of ordered pairs S_{s+1} at stage $s + 1$. We say that T_s *prefers* $\langle \alpha, i \rangle$ if $\langle \alpha, i \rangle$ is the first pair in S_{s+1} (under a fixed one-one recursive correspondence of N with $\mathscr{S}_f \times [0, 3]$ such that for all $\alpha, \beta \in \mathscr{S}_f$ and $i, j \leqslant 3$, if $\alpha \subset \beta$ then $\langle \alpha, i \rangle$ precedes $\langle \beta, j \rangle$ and if $i < j$ then $\langle \alpha, i \rangle$ precedes $\langle \alpha, j \rangle$) such that $i \leqslant 1$.

The following notation will be used for strings.

2.6 Notation. Let $\xi \in \mathscr{S}_f$ be given such that $\xi \neq \emptyset$. ξ^- will denote the unique $\lambda \in \mathscr{S}_f$ such that $\lambda \subset \xi$ and $\text{lh}(\lambda) + 1 = \text{lh}(\xi)$. Fix $i \in N$ such that $\xi = \xi^- * i$. $s(\xi)$ will denote the string $\xi^- * (i + 1)$, and if $i \neq 0$ then $p(\xi)$ will denote the string $\xi^- * (i - 1)$.

The identity tree will be the starting point for the construction. Since it is convenient to require that all trees contain infinitely many focal points, the first tree will be a partial subtree of the identity tree. This partial tree is defined through a recursive approximation which depends on the set of strings received by the tree. The tree is described in terms of its approximations.

2.7 Initial Tree Construction. Let $\{S_t \colon t > 0\}$ be a recursive sequence of finite subsets of $\mathscr{S}_f \times \{i \colon i \leqslant 3\}$. We construct a recursive sequence of finite trees, $\{\text{Init}_t(\{S_u \colon 0 < u \leqslant t\} \colon t \in N\}$, whose union is the *initial tree* specified by $\{S_t \colon t > 0\}$, $\text{Init}(\{S_t \colon t > 0\})$. For convenience, we use T_t to denote $\text{Init}_t(\{S_u \colon 0 < u \leqslant t\})$. T_{t-1} receives the set of strings S_t at stage t.

Let Id_f be the full identity tree as specified in VII.2.1, and let $\text{Id}_t = \{\sigma \subset \text{Id}_f \colon \text{lh}(\sigma) \leqslant t\}$.

We begin by setting $T_0 = \text{Id}_0$. Given T_t, fix $\langle \alpha, i \rangle \in S_{t+1}$ such that T_t prefers $\langle \alpha, i \rangle$ if such a pair $\langle \alpha, i \rangle$ exists. If no such pair $\langle \alpha, i \rangle$ exists or if $\langle \alpha, i \rangle$ does not satisfy 2.3(i)–(iv) for T_t, let $T_{t+1} = T_t$. Otherwise, let T_{t+1} be a type i extension of T_t for α of height $t + 1$ such that $T_{t+1} \subseteq \text{Id}_{t+1}$. Note that by Lemma 2.4, such an extension exists. No information is transmitted by T_t for any t.

The following remark summarizes the properties of $\text{Init}(\{S_t \colon t > 0\})$ used later in the construction. These properties follow easily from Definition 2.1 and 2.7.

2.8 Remark. Fix $\{S_t \colon t > 0\}$, let Init_t denote $\text{Init}_t(\{S_u \colon 0 < u \leqslant t\})$ and let Init denote $\text{Init}(\{S_t \colon t > 0\})$. Suppose that 2.3(i)–(iv) are satisfied at stage $t + 1$ by Init_t for all $\langle \alpha, i \rangle \in S_{t+1}$. Then:

(i) $\text{Init}_{t+1} \neq \text{Init}_t$ if, and only if Init_t prefers some $\langle \alpha, i \rangle$, in which case Init_{t+1} is a type i extension of Init_t for α of height $t + 1$.

(ii) If α is a potential focal point of Init_t and for all $\langle \beta, j \rangle$ received by Init_t with $j \leqslant 1$, either $j = 1$ and $\alpha \subseteq \beta$, or $j = 0$ and $\alpha \subseteq \beta$ and α is a focal point of Init_t, then α is a potential focal point of Init_{t+1}.

During the construction, it will not be enough to construct subtrees. Rather, all previously defined trees will play a role in determining the next tree. The new tree must be added to the end of a sequence of previously defined trees, and this new sequence must be *special*. Special sequences are used to insure that g, the limit of the strings constructed, has domain N. This is accomplished by forcing all trees to have infinitely many plateaus, with g as the unique infinite path through each of these trees.

2.9 Definition. Fix s^*, t, $k \in N$. For each $i \leqslant k$, let $\{T_{i,s}: s^* \leqslant s \leqslant t\}$ be a sequence of finite trees. The array $\{T_{i,s}: i \leqslant k \,\&\, s^* \leqslant s \leqslant t\}$ is *special* if it satisfies the following conditions:

(i) $\forall i < k \, \forall s \in [s^*, t](T_{i+1,s} \subseteq T_{i,s}) \,\&\, \forall i \leqslant k \, \forall s \in [s^*, t)(T_{i,s+1}$ extends $T_{i,s})$.

(ii) $\forall i \leqslant k \, \forall s \in [s^*, t] \, \forall \alpha \in \mathscr{S}_f$. (If α is a potential focal point of $T_{i,s}$ and for all $m \leqslant i$ and $\langle \beta, j \rangle$ received by $T_{m,s}$, either $j \in \{2, 3\}$, or $j = 1$ and $\alpha \subseteq \beta$, or $j = 0$ and $\alpha \subseteq \beta$ and α is a focal point of $T_{m,s}$, then α is a potential focal point of $T_{i,s+1}$.)

(iii) $\forall i \leqslant k \, \forall s \in [s^*, t)(T_{i,s+1} \neq T_{i,s} \rightarrow \forall j < i(\mathrm{ht}(T_{j,s+1}) = \mathrm{ht}(T_{i,s+1}))$ $\&\, (T_{i,s} \neq \emptyset \rightarrow \exists m \leqslant 1 \, \exists \alpha \in \mathscr{S}_f(T_{i,s}$ prefers $\langle \alpha, m \rangle \,\&\, T_{i,s+1}$ is a type m extension of $T_{i,s}$ for $\alpha)))$.

Conditions 2.9(ii) and (iii) tell us how to preserve focal points from tree to tree, and indicate the conditions under which trees will be extended. They reflect 2.8(ii) and (i) respectively, allowing us to immediately note the following fact.

2.10 Remark. Under the hypothesis of Remark 2.8, for all $t \in N$, $\{\mathrm{Init}_s: s \leqslant t\}$ is special.

Many of the trees which will be used during the construction will be defined by the Ext operation. These trees are defined through a recursive approximation. They do not process the pairs which they receive, but pass them on to the tree which they extend.

2.11 Extension Tree Construction. Let $s, s^* \in N$ be given such that $s^* \geqslant s$. Let T be a tree defined by $T = \cup\{T_t: t \geqslant s\}$, where $\{T_t: t \geqslant s\}$ is a recursive sequence of increasing finite trees. Let $\xi \in \mathscr{S}_f$ be given such that $T_{s^*}(\xi)\downarrow$. Define the tree $T^* = \mathrm{Ext}(T, \xi, s^*)$ as the union of the trees $\{T_t^*: t \geqslant s^*\}$, where $T_t^* = \mathrm{PExt}_f(T_t, \xi)$ (see VII.2.3). T_t^* will be denoted as $\mathrm{Ext}(T_t, \xi, s^*)$. (Note that s^* determines the stage at which the recursive approximation to T^* should begin.) T_t^* transmits exactly those pairs which it receives; these pairs are received by T_t.

The following properties of $\mathrm{Ext}(T, \xi, s^*)$ follow easily from its definition.

2.12 Remark. Let T, ξ, s and s^* be given as in Definition 2.11. Then the following conditions hold:

(i) For all $t \geqslant s^*$, $\mathrm{Ext}(T_t, \xi, s^*) \subseteq T_t$; and for all $t > s^*$, $\mathrm{Ext}(T_t, \xi, s^*)$ extends $\mathrm{Ext}(T_{t-1}, \xi, s^*)$.

(ii) For all $t \geqslant s^*$, $\mathrm{Ext}(T_t, \xi, s^*)$ transmits exactly those pairs which it receives.

(iii) For all $t \geqslant s^*$, if $T_t(\xi)$ is a potential focal point of T_t, then the (potential) focal points of $\mathrm{Ext}(T_t, \xi, s^*)$ are exactly those (potential) focal points α of T_t such that $T_t(\xi) \subseteq \alpha$.

There are two basic types of requirement which play a role in the construction of a function g for which $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$. The first type of requirement is a diagonalization requirement whose satisfaction produces a one-one map from $\mathscr{L}$ into $\mathscr{D}[\mathbf{0}, \mathbf{g}]$. Such a requirement is satisfied by forcing g to be on a differentiating tree. These trees are now introduced.

2.13 Differentiating Tree Construction. Let $k, s \in N$ be given, and let $\{T_{m,t}: m \leqslant k \mathbin{\&} t \geqslant s\}$ be an array of trees such that for all $m \leqslant k$ and $t \geqslant s$, $T_{m,t+1}$ extends $T_{m,t}$ and if $m \neq 0$ then $T_{m-1,t} \supseteq T_{m,t}$. For each $m \leqslant k$, let $T_m = \cup\{T_{m,t}: t \geqslant s\}$ and let $T_{m,t-1}$ receive $S_{m,t}$ at stage t. Fix e, $s^* \in N$ such that $s^* > s$ and $i, j \leqslant n$ such that $a_i \nleqslant a_j$. We construct an $\langle e, i, j\rangle$-differentiating tree

$$T_{k+1} = \mathrm{Diff}(\{T_{m,t}: m \leqslant k \mathbin{\&} t \geqslant s\}, e, i, j, s^*)$$

as the union of the increasing sequence of trees

$$\{T_{k+1,t} = \mathrm{Diff}_t(\{T_{m,r}: m \leqslant k \mathbin{\&} s \leqslant r \leqslant t\}, e, i, j, s^*): t \geqslant s^*\}.$$

We proceed by induction on $\{t: t \geqslant s^*\}$. By VII.1.1(i) and 1.2(ii), fix $p, q < f(0)$ such that $p \equiv_j q$ but $p \not\equiv_i q$, and the least x such that $T(p)(x) \not\equiv_i T(q)(x)$.

Stage t of the construction proceeds through the following sequence of steps. At the end of stage t, t is placed in some *state*. If $t > s^*$, proceed directly to the beginning of the step or substep of the construction in which a state was assigned at stage t. The reader may find Fig. 2.3 useful for following the construction. The first diagram is used if $r = p$ and the second if $r = q$.

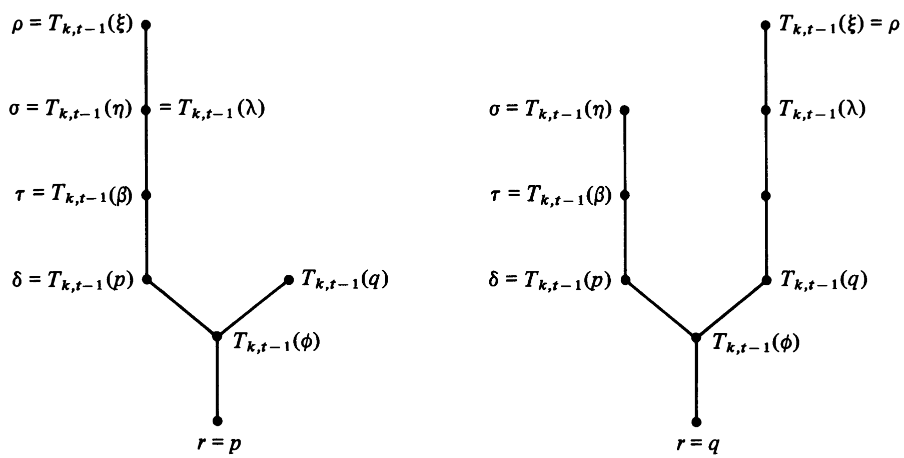

Fig. 2.3

Step 0. Begin T_{k+1}. If $T_{k,t-1}(\emptyset)\uparrow$ or if $\mathrm{ht}(T_{k,t-1}) \neq \mathrm{ht}(T_{0,t-1})$, set $T_{k+1,t} = \emptyset$. $T_{k+1,t-1}$ has no transmission. Place t in *state* $\langle 0, 0\rangle$ and proceed to the next stage. Otherwise, proceed to Step 1 if $T_{k,t-1}(p)\uparrow$, and to Step 2 if $T_{k,t-1}(p)\downarrow = \delta$.

Step 1. Begin branching on T_{k+1}. If $T_{k,t-1}$ is a type 1 extension of $T_{k,t-2}$ and $\mathrm{ht}(T_{k,t-1}) = \mathrm{ht}(T_{0,t-1})$, proceed to Step 3 letting $\delta = T_{k,t-1}(p)$. Otherwise, set

$T_{k+1,t} = \emptyset$ and let $T_{k+1,t-1}$ transmit $\langle T_{k,t-1}(\emptyset), 1\rangle$ to $T_{k,t-1}$. Place t in *state* $\langle 1, 0\rangle$ and proceed to the next stage.

Step 2. Obtain a type 0 extension. We want δ to be in the last plateau of $T_{k,t-1}$ with $\mathrm{ht}(T_{k,t-1}) = \mathrm{ht}(T_{0,t-1})$. If this is not the case, set $T_{k+1,t} = \emptyset$ and let $T_{k+1,t-1}$ transmit $\langle T_{k,t-1}(\emptyset), 0\rangle$ to $T_{k,t-1}$. Place t in *state* $\langle 2, 0\rangle$ and proceed to the next stage. Otherwise, go to Step 3.

Step 3. Force $\Phi_e^{g\langle j\rangle}(x)\downarrow$. Let $\tau = T_{k,t-1}(\beta)$ be the least string in $\mathscr{S}_f$ (under some fixed recursive one-one correspondence of N with $\mathscr{S}_f$) such that $\delta \subseteq \tau$ and τ is a potential focal point of $T_{k,v}$, where v is the first stage at which Step 1 or 2 is completed. Search for $\sigma \subset \mathrm{PExt}_f(T_{k,t-1}, \beta)$ such that $\tau \subseteq \sigma$ and $\Phi_e^{\sigma\langle j\rangle}(x)\downarrow$. If no such σ is found, set $T_{k+1,t} = \emptyset$ and let $T_{k+1,t-1}$ transmit $\langle \tau, 2\rangle$ to $T_{k,t-1}$. Place t in *state* $\langle 3, 0\rangle$ and proceed to the next stage. Otherwise, fix the least such $\sigma = T_{k,t-1}(\eta)$ (under some fixed recursive one-one correspondence of N with $\mathscr{S}_f$). Proceed to Step 4.

Step 4. Build the Diff tree. Before we define the Diff tree, we must make sure that the arrays of trees are special. This is accomplished in the first substep. The Diff tree is defined in the second substep.

Substep 0. Obtain a type 0 extension. We want δ to be in the last plateau of $T_{k,t-1}$ with $\mathrm{ht}(T_{k,t-1}) = \mathrm{ht}(T_{0,t-1})$. If this is not the case, set $T_{k+1,t} = \emptyset$ and let $T_{k+1,t-1}$ transmit $\langle T_{k,t-1}(\emptyset), 0\rangle$ to $T_{k,t-1}$. Place t in *state* $\langle 4, 0\rangle$ and proceed to the next stage. Otherwise, go to Substep 1.

Substep 1. Define the $\langle e, i, j\rangle$-differentiating tree. Let $z = \Phi_e^{\sigma\langle j\rangle}(x)$. Let r be the first of $\{p, q\}$ such that $T_{k,t-1}(r)^{\langle i\rangle}(x) \neq z$. If $r = p$ let $\lambda = \eta$ and if $r = q$ let $\lambda = \mathrm{tr}(p \to q; \eta)$. Let $\rho = T_{k,t-1}(\xi)$ be the first string (under some fixed recursive one-one correspondence of N with $\mathscr{S}_f$) such that $\lambda \subseteq \xi$ and $\mathrm{lh}(\rho) = \mathrm{ht}(T_{k,t-1})$. Set $T_{k+1,t}(\emptyset) = \rho$. $T_{k+1,t-1}$ has no transmission. For all stages $u \geqslant t$, let $T_{k+1,u} = \mathrm{Ext}(T_{k,u}, \xi, t)$ and place u in *state* $\langle 4, 1\rangle$. $T_{k+1,u}$ transmits all the pairs which it receives to $T_{k,u}$.

The next two lemmas specify properties of $\langle e, i, j\rangle$-differentiating trees which are important for the construction of the initial segment of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ which is isomorphic to $\mathscr{L}$. The first lemma specifies details of the construction, while the second lemma specifies properties which the final tree will have if suitable assumptions are made. The properties specified by the next lemma fall into four categories. The first three properties specify the type of tree which was defined. The next three properties aid with the verification of 2.3(i)–(v). We then have two properties dealing with the preservation of focal points and five properties specifying how information is processed.

2.14 Lemma. *Let e, i, j, k, s, $s^* \in N$ be given, and let*

$$T_{k+1} = \mathrm{Diff}(\{T_{m,t}\colon m \leqslant k \,\&\, t \geqslant s\}, e, i, j, s^*)$$

be defined as in 2.13 through the recursive approximation $\{T_{k+1,t}\colon t \geqslant s^*\}$. Then the following conditions hold:

(i) *For all* $t \geqslant s^*$, $T_{k+1,t} \subseteq T_{k,t}$; *and for all* $t > s^*$, $T_{k+1,t}$ *extends* $T_{k+1,t-1}$.

(ii) T_{k+1} *is recursive and weakly uniform, and if* $T_{k+1} \neq \emptyset$ *then* T_{k+1} *is* $\langle e, i, j\rangle$-*differentiating.*

(iii) *For all* $t \geqslant s^*$, *if* $T_{k+1,t}(\emptyset)\downarrow = T_{k,t}(\xi)$, *then* $T_{k+1,t} = \mathrm{PExt}_f(T_{k,t}, \xi)$ *and* $T_{k+1,t}$ *transmits exactly the pairs which it receives.*

(iv) *For all* $t \geqslant s^*$, *if* $T_{k+1,t} = \emptyset$ *then* $T_{k+1,t-1}$ *transmits at most one pair* $\langle \alpha, i\rangle$, *and* $\alpha \subset T_{k,t-1}$.

(v) *For all* $t \geqslant s^*$, *if* $T_{k+1,t} = \emptyset$ *and* $T_{k+1,t-1}$ *transmits* $\langle \alpha, 0\rangle$, *then* $\mathrm{lh}(\alpha) < \mathrm{ht}(T_{k,t-1})$.

(vi) *For all* $t \geqslant s^*$, *if* $T_{k+1,t} = \emptyset$ *and* $T_{k+1,t-1}$ *transmits* $\langle \alpha, i\rangle$ *with* $i \in \{1,2,3\}$ *and* t *and* $t-1$ *are in different states on* T_{k+1}, *then for all* $m \leqslant k$, α *is a potential focal point of* $T_{m,t-1}$ *which is not a focal point of* $T_{m,t-1}$.

(vii) *For all* $t \geqslant s^*$, *if* $T_{k+1,t} = \emptyset$ *and* $T_{k+1,t-1}$ *transmits* $\langle \alpha, i\rangle$ *with* $i \in \{2,3\}$, *then* $\mathrm{lh}(\alpha) > \mathrm{lh}(T_{k,t-1}(\emptyset))$.

(viii) *For all* $t \geqslant s^*$, *if* $T_{k+1,t} \neq \emptyset$ *then* $\mathrm{lh}(T_{k+1,t}(\emptyset)) > \mathrm{lh}(T_{k,t}(\emptyset))$.

(ix) *For all* $t \geqslant s^*$, *if* $T_{k+1,t} = \emptyset$ *and* $T_{k+1,t-1}$ *transmits* $\langle \alpha, i\rangle$, *then* t *and* $t+1$ *are in different states on* T_{k+1} *exactly when one of the following conditions holds*:

(a) $i \leqslant 1$ & $T_{k,t}$ *is a type* i *extension of* $T_{k,t-1}$ *for* α *such that* $\mathrm{ht}(T_{k,t}) = \mathrm{ht}(T_{0,t})$.

(b) $i = 2$ *and* $\alpha = T_{k,t-1}(\xi)$ *for a specified* ξ, *and for a specified* $x \in N$, there is a $\sigma \subset \mathrm{PExt}_f(T_{k,t}, \xi)$ *such that* $\Phi_e^{\sigma\langle j\rangle}(x)\downarrow$.

(x) *For all* $t \geqslant s^*$, *if* t *and* $t+1$ *are in the same state on* T_{k+1}, *then either* $T_{k+1,t-1}$ *and* $T_{k+1,t}$ *transmit the same pair, or neither tree transmits a pair.*

(xi) *For all* $t \geqslant s^*$, *if* t *and* $t+1$ *are in different states on* T_{k+1}, *then the state of* t *on* T_{k+1} *lexicographically precedes the state of* $t+1$ *on* T_{k+1}.

(xii) *If* $T_{k+1,t} = \emptyset$ *but* $T_{k+1,t+1} \neq \emptyset$, *then* $\mathrm{lh}(T_{k+1,t+1}(\emptyset)) = \mathrm{ht}(T_{k+1,t+1}) = \mathrm{ht}(T_{m,t})$ *for all* $m \leqslant k$ *and* $T_{k+1,t}$ *does not transmit any pairs.*

(xiii) *For all* $t \geqslant s^*$, *if* $T_{k+1,t} = \emptyset$ *then* $T_{k+1,t-1}$ *transmits a pair if and only if* t *is not in state* $\langle 0,0\rangle$ *on* T_{k+1}.

Proof. Immediate from the construction. ▯

In Sect. 4, we will construct a function g of degree $\leqslant \mathbf{0}'$ such that $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$. We will force g to have degree $\leqslant \mathbf{0}'$ by defining a recursive sequence of strings $\{\alpha_s \colon s \in N\}$ such that $g = \lim_s \alpha_s$. We will then have to show that $\mathrm{dom}(g) = N$. This will be accomplished by finding a path Γ through a tree of trees, and showing that $g = \bigcup\{T_\gamma(\emptyset) \colon \gamma \subset \Gamma\}$. The preservation of potential focal points is crucial for the verification of this fact, and the steps in tree constructions requiring that we take type 0 extensions with height restrictions related to the previous trees will allow us to preserve these focal points.

2.9(ii) is the central clause for the preservation of focal points, but in order to apply this clause, the process of transmission and reception of pairs must satisfy certain properties. 2.3(i)–(v) will yield these properties for reception of pairs. 2.9(iii) will allow us to show that the processing procedure for pairs leading to the transmission decision at a given stage has the right properties. It will then follow that we either define an $\langle e, i, j\rangle$-differentiating tree which contains g, or an extension tree containing g which is $\langle e, j\rangle$-divergent.

2.15 Lemma. *Fix e, i, j, k, s, $s^* \in N$, and let*

$$T_{k+1} = \operatorname{Diff}(\{T_{m,t}\colon m \leqslant k \,\&\, t \geqslant s\}, e, i, j, s^*)$$

be defined as in 2.13 through the recursive approximation $\{T_{k+1,t}\colon t \geqslant s^*\}$. For each $t \geqslant s^*$, assume that $\{T_{m,r}\colon m \leqslant k \,\&\, s^* \leqslant r \leqslant t\}$ is special. Then if $T_k \neq \emptyset$, $T_{k+1} = \emptyset$, and no sufficiently large t is in state $\langle 0, 0\rangle$ on T_{k+1}, then there are $\xi \in \mathscr{S}_f$ and $i \leqslant 2$ such that $T_{k+1,r}$ transmits $\langle T_k(\xi), i\rangle$ for all sufficiently large r and $\operatorname{PExt}_f(T_k, \xi)$ is either finite or $\langle e, j\rangle$-divergent.

Proof. We note that there are only finitely many states which $t \geqslant s^*$ can occupy on T_{k+1}. Furthermore, if $t \geqslant s^*$ and $T_{k+1,t} = \emptyset$, then t is in some state on T_{k+1}, and by 2.14(xiii), $T_{k+1,t-1}$ transmits some pair unless t is in state $\langle 0, 0\rangle$ on T_{k+1}. Hence by 2.14(x) and (xi), there is a stage r and a pair $\langle \alpha, i\rangle$ with $i \leqslant 2$ such that every stage $t \geqslant r$ is in a fixed state and for each such t, $T_{k+1,t-1}$ transmits $\langle \alpha, i\rangle$.

First assume that $i \leqslant 1$. Fix $t \geqslant r$. If $T_{k,t} \neq T_{k,t-1}$, then by 2.9(iii), $T_{k,t}$ is a type i extension of $T_{k,t-1}$ for α and $\operatorname{ht}(T_{k,t}) = \operatorname{ht}(T_{m,t})$ for all $m \leqslant k$. Hence by 2.14(ix), t and $t+1$ are in different states on T_{k+1}, contradicting the choice of r. Hence $T_{k,t} = T_{k,r}$ for all $t \geqslant r$. By 2.14(iv), there is a $\xi \in \mathscr{S}_f$ such that $T_k(\xi) = \alpha$. $\operatorname{PExt}_f(T_k, \xi)$ is now seen to be finite.

Assume that $i \in \{2, 3\}$. Then $i = 2$. By 2.14(ixb), there are $\xi \in \mathscr{S}_f$ and $x \in N$ such that $\alpha = T_{k,r}(\xi)$, and for all $\sigma \subset \operatorname{Ext}(T_k, \xi, s^*)$, $\Phi_e^{\sigma^{\langle j\rangle}}(x)\uparrow$. Hence $\operatorname{PExt}_f(T_k, \xi)$ is $\langle e, j\rangle$-divergent. ▯

The other trees needed for the construction of g are weak e-splitting trees. These trees are constructed in the next section.

3. Splitting Trees

The remaining type of requirement which will have to be satisfied deals with controlling the degree of Φ_e^g where g is constructed so that $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$. Such requirements are satisfied through the use of splitting trees. As in Chap. VII, we will prove interpolation lemmas which will enable us to construct splitting trees. We restate Lemma VII.3.2, the GLB Interpolation Lemma, for the reader's convenience.

3.1 GLB Interpolation Lemma. *Let $i, j, k \leqslant n$ and $\sigma, \tau, \rho \in \mathscr{S}_f$ be given such that $u_i \wedge u_j = u_k$, $\operatorname{lh}(\sigma) > 0$, $\operatorname{lh}(\tau) = \operatorname{lh}(\rho)$, and $\tau \equiv_k \rho$. Then there is a sequence $\tau = \tau_0, \ldots, \tau_m = \rho$ such that for all $p \leqslant m$, $\operatorname{lh}(\tau_p) = \operatorname{lh}(\tau)$, $\sigma * \tau_p \in \mathscr{S}_f$, and $\tau_0 \equiv_i \tau_1 \equiv_j \tau_2 \equiv_i \cdots \equiv_j \tau_m$.*

As we try to build a weak e-splitting subtree T^* of T, we will face the following situation. We will be given strings $\mu, \nu \in \operatorname{dom}(T)$ such that $\operatorname{lh}(\mu) = \operatorname{lh}(\nu)$, and will want to find strings σ and τ such that $\langle T(\mu * \sigma), T(\nu * \tau)\rangle$ forms an e-splitting; we wish to erect these strings on T^* while preserving the weak uniformity of T^*. Since we will have to combine old plateaus to form new plateaus, we cannot succeed as in Chap. VII merely by considering the case in which $\mathrm{lh}(\mu) = 1$. We thus try to reduce the problem which we now face to the situation of Chap. VII by fixing the least y such that $\mu(y) \neq \nu(y)$, fixing i and j such that $\mu(y) = i$ and $\nu(y) = j$, and requiring, for all k, that if $i \equiv_k j$ then $\sigma \equiv_k \tau$. The extension maps will then depend on the value of the corresponding string at y. We thus need a new definition of extendibility, and must prove a new Extendibility Interpolation Lemma.

3.2 Definition. Let $i, j, m \in N$, $\sigma, \tau \in \mathscr{S}$ and $\mu, \nu \in \mathscr{S}_f$ be given such that $m + 1 = \mathrm{lh}(\mu) = \mathrm{lh}(\nu)$. Fix the least y such that $\mu(y) \neq \nu(y)$. Assume that:

(i) $\mathrm{lh}(\sigma) = \mathrm{lh}(\tau)$.

(ii) $\mu(y) = i \,\&\, \nu(y) = j$.

(iii) $\forall k \leqslant n(i \equiv_k j \to \sigma \equiv_k \tau)$.

Let $\mathscr{U} = \{\alpha \in \mathscr{S}_f : \mathrm{lh}(\alpha) = \mathrm{lh}(\mu)\}$. We say that $\langle \sigma, \tau \rangle$ is *extendible for* $\langle \mu, \nu \rangle$ if there is a map $\theta: \mathscr{U} \to \{\xi : \mathrm{lh}(\xi) = \mathrm{lh}(\sigma) \,\&\, 0_{m+1} * \xi \in \mathscr{S}_f\}$ such that the following conditions hold:

(iv) $\theta(\mu) = \sigma \,\&\, \theta(\nu) = \tau$.

(v) $\forall k \leqslant n \forall \beta, \gamma \in \mathscr{U}(\beta \equiv_k \gamma \to \theta(\beta) \equiv_k \theta(\gamma))$.

The following lemma enables us to find extendible branchings under suitable hypotheses.

3.3 Extendibility Interpolation Lemma. *Let i, j, $m \in N$ and σ, τ, μ, $\nu \in \mathscr{S}_f$ be given satisfying* 3.2(i)–(iii) *with* $m + 1 = \mathrm{lh}(\mu) = \mathrm{lh}(\nu)$. *Then there is a λ such that* $\mathrm{lh}(\lambda) = \mathrm{lh}(\sigma)$, $0_{m+1} * \lambda \in \mathscr{S}_f$ *and both $\langle \sigma, \lambda \rangle$ and $\langle \lambda, \tau \rangle$ are extendible for $\langle \mu, \nu \rangle$.*

Proof. Fix $i, j, y, \sigma, \tau, \mu, \nu$ as in the hypothesis of Lemma 3.3. By Lemma VII.3.8 (the previous Extendibility Interpolation Lemma), there is a λ such that $\mathrm{lh}(\lambda) = \mathrm{lh}(\sigma)$, $0_{m+1} * \lambda \in \mathscr{S}_f$ and both $\langle \sigma, \lambda \rangle$ and $\langle \lambda, \tau \rangle$ are y-extendible for $\langle i, j \rangle$. Let ψ_0 and ψ_1 be the corresponding extension maps. Let $\mathscr{U} = \{\alpha \in \mathscr{S}_f : \mathrm{lh}(\alpha) = \mathrm{lh}(\mu)\}$. For $m \leqslant 1$ and $\delta \in \mathscr{U}$, define $\theta_m(\delta) = \psi_m(\delta(y))$. For all $\beta, \gamma \in \mathscr{U}$ and $k \leqslant n$,

$$\beta \equiv_k \gamma \to \beta(y) \equiv_k \gamma(y) \to \psi_m(\beta(y)) \equiv_k \psi_m(\gamma(y)) \leftrightarrow \theta_m(\beta) \equiv_k \theta_m(\gamma).$$

It is now easily verified that θ_0 and θ_1 witness the extendibility of $\langle \sigma, \lambda \rangle$ and $\langle \lambda, \tau \rangle$ respectively for $\langle \mu, \nu \rangle$. ▯

We now construct a typical splitting tree. The reader will find a thorough understanding of Lemma VII.3.10 to be very helpful. We try to motivate each step of the construction in detail, naming each step and keeping track of the progress of the construction at stage t.

3.4 Splitting Tree Construction. Let $m', s' \in N$ be given, and let $\{T_{m,t} : m < m' \,\&\, t \geqslant s'\}$ be an array of trees. Fix $k \leqslant n$ and e, $s^* \in N$ such that $s^* \geqslant s'$ and fix $\beta \subset T_{m'-1}$. Let $T_m = \cup\{T_{m,t} : t \geqslant s'\}$ for all $m < m'$, and let $T_{m,t}$ receive the set

$S_{m,t+1}$. We construct a tree

$$T_{m'} = \mathrm{Sp}(\{T_{m,t}: m < m' \,\&\, t \geqslant s'\}, e, k, \beta, s^*, \{S_{m,t}: m \leqslant m' \,\&\, t > s^*\})$$

as the union of an increasing sequence of trees $\{T_{m',t}: t \geqslant s^*\}$ where

$$T_{m',t} = \mathrm{Sp}_t(\{T_{m,r}: m < m' \,\&\, s' \leqslant r \leqslant t\}, e, k, \beta, s^*, \{S_{m,r}: m \leqslant m' \,\&\, s^* < r \leqslant t\}).$$

$T_{m'}$ will be a weak e-splitting subtree of $T_{m'-1}$ for k above β whose construction begins at stage s^* and which receives $S_{m',t}$ at stage $t > s^*$.

We will proceed by induction on the set of stages $\{t: t \geqslant s^* - 1\}$. Stages will be placed in *states* according to the progress being made in the construction of a given level of the splitting tree. States will be triples $\langle r_1, r_2, r_3\rangle$ ordered lexicographically. Unless either t is in state $\langle 0, 0, 0\rangle$ or all steps in the construction are completed at the end of stage t, we will set $T_{m',t} = T_{m',t-1}$.

We begin by placing $s^* - 1$ in *state* $\langle 0,0,0\rangle$ and setting $T_{m',s^*-1} = \emptyset$. Fix $t + 1 \geqslant s^*$, and assume by induction that $T_{m',t}$ has been defined. We indicate how to define $T_{m',t+1}$. We begin stage $t + 1$ with Step 0 if t is in state $\langle 0, 0, 0\rangle$. Otherwise, we begin state $t + 1$ with Step 1.

Step 0. Define $T_{m'}(\emptyset)$. Define

$$T_{m',t+1}(\delta) = \begin{cases} \beta & \text{if } \beta \subset T_{m'-1,t} \,\&\, \delta = \emptyset \,\&\, \mathrm{lh}(\beta) = \mathrm{ht}(T_{0,t}) \,\&\, t + 1 = s^* \\ \uparrow & \text{otherwise.} \end{cases}$$

If $T_{m',t+1} = \emptyset$, t is placed in *state* $\langle 0,0,0\rangle$ and $T_{m',t}$ does not transmit any strings to $T_{m'-1,t}$. Proceed to the next stage. Otherwise, proceed to Step 1.

Step 1. Express Preference. If $S_{m',t+1}$ does not satisfy 2.3(i)–(v) or if $T_{m',t}$ does not prefer any element of $S_{m',t+1}$, place t in *state* $\langle 0,1,0\rangle$. $T_{m',t}$ does not transmit any strings to $T_{m'-1,t}$. Proceed to the next stage. Otherwise, fix $\langle \alpha, i^*\rangle \in S_{m',t+1}$ such that $T_{m',t}$ prefers $\langle \alpha, i^*\rangle$. If $T_{m',t-1}$ also preferred $\langle \alpha, i^*\rangle \in S_{m',t}$, proceed directly to the point in the construction at which stage t ended. (Thus if stage t ended within a last step, substep, or subsubstep, we proceed directly to the beginning of that step, substep or subsubstep, with everything in the construction which has been defined at stage t unchanged at stage $t + 1$.) Otherwise, let $\alpha^* = \alpha$ if $i^* = 1$, and let α^* be the longest focal point of $T_{m',t}$ such that $\alpha^* \subseteq \alpha$ if $i^* = 0$. (In the latter case, 2.3(ii) implies the existence of such a focal point.) α^* is tentatively designated as the next focal point of $T_{m'}$. We now begin a new splitting level for $T_{m'}$. We thus require that $T_{m'-1,t}$ be a type i^* extension of $T_{m'-1,t-1}$ for α^* with $\mathrm{ht}(T_{m'-1,t}) = \mathrm{ht}(T_{0,t})$. If this is not the case, place t in *state* $\langle 1,0,0\rangle$, let $T_{m',t}$ transmit $\langle \alpha^*, i^*\rangle$ to $T_{m'-1,t}$, and proceed to the next stage. Otherwise, we continue stage $t + 1$ of the construction, letting T' be a type 2 extension of $T_{m',t}$ for α^* such that $T' \subseteq T_{m'-1,t}$. Such an extension must exist inside a type i^* extension of $T_{m'-1,t-1}$.

If t is not placed in state $\langle 1, 0, 0\rangle$, let $\{\xi_i': i \leqslant p\}$ be the set of all strings ξ such that $\alpha^* \subseteq T'(\xi)$ and $T'(\xi)\downarrow$ and is terminal on T'. For each $i \leqslant p$, fix ξ_i^+ such that $T_{m'-1,t}(\xi_i^+) = T'(\xi_i')$. Let $\{\eta_i^0: i \leqslant v\}$ be a list of all $\eta \in \mathscr{S}_f$ such that $\eta = \xi_i^+ * j$ for some $i \leqslant p$ and $j < f(\mathrm{lh}(\xi_i'))$. For each $i \leqslant v$, let $\beta_i^0 = T_{m'-1,t}(\eta_i^0)$. $\{\beta_i^0 : i \leqslant v\}$ starts the splitting level for $T_{m'}$, but must be extended to insure that we have the desired e-splittings. We will do this as follows. Following a fixed recursive procedure, form a list of all pairs $\{\langle \beta_{i_u}^0, \beta_{j_u}^0 \rangle : \beta_{i_u}^0 \not\equiv_k \beta_{j_u}^0 \;\&\; u \leqslant q\}$. We will proceed inductively through steps $\{u+2 : u \leqslant q\}$, building $\{\beta_i^{u+1} = T_{m'-1,t}(\eta_i^{u+1}) : i \leqslant v\}$ at step $u+2$ such that:

(1) $\quad \forall i \leqslant v(\beta_i^{u+1} \supseteq \beta_i^u)$.

(2) $\quad \forall i,j \leqslant v(\mathrm{lh}(\beta_i^{u+1}) = \mathrm{lh}(\beta_j^{u+1}))$.

(3) $\quad \langle \beta_{i_u}^{u+1}, \beta_{j_u}^{u+1} \rangle$ form an e-splitting.

(4) $\quad \forall i,j \leqslant v \,\forall m \leqslant n(\eta_i^0 \equiv_m \eta_j^0 \rightarrow \beta_i^{u+1} \equiv_m \beta_j^{u+1})$.

Note that conditions (2) and (4) will hold with 0 in place of $u+1$ once we show that $T_{m'-1}$ is weakly uniform. We now proceed to the next step.

Step $u+2$. *Define* $\{\beta_i^{u+1} : i \leqslant v\}$ *satisfying* (1)–(4). This step has several substeps. Two interpolations may be needed, so we must always work above level 1 of $T_{m'-1}$. So far, we have only guaranteed that level 0 of $T_{m'-1}$ has been defined. Thus we begin with Substeps 0 and 1.

Substep 0. *Specialize the sequence of trees for* $\beta_{i_u}^u$. If there is a $\hat{\beta}_{i_u}^u \supseteq \beta_{i_u}^u$ such that $\hat{\beta}_{i_u}^u \subset T_{m'-1,t}$ and $\mathrm{lh}(\hat{\beta}_{i_u}^u) = \mathrm{ht}(T_{0,t})$, fix the least such $\hat{\beta}_{i_u}^u = T_{m'-1,t}(\hat{\eta}_{i_u}^u)$ under some fixed recursive one-one correspondence of N with $\mathscr{S}_f$ and proceed to the next substep. (Note that if $T_{m'-1,t}$ is a type 0 extension of $T_{m'-1,t-1}$ of the same height as $T_{0,t}$, then $\hat{\beta}_{i_u}^u$ will exist.) Otherwise, place t in *state* $\langle u+2, 0, 0\rangle$, let $T_{m',t}$ transmit $\langle \alpha^*, 0\rangle$ to $T_{m'-1,t}$, and proceed to the next stage.

Substep 1. Define level 1 of $\mathrm{PExt}_f(T_{m'-1,t}, \hat{\eta}_{i_u}^u)$. If $T_{m'-1,t}$ is a type 1 extension of $T_{m'-1,t-1}$ for $\hat{\beta}_{i_u}^u$ and $\mathrm{ht}(T_{m'-1,t}) = \mathrm{ht}(T_{0,t})$, go to the next substep. In this case, $T_{m'-1,t}(\hat{\eta}_{i_u}^u * 0)\downarrow$ and is a potential focal point of $T_{m'-1,t}$ which is not a focal point of $T_{m'-1,t}$. Otherwise, place t in *state* $\langle u+2, 1, 0\rangle$, let $T_{m',t}$ transmit $\langle \hat{\beta}_{i_u}^u, 1\rangle$ to $T_{m'-1,t}$, and proceed to the next stage.

Substep 2. Find an e-splitting of $\beta_{i_u}^u$. Fix $\hat{\eta} \in \mathscr{S}_f$ such that $\hat{\eta}_{i_u}^u = \eta_{i_u}^u * \hat{\eta}$, and for all $j \leqslant v$, define $\hat{\eta}_j^u = \eta_j^u * \hat{\eta}$. Since $\beta_{i_u}^0 \not\equiv_k \beta_{j_u}^0$, there is a least $y < \mathrm{lh}(\eta_{i_u}^0)$ such that $\eta_{i_u}^0(y) \not\equiv_k \eta_{j_u}^0(y)$. Fix this y and fix the greatest element $u_m \in L$ for which $\eta_{i_u}^0(y) \equiv_m \eta_{j_u}^0(y)$. Let $u_b = u_m \wedge u_k$. Search for an e-splitting mod b on $\mathrm{PExt}_f(T_{m'-1,t}, \hat{\eta}_{i_u}^u * 0)$. (We will then interpolate to get an e-splitting mod m or an e-splitting mod k.) If no such e-splitting exists, place t in *state* $\langle u+2, 2, 0\rangle$, let $T_{m',t}$ transmit $\langle T_{m'-1,t}(\hat{\eta}_{i_u}^u * 0), 3\rangle$ to $T_{m'-1,t}$, and proceed to the next stage.

Suppose that e-splittings mod b exist. Let $\langle \gamma_0'', \gamma_1''\rangle$ be the least e-splitting mod b found at stage t (under some fixed recursive one-one correspondence of N with $\mathscr{S}_f^2$). Fix x such that $\Phi_e^{\gamma_0''}(x)\downarrow \neq \Phi_e^{\gamma_1''}(x)\downarrow$ and go to the next substep.

Substep 3. Place $T_{m'-1,t}(\hat{\eta}_{i_u}^u)$ *in the last plateau of* $T_{m'-1,t}$. We wish to use the GLB Interpolation Lemma to transform $\langle \gamma_0'', \gamma_1''\rangle$ into an e-splitting mod k or an e-splitting mod m. The procedure for obtaining such an e-splitting involves searching through an extension tree T^* of $T_{m'-1,t}$ for e-splittings. In order for the sequence of trees to remain special, we must define $T^*(\emptyset)$ to be a potential focal point of $T_{m'-1,t}$.

Thus we require that $T_{m'-1,t}(\hat{\eta}^u_{i_u})$ be in the last plateau of $T_{m'-1,t}$ with $\mathrm{ht}(T_{m'-1,t}) = \mathrm{ht}(T_{0,t})$. If this is the case, then there are $\gamma'_i \supseteq \gamma''_i$ for $i \leqslant 1$ such that $\mathrm{lh}(\gamma'_0) = \mathrm{lh}(\gamma'_1) = \mathrm{ht}(T_{0,t})$, $\gamma'_0 \equiv_b \gamma'_1$, and $\gamma'_1 \subset \mathrm{PExt}_f(T_{m'-1,t}, \hat{\eta}^u_{i_u} * 0)$. Fix such $\gamma'_i = T_{m'-1,t}(\zeta'_1)$ and proceed to the next substep. Otherwise, place t in *state* $\langle u+2, 3, 0\rangle$, let $T_{m',t}$ transmit $\langle \alpha^*, 0\rangle$ to $T_{m'-1,t}$, and proceed to the next stage.

Substep 4. Interpolate to get an e-splitting mod m. By the GLB Interpolation Lemma (3.1), there are arrays $\{v^j_i : i \leqslant w \,\&\, j \leqslant w-1\}$ and $\{\delta^j_i : i \leqslant w \,\&\, j \leqslant w-1\}$ satisfying (5)–(9) below for $j = 0$.

(5) $\quad \forall i, i' < w(\mathrm{lh}(v^j_i) = \mathrm{lh}(v^j_{i'}))$.

(6) $\quad \zeta'_0 = \hat{\eta}^u_{i_u} * 0 * v^0_0 \,\&\, \zeta'_1 = \hat{\eta}^u_{i_u} * 0 * v^0_w \,\&\, \forall i \leqslant w(v^j_i \supseteq v^{j-1}_i) \,\&\, \delta^j_i =$
$T_{m'-1,t}(\hat{\eta}^u_{i_u} * 0 * v^j_i)$.

(7) $\quad \forall i \leqslant w(0 * v^j_i \in \mathscr{S}_f)$.

(8) $\quad \Phi_e^{\delta^j_j}(x)\downarrow$.

(9) $\quad v^j_0 \equiv_m v^j_1 \equiv_k v^j_2 \equiv_m \cdots \equiv_k v^j_w$.

We proceed through the sequence of subsubsteps $\{j : 1 \leqslant j \leqslant 2w-2\}$, constructing $\{v^j_i : i \leqslant w\}$ and $\{\delta^j_i : i \leqslant w\}$ satisfying (5)–(9) at subsubsteps $2j-1$ and $2j$. Note that it is not necessary to follow this procedure for $j = w$ since $\Phi_e^{\delta^0_w}(x)\downarrow$. At the end of subsubstep $2w-2$, we will either have an e-splitting mod k or an e-splitting mod m on $\mathrm{PExt}_f(T_{m'-1,t}, \hat{\eta}^u_{i_u})$.

Subsubstep $2j-1$. Define δ^j_j satisfying (8). Institute a search in $\mathrm{PExt}_f(T_{m'-1,t}, \hat{\eta}^u_{i_u} * 0 * v^{j-1}_j)$ for $v \in \mathscr{S}_f$ such that

$$\Phi_e^{T_{m'-1,t}(\hat{\eta}^u_{i_u} * 0 * v^{j-1}_j * v)}(x)\downarrow.$$

If no such v exists, place t in *state* $\langle u+2, 4, 2j-1\rangle$, let $T_{m',t}$ transmit $\langle \delta^{j-1}_j, 2\rangle$ to $T_{m'-1,t}$, and proceed to the next stage. Otherwise, fix the least such v (under some fixed recursive one-one correspondence of N with $\mathscr{S}_f$). For all $i \leqslant w$, let $\hat{v}^j_i = v^{j-1}_i * v$. Proceed to the next subsubstep.

Subsubstep 2j. Place α^* in the last plateau of $T_{m'-1,t}$. We will want to define δ^{j+1}_{j+1} satisfying (8) with $j+1$ in place of j. Thus we will search for such a $\delta = \delta^{j+1}_{j+1}$ on an extension tree T^* of $T_{m'-1}$, and, failing to find δ, we will use T^* as the next tree in our sequence of trees. Since this new sequence will have to be special, $T^*(\emptyset)$ will have to be a potential focal point of $T_{m'-1}$. We insure that this is the case by taking a type 0 extension of $T_{m'-1,t}$ if necessary. Thus if α^* is not in the last plateau of $T_{m'-1,t}$ or if $\mathrm{ht}(T_{m'-1,t}) \neq \mathrm{ht}(T_{0,t})$, we place t in *state* $\langle u+2, 4, 2j\rangle$, let $T_{m',t}$ transmit $\langle \alpha^*, 0\rangle$ to $T_{m'-1,t}$, and proceed to the next stage. Otherwise, fix the least $v^* \in \mathscr{S}_f$ (under some fixed recursive one-one correspondence of N with $\mathscr{S}_f$) such that $v^* \supseteq v$ and $\mathrm{lh}(T_{m'-1,t}(\hat{\eta}^u_{i_u} * 0 * \hat{v}^j_j * v^*) = \mathrm{ht}(T_{0,t})$. For all $i \leqslant w$, let $v^j_i = \hat{v}^j_i * v^*$ and $\delta^j_i = T_{m'-1,t}(\hat{\eta}^u_{i_u} * 0 * v^j_i)$. Proceed to the next subsubstep if $j < w-1$.

Suppose that $j = w-1$. Then there is a least $i \leqslant w$ such that $\langle \delta^{w-1}_i, \delta^{w-1}_{i+1}\rangle$ e-splits on x, since $\langle \delta^{w-1}_0, \delta^{w-1}_w\rangle$ e-splits on x and $\Phi_e^{\delta^{w-1}_j}(x)\downarrow$ for all $j \leqslant w$. Fix this i.

If i is odd, then we have found an e-splitting mod k on $\mathrm{PExt}_f(T_{m'-1,t}, \hat{\eta}^u_{i_u})$ and the construction of $T_{m'}$ is terminated at this point. If i is even, let $\hat{\gamma}_0 = \delta^{w-1}_i$ and $\hat{\gamma}_1 = \delta^{w-1}_{i+1}$. Note that $\langle \hat{\gamma}_0, \hat{\gamma}_1 \rangle$ e-splits $T_{m'-1,t}(\hat{\eta}^u_{i_u}) \bmod m$, and that, by (8) and (7), $0 * v^{w-1}_i$, $0 * v^{w-1}_{i+1} \in \mathscr{S}_f$. Hence $\hat{\gamma}_0, \hat{\gamma}_1 \subset \mathrm{PExt}_f(T_{m'-1,t}, \hat{\eta}^u_{i_u})$. By the last subsubstep, $\mathrm{lh}(\hat{\gamma}_0) = \mathrm{lh}(\hat{\gamma}_1) = \mathrm{ht}(T_{0,t})$. Proceed to the next substep, defining $\hat{\zeta}_j \in \mathscr{S}_f$ by $T_{m'-1,t}(\hat{\zeta}_j) = \hat{\gamma}_j$ for $j \leqslant 1$.

Substep 5. Interpolate to get an extendible e-splitting mod m. The e-splitting mod m $\langle \hat{\gamma}_0, \hat{\gamma}_1 \rangle$ of $T_{m'-1,t}(\hat{\eta}^u_{i_u})$ will only be useful if it is appropriately extendible above $\beta^u_{j_u}$. Let λ_0, $\lambda_2 \in \mathscr{S}_f$ be defined by $\hat{\zeta}_0 = \hat{\eta}^u_{i_u} * \lambda_0$ and $\hat{\zeta}_1 = \hat{\eta}^u_{i_u} * \lambda_2$. Since $\mathrm{lh}(\hat{\eta}^u_{i_u}) > 0$, it follows from the Extendibility Interpolation Lemma (3.3) that there is a λ_1 such that $\hat{\eta}^u_{i_u} * \lambda_1 \in \mathscr{S}_f$, $\mathrm{lh}(\lambda_1) = \mathrm{lh}(\lambda_0)$, and both $\langle \lambda_0, \lambda_1 \rangle$ and $\langle \lambda_1, \lambda_2 \rangle$ are extendible for $\langle \hat{\eta}^u_{i_u}, \hat{\eta}^u_{j_u} \rangle$. Note that $T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_1)\downarrow$. In order to make use of these extendible branchings, we must force

$$\langle T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_j), T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_{j+1}) \rangle$$

to be an e-splitting for $j = 0$ or $j = 1$. This will be the case if

$$\Phi_e^{T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_1)}(x)\downarrow.$$

We try to achieve this last condition for some $\lambda \supseteq \lambda_1$ in the first subsubstep of Substep 5.

Subsubstep 0. Extend the interpolant to get a convergent computation on x. Search for $\sigma \subset \mathrm{PExt}_f(T_{m'-1,t}, \hat{\eta}^u_{i_u} * \lambda_1)$ such that $\Phi^\sigma_e(x)\downarrow$. If no such σ is found, place t in *state* $\langle u + 2, 5, 0 \rangle$, let $T_{m',t}$ transmit $\langle T_{m',t}(\hat{\eta}^u_{i_u} * \lambda_1), 2 \rangle$ to $T_{m'-1,t}$, and proceed to the next stage. Otherwise, fix the least such σ (under some recursive one-one correspondence of N with $\mathscr{S}_f$). Let $\sigma = T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_1 * \hat{\lambda})$. Let $j = 0$ if $\langle T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_0), \sigma \rangle$ e-splits on x, and let $j = 1$ otherwise. Note that if $j = 1$, then $\langle \sigma, T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_2) \rangle$ e-splits on x. Before we can define the extension map above $T_{m'-1,t}(\hat{\eta}^u_{j_u})$, we will need $T_{m'-1,t}(\hat{\eta}^u_{i_u})$ to be in the last plateau of $T_{m'-1,t}$ with $\mathrm{ht}(T_{m'-1,t}) = \mathrm{ht}(T_{0,t})$. This is achieved in the next subsubstep.

Subsubstep 1. Place $\hat{\beta}^u_{j_u}$ in the last plateau of $T_{m'-1,t}$. If $\hat{\beta}^u_{j_u}$ is not in the last plateau of $T_{m'-1,t}$ or if $\mathrm{ht}(T_{m'-1,t}) \neq \mathrm{ht}(T_{0,t})$, place t in *state* $\langle u + 2, 5, 1 \rangle$, let $T_{m',t}$ transmit $\langle \alpha^*, 0 \rangle$ to $T_{m'-1,t}$, and proceed to the next stage. Otherwise, there is a $\lambda' \supseteq \hat{\lambda}$ such that $\mathrm{ht}(T_{0,t}) = \mathrm{lh}(T_{m'-1,t}(\hat{\eta}^u_{j_u} * \lambda_{j+1} * \lambda')\downarrow)$. Fix the least such λ' (under some fixed recursive one-one correspondence of N with $\mathscr{S}_f$) and proceed to the next subsubstep.

Subsubstep 2. Force convergence to enable transfer of e-splittings. Search for $\tau \subset \mathrm{PExt}_f(T_{m'-1,t}, \hat{\eta}^u_{j_u} * \lambda_{j+1} * \lambda')$ such that $\Phi^\tau_e(x)\downarrow$. If no such τ is found, place t in *state* $\langle u + 2, 5, 2 \rangle$, let $T_{m',t}$ transmit $\langle T_{m'-1,t}(\hat{\eta}^u_{j_u} * \lambda_{j+1} * \lambda'), 2 \rangle$ to $T_{m'-1,t}$, and proceed to the next stage. Otherwise, fix the least such $\tau = T_{m'-1,t}(\hat{\eta}^u_{j_u} * \lambda_{j+1} * \lambda'')$ (under a fixed recursive one-one correspondence of N with $\mathscr{S}_f$). Let $j^* = 0$ if either $j = 0$ and $\Phi^\sigma_e(x) = \Phi^\tau_e(x)$ or if $j = 1$ and $\Phi^\sigma_e(x) \neq \Phi^\tau_e(x)$, and let $j^* = 1$ otherwise. (j^* is used to indicate that we have an e-splitting extending $\langle T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_{j+j^*}), \tau \rangle$.) The extension map will be defined once we have an appropriate e-splitting in the last plateau of $T_{m'-1,t}$. This is achieved in the next two subsubsteps.

Subsubstep 3. Place $\hat{\beta}^u_{j_u}$ in the last plateau of $T_{m'-1,t}$. If $\hat{\beta}^u_{j_u}$ is not in the last plateau of $T_{m'-1,t}$ or if $\mathrm{ht}(T_{m'-1,t}) \neq \mathrm{ht}(T_{0,t})$, place t in *state* $\langle u+2, 5, 3\rangle$, let $T_{m',t}$ transmit $\langle \alpha^*, 0\rangle$ to $T_{m'-1,t}$, and proceed to the next stage. Otherwise, proceed to the next subsubstep in order to define $\{\beta^{u+1}_c : c \leqslant v\}$.

Subsubstep 4. Define $\{\beta^{u+1}_c : c \leqslant v\}$. There are two cases, depending on the values of j and j^*. The e-splitting on x which we have obtained as the beginning of an extension of $\langle \hat{\beta}^u_{i_u}, \hat{\beta}^u_{j_u}\rangle$ extends $\langle T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_c), T_{m'-1,t}(\hat{\eta}^u_{j_u} * \lambda_d)\rangle$ for some c and d. j and j^* determine λ_c and λ_d according to the following table:

		j	
	$\langle\lambda_c,\lambda_d\rangle$	0	1
j^*	0	$\langle\lambda_0,\lambda_1\rangle$	$\langle\lambda_1,\lambda_1\rangle$
	1	$\langle\lambda_2,\lambda_2\rangle$	$\langle\lambda_1,\lambda_2\rangle$

Fig. 3.1

Case 1. $j = j^*$. If $j = 0$ then $\langle T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_0 * \lambda''), T_{m'-1,t}(\hat{\eta}^u_{j_u} * \lambda_1 * \lambda'')\rangle$ is an e-splitting mod m, and if $j = 1$, then $\langle T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_1 * \lambda''), T_{m'-1,t}(\hat{\eta}^u_{j_u} * \lambda_2 * \lambda'')\rangle$ is an e-splitting mod m, and for $k^* \in \{0,1\}$, $\langle T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_{k^*} * \lambda''), T_{m'-1,t}(\hat{\eta}^u_{j_u} * \lambda_{k^*+1} * \lambda'')\rangle$ is extendible for $\langle \hat{\eta}^u_{i_u}, \hat{\eta}^u_{j_u}\rangle$. By Subsubstep 3, we may assume that $\mathrm{ht}(T_{0,t}) = \mathrm{lh}(T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_0 * \lambda''))$ else we replace λ'' with some $\lambda''' \supseteq \lambda''$ having this property. Let θ be the extension map for $\langle \hat{\eta}^u_{i_u}, \hat{\eta}^u_{j_u}\rangle$.

Case 2. $j \neq j^*$. In this case, $\langle T_{m'-1,t}(\hat{\lambda}^u_{i_u} * \lambda_d * \lambda''), T_{m'-1,t}(\hat{\eta}^u_{j_u} * \lambda_d * \lambda'')\rangle$ is an e-splitting mod m for some $d \in \{1,2\}$. Fix this d. Again by Subsubstep 3, we may assume that $\mathrm{lh}(T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_1 * \lambda'')) = \mathrm{ht}(T_{0,t})$. We note that if we set $\theta(\hat{\eta}^u_c) = \lambda_d * \lambda''$ for all $c \leqslant v$, then θ is an extension map of $\langle T_{m'-1,t}(\hat{\eta}^u_{i_u} * \lambda_d * \lambda''), T_{m'-1,t}(\hat{\eta}^u_{j_u} * \lambda_d * \lambda'')\rangle$ for $\langle \hat{\eta}^u_{i_u}, \hat{\eta}^u_{j_u}\rangle$.

In both cases, let $\eta^{u+1}_c = \hat{\eta}^u_c * \theta(\hat{\eta}^u_c)$ and $\beta^{u+1}_c = T_{m'-1,t}(\eta^{u+1}_c)$ for all $c \leqslant v$. Proceed to the next step if $u < q$. If $u = q$, place t in *state* $\langle 0, 1, 0\rangle$, with $T_{m',t}$ having no transmission. Extend T' to $T_{m',t+1}$ as follows:

$$T_{m',t+1}(\delta) = \begin{cases} T'(\delta) & \text{if } T'(\delta)\downarrow \\ \beta^{v+1}_c & \text{if } \delta = \xi'_{c'} * d' \text{ for some } d' \in N \;\&\; \eta^0_c = \xi^+_{c'} * d' \\ \uparrow & \text{otherwise.} \end{cases}$$

Proceed to the next stage. ▯

We have tried to make the construction of a weak e-splitting tree for k above follow, as closely as possible, the scheme used to construct e-splitting trees in Chap. VII. There are certain key differences which arise when we work below $\mathbf{0}'$, which we note.

Since $T_{m'-1}$ is not total, we must try to take type 1 extensions of $T_{m'-1,t}$ in order to make the domain of $T_{m'-1,t}$ sufficiently large. Type 0 extensions are taken either in order to be able to interpolate or in order to have $T_{m'-1,t}$ defined on the domain of an extension map, or to force all trees to have infinitely many plateaus. The accomplishment of this last goal will not become evident until the next section, since it depends on the mechanics of the transmission of strings. This goal seems to be necessary in order to produce a function g whose domain is *all* of N and such that $\mathscr{D}[\mathbf{0},\mathbf{g}] \simeq \mathscr{L}$. Of course, we have already commented on the need to make $T_{m'}$ weakly e-splitting rather than fully e-splitting. Two other differences should be noted. The GLB Interpolation Lemma is applied within the construction of the tree. This is done since we cannot effectively know whether or not there are e-splittings mod k on $T_{m'-1}$, and if such e-splittings are found, k was chosen incorrectly. Thus k must be chosen during the construction, and we try to produce e-splittings mod k whenever possible. Finally, the Extendability Interpolation Lemma produces an extension map θ which does not generally give rise to a uniform tree because of the way in which y was chosen. This choice was dictated by the need to produce an e-splitting level for k on $T_{m'}$.

The next two lemmas specify properties of the weak e-splitting trees for k which we have just constructed. These properties will be used in our construction of an initial segment of $\mathscr{D}[\mathbf{0},\mathbf{0}']$ which is isomorphic to $\mathscr{L}$. The first lemma specifies details of the construction of the weak e-splitting trees, while the second lemma specifies properties which will hold if suitable assumptions are made about constructions of other trees. The properties specified by the next lemma fall into four categories. The first two properties specify the type of tree which was defined. The next three properties aid with the verification of 2.3(i)–(v). We then have two properties dealing with the preservation of focal points and six properties which specify how information is processed. These latter properties will be useful in verifying that sequences of trees are special.

3.5 Lemma. *Let m', s', k, e, $s^* \in N$, $\beta \in \mathscr{S}_f$, $\{T_{m,t}: m < m' \,\&\, t \geqslant s'\}$ and $\{S_{m,t}: m \leqslant m' \,\&\, t > s^*\}$ be as in the hypothesis of 3.4. For all $t \geqslant s^*$, let $\alpha^*(t)$ be the α^* chosen at stage t, Step 1 of 3.4. For all $t \geqslant s^*$, let*

$$T_{m',t} = \mathrm{Sp}_t(\{T_{m,r}: m < m' \,\&\, s' \leqslant r \leqslant t\}, e, k, \beta, s^*, \{S_{m,r}: m \leqslant m' \,\&\, s^* < r \leqslant t\})$$

and $T_{m'} = \cup\{T_{m,t}: t \geqslant s^*\}$. Then the following conditions hold:

(i) *For all $t \geqslant s^*$, $T_{m',t} \subseteq T_{m'-1,t}$; and for all $t > s^*$, $T_{m',t}$ extends $T_{m',t-1}$.*

(ii) *$T_{m'}$ is recursive and weakly uniform; and if $T_{m'}$ has no e-splittings* mod k *then $T_{m'}$ is weak e-splitting for k.*

(iii) *For all $t \geqslant s^*$ there is at most one pair $\langle\alpha, i\rangle$ such that $T_{m',t}$ transmits $\langle\alpha, i\rangle$. For this pair, $\alpha \subset T_{m'-1,t}$ and $\alpha^*(t) \subseteq \alpha$, and if $i = 0$ then $\alpha^*(t) = \alpha$.*

(iv) *For all $\alpha \in \mathscr{S}_f$ and $t \geqslant s^*$, if $T_{m',t}$ transmits $\langle\alpha, 0\rangle$ then* $\mathrm{lh}(\alpha) < \mathrm{ht}(T_{m',t}) \leqslant \mathrm{ht}(T_{m'-1,t})$.

(v) *For all $\alpha \in \mathscr{S}_f$, $i \in \{1,2,3\}$ and $t \geqslant s^*$, if $T_{m',t}$ transmits $\langle\alpha, i\rangle$ and t and $t+1$ are in different states on $T_{m'}$, then for all $m < m'$, α is a potential focal point of $T_{m,t}$ which is not a focal point of $T_{m,t}$.*

(vi) *For all $t \geqslant s^*$, $\alpha \in \mathscr{S}_f$ and $i \in \{2,3\}$, if $T_{m',t}$ transmits $\langle\alpha, i\rangle$ then* $\mathrm{lh}(\alpha) > \mathrm{ht}(T_{m',t})$.

(vii) *For all $t \geqslant s^*$, if $T_{m',t}(\emptyset)\downarrow$ then $T_{m',t}(\emptyset) = \beta \subset T_{m'-1,t}$.*

(viii) *For all* $t \geqslant s^*$, $\alpha \in \mathscr{S}_f$ *and* $i \leqslant 3$, *if* $T_{m',t}$ *and* $T_{m',t-1}$ *prefer the same pair and* $T_{m',t-1}$ *transmits* $\langle \alpha, i \rangle$, *then* t *and* $t+1$ *will be in different states on* $T_{m'}$ *exactly when one of the following conditions holds*:

(a) $i \leqslant 1$ *and* $T_{m'-1,t}$ *is a type* i *extension of* $T_{m'-1,t-1}$ *for* α *such that* $\mathrm{ht}(T_{m'-1,t}) = \mathrm{ht}(T_{0,t})$.

(b) $i = 2$, $\alpha = T_{m'-1,t-1}(\xi)$, *and for a specified* $x \in N$, *there is a* $\delta \subset \mathrm{PExt}_f(T_{m'-1,t}, \xi)$ *such that* $\Phi_e^\delta(x)\downarrow$.

(c) $i = 3$, $\alpha = T_{m'-1,t-1}(\xi)$, b *is as in Step* $u+2$, *Substep 2 where it is decided to transmit* $\langle \alpha, i \rangle$, *and there is an e-splitting* $\bmod b$ *on* $\mathrm{PExt}_f(T_{m'-1,t}, \xi)$.

(ix) *For all* $t \geqslant s^*$, *if* $T_{m',t+1} \neq T_{m',t}$ *then* $\mathrm{ht}(T_{m',t+1}) = \mathrm{ht}(T_{m'-1,t}) = \mathrm{ht}(T_{0,t})$ *and* $T_{m',t}$ *does not transmit any pairs. If, in addition,* $T_{m',t} \neq \emptyset$, *then* $T_{m',t}$ *prefers some* $\langle \alpha, i \rangle$, $T_{m',t+1}$ *is a type* i *extension of* $T_{m',t}$ *for* α, *and for all* $\delta \subset T_{m',t+1} - T_{m',t}$, $\delta \supseteq \alpha^*(t)$.

(x) *If* $T_{m',t}$ *transmits a pair, then* $T_{m',t}$ *prefers a pair.*

(xi) *For all* $t \geqslant s^*$, *if* $T_{m',t-1}$ *and* $T_{m',t}$ *prefer the same pair and* t *and* $t+1$ *are in the same state on* $T_{m'}$, *then either* $T_{m',t-1}$ *and* $T_{m',t}$ *transmit the same pair, or neither tree transmits a pair.*

(xii) *For all* $t \geqslant s^*$, *if* $T_{m',t-1}$ *and* $T_{m',t}$ *prefer the same pair and* t *and* $t+1$ *are in different states on* $T_{m'}$, *then either the state of* t *on* $T_{m'}$ *lexicographically precedes the state of* $t+1$ *on* $T_{m'}$ *or* $T_{m',t+1} \neq T_{m',t}$.

(xiii) *For all* $t > s^*$, *if* $\beta \subset T_{m'-1,t}$, $T_{m',t-1}$ *prefers some pair, reception of pairs by* $T_{m',t-1}$ *satisfies* 2.3(i)–(v), *and* $T_{m',t-1}$ *has no transmission, then* $T_{m',t} \neq T_{m',t-1}$.

Proof. The proof is a routine but tedious check of the construction of 3.4 which we leave to the reader. ▯

3.6 Lemma. *Let* m', s', k, e, $s^* \in N$, $\beta \in \mathscr{S}_f$, $\{T_{m,t}\colon m < m' \,\&\, t \geqslant s'\}$ *and* $\{S_{m,t}\colon m \leqslant m' \,\&\, t > s^*\}$ *be as in the hypothesis of 3.4. For all* $t \geqslant s^*$, *let* $\alpha^*(t)$ *be the* α^* *chosen at stage* $t+1$, *Step 1 of 3.4. For all* $t \geqslant s^*$, *let*

$$T_{m',t} = \mathrm{Sp}_t(\{T_{m,r}\colon m < m' \,\&\, s' \leqslant r \leqslant t\}, e, k, \beta, s^*, \{S_{m,r}\colon m \leqslant m' \,\&\, s^* < r \leqslant t\})$$

and $T_{m'} = \bigcup\{T_{m,t}\colon t \geqslant s^*\}$. *Assume that for all* $t \geqslant s^*$, $\{T_{m,r}\colon m \leqslant m' \,\&\, s^* \leqslant r \leqslant t\}$ *is special. Also assume that for all sufficiently large* t, *reception of pairs by* $T_{m',t}$ *satisfies* 2.3(i)–(v), *that* $T_{m',t}$ *prefers* $\langle \alpha, i \rangle$, *and that* $T_{m'}$ *is finite and has no e-splittings* $\bmod k$. *Then there are* $\lambda \in \mathscr{S}_f$ *and* $j \leqslant 3$ *such that* $T_{m',r}$ *transmits* $\langle T_{m'-1}(\lambda), j \rangle$ *for all sufficiently large* r *and* $\mathrm{PExt}_f(T_{m'-1}, \lambda)$ *is either finite, or* $\langle e, n \rangle$-*divergent, or has no e-splittings* $\bmod b$ *for some* b *such that* $u_b < u_k$.

Proof. We note that there are only finitely many states which t can occupy on $T_{m'}$ if $T_{m',t}$ prefers $\langle \alpha, i \rangle$ for all sufficiently large t. By the hypotheses, every sufficiently large t occupies some state on $T_{m'}$, and transmits some pair. Hence by 3.5(xi) and (xii), there is a stage r and a pair $\langle \beta, j \rangle$ such that every stage $t \geqslant r$ is in a fixed state and for each such t, $T_{m',t-1}$ transmits $\langle \beta, j \rangle$. The Lemma now follows from 3.5(viii) and 2.9(iii) in a way similar to the proof of Lemma 2.15. ▯

We now have the trees needed to construct an initial segment of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ which is isomorphic to $\mathscr{L}$. The construction of the initial segment is presented in the next section.

4. The Construction

In this section, we present a full approximation construction which produces a function g such that $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$. The basic outline of the construction in terms of priorities is similar to that of the construction given in the proof of Theorem XI.2.2. Thus we will have a tree of trees $\mathscr{T}$, and will choose, at stage s of the construction, a string $\gamma_s \subset \mathscr{T}$. $\mathscr{T}$ will be finite branching, and priorities will be used to show that the path Γ through $\mathscr{T}$ defined by $\Gamma = \limsup_s \gamma_s$ (the sup is taken in terms of the priority ordering) has infinite length. Strings on $\mathscr{T}$ which are of higher priority than those contained in Γ will be chosen only finitely often. Strings on $\mathscr{T}$ which are of lower priority than those contained in Γ will not be allowed to have much of an influence on the construction of g, as they will be cancelled when a higher priority path is later followed, and the action they have caused within the construction will be masked. Thus g will be viewed in terms of how it sits within the trees T_γ for $\gamma \subset \Gamma$. We now define priorities.

4.1 Definition. Let $h: N \to N$ be defined by $h(2i) = n + 1$ and $h(2i + 1) = 2$ for all $i \in N$. Given $\gamma, \delta \in \mathscr{S}_h$, we say that γ *has higher priority than* δ if either there is a least x such that $\gamma(x) \neq \delta(x)$ and for this x, $\gamma(x) < \delta(x)$, or $\gamma \subset \delta$. If Γ is given such that $\Gamma \upharpoonright k \in \mathscr{S}_h$ for all $k \in N$, then we extend priorities to Γ, letting δ *have higher* (*lower*) *priority than* Γ if for all but finitely many k, δ has higher (lower) priority than $\Gamma \upharpoonright k$. Trees are indexed by elements of $\mathscr{S}_h$, and we let the *priority* of a tree be the priority of its index.

The trees $\{T_\gamma\colon \gamma \subset \Gamma\}$ are defined to satisfy certain requirements. Thus if $\mathrm{lh}(\gamma) = 2e + 1$, then T_γ will either be e-splitting for some $k \leqslant n$ or $\langle e, n\rangle$-divergent. And if $\mathrm{lh}(\gamma) = 2e + 2$ and $\langle m, i, j\rangle$ is the eth triple under some recursive one-one correspondence of N with $N \times \{\langle i, j\rangle\colon i, j \leqslant n \,\&\, u_i \nleqslant u_j\}$ which we now fix for the remainder of this section, then T_γ will either be $\langle m, i, j\rangle$-differentiating or $\langle m, j\rangle$-divergent.

Trees will be designated at certain stages of the construction as either initial trees, extension trees, differentiating trees, or splitting trees for some $k \leqslant n$. Designations may be cancelled, so that T_γ may be designated differently at stage s and stage t. At a given stage s, the designation of T_γ is the last previous designation given to T_γ which has not yet been cancelled.

We recall the mechanics of the reception and transmission of pairs from Sects. 2 and 3. Initial trees transmit nothing, and respond immediately to the information which they receive. Extension trees transmit exactly what they receive. As long as a differentiating tree is empty, it ignores what it receives, but triggers new transmissions; once it becomes non-empty, it behaves like an extension tree. Splitting trees process what they receive, and transmit at most one pair; the pair transmitted is determined by the processing procedure. Reception and transmission of information is arranged so that $\{T_\gamma\colon \gamma \subset \Gamma\}$ will be a special sequence which satisfies 2.3(i)–(v). The lemmas of Sects. 2 and 3 will then be used to show that these trees have the desired properties.

The choice of $\gamma_s \in \mathscr{S}_h$ depends on the reception and transmission of information. Strings originating on a differentiating tree must be traced through the processing mechanism of splitting trees in order to determine γ_s. The following definition is used in that determination.

4.2 Definition. Let β, $\delta \in \mathscr{S}_h$ be given such that $\beta \subseteq \delta$. Let $s \in N$ be given such that $T_{\delta,s} = \emptyset$ and $T_{\delta,s}$ is designated as a differentiating tree. The sequence $\{\langle \sigma_\gamma, i_\gamma \rangle\colon \beta \subseteq \gamma \subset \delta\}$ is a *transmission sequence at stage* $s + 1$ if the following conditions hold:

(i) $T_{\delta,s}$ transmits $\langle \sigma_{\delta^-}, i_{\delta^-} \rangle$ and $i_{\delta^-} \in \{0, 1\}$.

(ii) For all γ such that $\beta \subset \gamma \subset \delta$, if $T_{\gamma,s}$ is designated as a splitting tree, then:

(a) $T_{\gamma,s}$ transmits $\langle \sigma_{\gamma^-}, i_{\gamma^-} \rangle$ and $i_{\gamma^-} \in \{0, 1\}$.

(b) $T_{\gamma,s}$ prefers $\langle \sigma_\gamma, i_\gamma \rangle$ and $\sigma_\gamma \subset T_{\gamma,s}$.

(iii) For all γ such that $\beta \subset \gamma \subset \delta$, if $T_{\gamma,s}$ is designated as either an extension or a differentiating tree, then $T_{\gamma,s}$ satisfies (iia) and (iib) and $\langle \sigma_{\gamma^-}, i_{\gamma^-} \rangle = \langle \sigma_\gamma, i_\gamma \rangle$.

The transmission sequence $\{\langle \sigma_\gamma, i_\gamma \rangle\colon \beta \subseteq \gamma \subset \delta\}$ at stage $s + 1$ is a *triggering sequence at stage* $s + 1$ if it satisfies:

(iv) Either $\beta = \emptyset$ or for some ξ such that $\beta \subseteq \xi \subseteq \delta$, s and $s + 1$ are in different states on T_ξ and $T_{\beta,s}$ does not transmit any pair $\langle \alpha, i \rangle$ with $i \leqslant 1$.

We say that T_δ *triggers* T_β *at stage* $s + 1$ if there is a triggering sequence $\{\langle \sigma_\gamma, i_\gamma \rangle\colon \beta \subseteq \gamma \subset \delta\}$ at stage $s + 1$. We call T_δ a *trigger at stage* $s + 1$ if there is a $\beta \subseteq \delta$ such that T_δ triggers T_β at stage $s + 1$.

A transmission sequence is a sequence of pairs transmitted down through the tree of trees, each pair instructing the tree which receives it to take a type i extension of itself for some $i \leqslant 1$. Such a sequence triggered by T_δ triggers T_β if one of three situations occurs. The first situation is that $\beta = \emptyset$. Thus the transmission sequence can be extended no further. However, $T_\emptyset$ immediately responds by producing the desired extension, so the construction of an earlier T_ξ with $\xi \subseteq \delta$ can proceed beyond the point at which it had stalled, thus allowing us to make progress towards defining $T_\delta(\emptyset)$. The second situation in which T_δ triggers T_β is if T_β has no transmission. This will only occur because T_β has responded to the instruction it received at the previous stage by producing the desired extension. Thus progress is made towards defining $T_\delta(\emptyset)$ as in the previous case. The final situation in which T_δ triggers T_β is when T_β transmits $\langle \alpha, i \rangle$ with $i \in \{2, 3\}$. In this case, $\alpha = T_{\beta^-}(\xi)$, and we are prevented from making more progress towards defining $T_\delta(\emptyset)$ by the need to find either a $\tau \subset \mathrm{PExt}_f(T_{\beta^-}, \xi)$ such that $\Phi_e^\tau(x)\downarrow$ for some specified e and x, or an e-splitting on $\mathrm{PExt}_f(T_{\beta^-}, \xi)$ for k for a specified e and k. If such strings are found at a later stage, then the transmission of T_β changes, and we continue making progress towards defining $T_\delta(\emptyset)$. If no such strings are found, we need not define $T_\delta(\emptyset)$, but must insure that our sequence of trees satisfying the various requirements contains $\mathrm{PExt}_f(T_{\beta^-}, \xi)$. We will therefore change our guess at the path through the tree of trees when this happens, directing it along a path inhabited by $\mathrm{PExt}_f(T_{\beta^-}, \xi)$.

We are now ready to construct a function g such that $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$. Along with the trees and their designations, a string $\gamma_s \in \mathscr{S}_h$ and an approximation α_s to g will be specified at stage s.

4.3 The Construction. *Stage 0*: Designate $T_{\emptyset,0}$ as $\mathrm{Init}(\{S_{\emptyset,t}\colon t > 0\})$. Let $\alpha_0 = \gamma_0 = \emptyset$.

Stage $s + 1$. For all β such that $T_{\beta,s}$ is designated, the information received by $T_{\beta,s}$ at stage $s + 1$ is the information transmitted by those $T_{\delta,s}$ such that $\beta = \delta^-$ at stage

s. Fix the tree T_δ of highest priority such that T_δ is a trigger at stage $s+1$. If no such T_δ exists, proceed to Case 2. Otherwise, fix $\beta \subseteq \delta$ such that T_δ triggers T_β at stage $s+1$, and let $\{\langle \sigma_\gamma, i_\gamma \rangle : \beta \subseteq \gamma \subset \delta\}$ be the corresponding triggering sequence. Note that there is only one possible choice for T_β. Proceed to Case 1.

Case 1. Cancel all trees of lower priority than T_δ and their designations. Also cancel all information transmitted by trees of lower priority than T_δ. Those γ for which T_γ is still designated at stage $s+1$ are called *active at stage* $s+1$. For all γ which are active at stage $s+1$, let $S_{\gamma,s+1}$ be the information received by $T_{\gamma,s}$ at stage $s+1$ from non-cancelled trees. For these γ, $T_{\gamma,s+1}$ will have the same designation as $T_{\gamma,s}$, and will be defined as the next step in the approximation to the tree so designated as in the definitions of these trees in Sects. 2 and 3. Note that the triggering sequence $\{\langle \sigma_\gamma, i_\gamma \rangle : \beta \subseteq \gamma \subset \delta\}$ has not been cancelled. We proceed by subcases, depending on how T_δ triggers T_β at stage $s+1$. If $T_{\beta,s}$ is designated as a splitting tree or if $\beta = \delta$, let $\langle \alpha^*, i^* \rangle$ be the information transmitted by $T_{\beta,s}$ if any information is transmitted. Note that if i^* is defined, then $i^* \in \{2,3\}$.

Subcase 1. $i^* = 2$. Let $\alpha_{s+1} = \alpha^* = T_{\beta^-,s}(\xi^*)$. Let $\gamma_{s+1} = s(\beta)$ and designate $T_{\gamma_{s+1}}$ as the following extension tree:

$$T_{\gamma_{s+1}} = \mathrm{Ext}(T_{\beta^-}, \xi^*, s+1).$$

Begin building $T_{\gamma_{s+1}}$.

Subcase 2. $i^* = 3$. In this case, $T_{\beta,s+1}$ is designated as an e-splitting tree for k for some $e \in N$, $k \leqslant n$, and $s+1$ is in state $\langle u+2, 2, 0 \rangle$ on T_β searching for an e-splitting mod b for some $b \leqslant n$ such that $u_b < u_k$. Let $\alpha_{s+1} = \alpha^* = T_{\beta^-,s}(\xi^*)$. Let $\gamma_{s+1} = s(\beta)$. If $b = 0$, designate $T_{\gamma_{s+1}}$ as in Subcase 1; and if $b \neq 0$, designate $T_{\gamma_{s+1}}$ as the following e-splitting tree for b:

$$T_{\gamma_{s+1}} = \mathrm{Sp}(\{T_{\xi,t} : \xi \subset \gamma_{s+1} \,\&\, t \geqslant s+1\}, e, b, \alpha^*, s+1, \{S_{\xi,t} : \xi \subseteq \gamma_{s+1} \,\&\, t > s+1\}).$$

Begin building $T_{\gamma_{s+1}}$.

Subcase 3. $\beta = \delta$ and $T_{\delta,s}$ has no transmission. In this case, $s+1$ will be in state $\langle 3,1 \rangle$ on T_δ. Let $\alpha_{s+1} = T_{\delta,s+1}(\emptyset)$ and $\gamma_{s+1} = \delta$.

Subcase 4. Otherwise. Then $T_{\beta,s}$ will be designated either as the initial tree or as a splitting tree, and has no transmission. Let $\alpha_{s+1} = \sigma_\beta$ and $\gamma_{s+1} = \delta$.

Case 2. For all γ such that $T_{\gamma,s}$ is designated, let $T_{\gamma,s+1}$ have the same designation as $T_{\gamma,s}$, and define $T_{\gamma,s+1}$ as the next step in the approximation to the tree so designated as in the definitions of Sects. 2 and 3; let $S_{\gamma,s+1}$ be the information received by $T_{\gamma,s}$ at stage $s+1$. There are two subcases.

Subcase 1. $\mathrm{lh}(\gamma_s) = 2e$. Let $\alpha^* = \alpha_{s+1} = T_{\gamma_s,s}(\emptyset)$ and $\gamma_{s+1} = \gamma_s * 0$. Set $b = n$ and designate $T_{\gamma_{s+1}}$ as in Case 1, Subcase 2. Begin building $T_{\gamma_{s+1}}$.

Subcase 2. $\mathrm{lh}(\gamma_s) = 2e+1$. Let $\langle m, i, j \rangle$ be the eth triple under the ordering fixed earlier. Set $\gamma_{s+1} = \gamma_s * 0$ and designate $T_{\gamma_{s+1}}$ as the following $\langle m, i, j \rangle$-

differentiating tree:

$$T_{\gamma_{s+1}} = \mathrm{Diff}(\{T_{\xi,t} : \xi \subseteq \gamma_s \,\&\, t \geqslant s+1\}, m, i, j, s+1).$$

Let $\alpha_{s+1} = T_{\gamma_s,s}(\emptyset)$. Note that $T_{\gamma_{s+1},s+1} = \emptyset$.

In both cases, we say that $T_{\beta,s+1}$ is *newly designated* if $T_{\beta,s+1}$ is designated and either $T_{\beta,s}$ is not designated or $T_{\beta,s}$ is cancelled at stage $s+1$.

This completes the construction. Define Γ by $\Gamma = \limsup_s \gamma_s$ where the sup is taken over the priority ordering, and $g = \lim_s \alpha_s$. We will need to prove the following facts about Γ and g:

(1) $\mathrm{lh}(\Gamma) = \infty$.

(2) $\mathrm{dom}(g) = \mathrm{N}$.

From (2) and the definition of g, it will then follow that $\mathbf{g} \leqslant \mathbf{0}'$. (1) will allow us to use the trees along the path specified by Γ to show that $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$. (2) will follow fairly easily once we prove

(3) $g = \cup\{T_\gamma(\emptyset) : \gamma \subset \Gamma\}$ where $T_\gamma = \lim_s T_{\gamma,s}$.

For we will show that if $\mathrm{lh}(\gamma) \geqslant 2$, then $\mathrm{lh}(T_{\gamma^{--}}(\emptyset)) < \mathrm{lh}(T_\gamma(\emptyset))$.

The proof of (3) will involve an analysis of transmission sequences. When a tree is newly designated, its definition depends on the trees defined at the previous stage of the construction. In order to preserve specialness of sequences of trees, we will have to show that trees remain unchanged except through the direct action of triggering sequences.

4.4 Lemma. *Fix λ, $\delta \in \mathscr{S}_f$ such that $\lambda \subseteq \delta$ and $T_{\delta,t}$ is designated. Then:*

(i) *If $T_{\lambda,t}$ receives $\langle \alpha, i\rangle$ with $i \leqslant 1$, then there is an $\eta \supset \lambda$ and a transmission sequence $\{\langle \sigma_\beta, i_\beta\rangle : \lambda \subseteq \beta \subset \eta\}$ at stage $t+1$.*

(ii) *If $\xi^- = \lambda^-$, $T_{\lambda,t}$ is not cancelled at stage $t+1$, ξ has higher priority than λ, and $T_{\xi,t}$ transmits $\langle \alpha, i\rangle$, then $i \in \{2, 3\}$.*

(iii) *If $T_{\lambda,t}$ is not cancelled at stage $t+1$, $\langle \alpha, i\rangle \in S_{\lambda,t+1}$ and $i \leqslant 1$, then $T_{\lambda,t}$ prefers $\langle \alpha, i\rangle$.*

(iv) *Suppose that $T_{\delta,t-1}$ is designated but not cancelled at stage t, that $S = \{\langle \sigma_\beta, i_\beta\rangle : \lambda \subseteq \beta \subset \delta\}$ is a transmission sequence at stage t, and that for all ξ such that $\lambda \subset \xi \subseteq \delta$, t and $t-1$ are in the same state on T_ξ. Then S is a transmission sequence at stage $t+1$.*

(v) *Suppose that $T_{\delta,t-1}$ is designated but not cancelled at stage t, that $S = \{\langle \sigma_\beta, i_\beta\rangle : \lambda \subseteq \beta \subset \delta\}$ is a transmission sequence at stage t, and that $T_{\lambda,t}$ does not transmit any $\langle \alpha, i\rangle$ with $i \leqslant 1$. Let $\xi \in \mathscr{S}_h$ be given such that $\lambda \subset \xi \subseteq \delta$, $T_{\xi,t}$ is designated as a splitting tree or as an empty differentiating tree, and suppose that either t and $t+1$ are in different states on T_ξ or $T_{\xi,t+1} \neq T_{\xi,t}$. Then $T_{\lambda,t}$ is a type i_λ extension of $T_{\lambda,t-1}$ for σ_λ and* $\mathrm{ht}(T_{\lambda,t}) = \mathrm{ht}(T_{\emptyset,t-1})$.

(vi) *If $T_{\delta,t}$ is not cancelled at stage $t+1$, $S = \{\langle \sigma_\beta, i_\beta\rangle : \lambda \subseteq \beta \subset \delta\}$ is a transmission sequence at stage t, and $T_{\lambda,t}$ is a type i_λ extension of $T_{\lambda,t-1}$ for σ_λ with* $\mathrm{ht}(T_{\lambda,t}) = \mathrm{ht}(T_{\emptyset,t-1})$, *then either $\delta = \lambda$ and $T_{\lambda,t}$ has no transmission, or there is a ξ such that T_δ triggers T_ξ at stage $t+1$.*

(vii) *If* $T_{\lambda,t}$ *is not cancelled at stage* $t+1$ *and* $T_{\lambda,t+1} \neq T_{\lambda,t}$, *then there are* $\eta \supseteq \lambda$ *and* $\xi \subseteq \lambda$ *such that* T_η *triggers* T_ξ *at stage* $t+1$ *and* $T_{\eta,t}$ *is not cancelled at stage* $t+1$; *and for all* v *of lower priority than* η, $T_{v,t+1}$ *is not designated.*

Proof. We proceed by induction on t. The lemma follows easily for $t = -1$.

(i) We proceed by induction on those λ such that $T_{\lambda,t}$ is designated, with lower priority strings coming first. Since only finitely many trees are designated at stage t, the ordering on which the induction is carried out is well-founded. Assume that $T_{\lambda,t}$ receives $\langle \alpha, i \rangle$ with $i \leqslant 1$. Then $T_{\lambda,t}$ receives and prefers some pair $\langle \sigma_\lambda, i_\lambda \rangle$, and there is a $\xi \in \mathscr{S}_h$ such that $\xi^- = \lambda$ and $T_{\xi,t}$ transmits $\langle \sigma_\lambda, i_\lambda \rangle$. If $T_{\xi,t}$ is designated as an empty differentiating tree, then $\{\langle \sigma_\lambda, i_\lambda \rangle\}$ is a transmission sequence at stage $t+1$. If $T_{\xi,t}$ is designated as an extension tree or as a non-empty differentiating tree, then by 2.12(ii) and 2.14(iii) respectively, $T_{\xi,t}$ receives $\langle \sigma_\lambda, i_\lambda \rangle$. And if $T_{\xi,t}$ is designated as a splitting tree, then by 3.5(x), $T_{\xi,t}$ must receive and prefer some $\langle \beta, j \rangle$ with $j \leqslant 1$. Thus in all cases, it follows by induction that there is an $\eta \supseteq \xi$ and a transmission sequence $\{\langle \sigma_\beta, i_\beta \rangle : \xi \subseteq \beta \subset \eta\}$ at stage $t+1$. Hence $\{\langle \sigma_\beta, i_\beta \rangle : \lambda \subseteq \beta \subset \eta\}$ will be a transmission sequence at stage $t+1$.

(ii) Assume that $\xi^- = \lambda^-$, ξ has higher priority than λ, and $T_{\xi,t}$ transmits $\langle \alpha, i \rangle$. Without loss of generality, we may assume that $\lambda = s(\xi)$, since if $T_{\lambda,t}$ is designated, then $T_{s(\xi),t}$ must also be designated. Fix the greatest $r \leqslant t$ such that $T_{\lambda,r}$ is newly designated. Since $\lambda \neq \lambda^- * 0$, Case 1, Subcase 1 or 2 of the construction of 4.3 must be followed, $T_{\xi,r}$ transmits a unique $\langle \alpha, i \rangle$ with $i \in \{2, 3\}$, and there is a transmission sequence $S = \{\langle \sigma_\beta, i_\beta \rangle : \xi \subseteq \beta \subset \eta\}$ for some $\eta \supseteq \xi$ at stage r. Furthermore, $T_{\xi,r}$ is designated as an empty differentiating tree or as a splitting tree. If $t = r$, then we are done. Otherwise, we assume by induction that for all v such that $\xi \subseteq v \subseteq \eta$ and all s such that $r \leqslant s < t$, s and $s+1$ are in the same state on T_v and $T_{v,s}$ is not cancelled at stage $s+1$. Applying (iv) inductively, we see that S is a transmission sequence at stage $t+1$, hence for all v such that $\xi \subset v \subseteq \eta$, t and $t+1$ are in the same state on T_v. If t and $t+1$ are in the same state on T_ξ, then by 2.14(x) and 3.5(xi), $T_{\xi,t}$ transmits $\langle \alpha, i \rangle$. Otherwise, since t and $t+1$ are in different states on T_ξ and S is a transmission sequence at stage $t+1$, T_η is a trigger at stage $t+1$. Hence $T_{\lambda,t}$ is cancelled at stage $t+1$, contrary to the hypothesis of (ii).

(iii) We proceed by induction on those λ such that $T_{\lambda,t}$ is designated, with lower priority strings coming first. Let $\langle \alpha, i \rangle \in S_{\lambda,t+1}$ be given, with $i \leqslant 1$. Fix the lowest priority η such that $T_{\eta,t}$ is designated and not cancelled at stage $t+1$, with $\eta^- = \lambda$. By (ii), $T_{\eta,t}$ transmits $\langle \alpha, i \rangle$. If $T_{\eta,t}$ is designated as an empty differentiating tree or as a splitting tree, then by 2.14(iv) and 3.5(iii), $\langle \alpha, i \rangle$ is the unique transmission of $T_{\eta,t}$, so $T_{\lambda,t}$ must prefer $\langle \alpha, i \rangle$. And if $T_{\eta,t}$ is designated as an extension tree or as a non-empty differentiating tree, then by 2.12(ii) and 2.14(iii), $\langle \alpha, i \rangle \in S_{\eta,t+1}$, so by induction, $T_{\eta,t}$ prefers $\langle \alpha, i \rangle$. Hence $T_{\lambda,t}$ must prefer $\langle \alpha, i \rangle$. (Since $\langle \alpha, i \rangle$ was arbitrary, we have shown that there is at most one $\langle \alpha, i \rangle \in S_{\lambda,t+1}$ such that $i \leqslant 1$.)

(iv) Suppose that $T_{\delta,t-1}$ is designated but not cancelled at stage t, that $S = \{\langle \sigma_\beta, i_\beta \rangle : \lambda \subseteq \beta \subset \delta\}$ is a transmission sequence at stage t, and that for all ξ such that $\lambda \subset \xi \subseteq \delta$, t and $t-1$ are in the same state on T_ξ. Then an induction argument on $\{\xi : \lambda \subset \xi \subseteq \delta\}$, longer strings first, using (iii), 2.12(ii), 2.14(iii) and (x) and 3.5(xi) shows that for all ξ such that $\lambda \subset \xi \subseteq \delta$, $T_{\xi,t}$ and $T_{\xi,t-1}$ transmit the same pair. Hence S is a transmission sequence at stage $t+1$.

(v) Suppose that $T_{\delta,t-1}$ is designated but not cancelled at stage t, and that $S = \{\langle \sigma_\beta, i_\beta \rangle : \lambda \subseteq \beta \subset \delta\}$ is a maximal transmission sequence at stage t. Let $\xi \in \mathscr{S}_h$ be given such that $\lambda \subset \xi \subseteq \delta$, $T_{\xi,t}$ is designated as a splitting tree or as an empty differentiating tree, and either t and $t+1$ are in different states on T_ξ or $T_{\xi,t+1} \neq T_{\xi,t}$. Fix the longest η such that $\lambda \subseteq \eta \subset \xi$ and $T_{\eta,t-1}$ is designated as an initial tree or as a splitting tree. Note that by choice of ξ, $T_{\lambda,t-1}$ is designated as an initial tree or as a splitting tree, so η exists. By 2.14(xii) and 3.5(ix), $T_{\xi,t} = T_{\xi,t-1}$. By 2.14(ixa), 3.5(viiia), 2.12(ii) and 2.14(iii), $T_{\eta,t}$ is a type i_ξ extension of $T_{\eta,t-1}$ for σ_ξ and $\mathrm{ht}(T_{\eta,t}) = \mathrm{ht}(T_{\emptyset,t-1})$. Furthermore, $\eta = \lambda$, else $\langle \sigma_\eta, i_\eta \rangle$ would be defined, and so by 3.5(ix), we would have $T_{\eta,t} = T_{\eta,t-1}$.

(vi) Let $S = \{\langle \sigma_\beta, i_\beta \rangle : \lambda \subseteq \beta \subset \delta\}$ be a transmission sequence at stage t, and assume that $T_{\lambda,t}$ is a type i_λ extension of $T_{\lambda,t-1}$ for σ_λ with $\mathrm{ht}(T_{\lambda,t}) = \mathrm{ht}(T_{\emptyset,t-1})$. If $\lambda = \delta$, then by 2.14(xii), $T_{\lambda,t}$ has no transmission. Assume that $\lambda \neq \delta$, and fix the shortest $\eta \supset \lambda$ for which T_η is designated as an empty differentiating tree or as a splitting tree. By 4.2(iii), $T_{\eta,t-1}$ transmits $\langle \sigma_\lambda, i_\lambda \rangle$, so by 2.14(ixa), 3.5(viiia), 2.12(ii), 2.14(iii) and since $T_{\eta,t}$ is not cancelled at stage $t+1$, t and $t+1$ are in different states on T_η as long as $T_{\eta,t}$ receives $\langle \sigma_\eta, i_\eta \rangle$. In any case, it follows from (iv) that there is a longest v such that $\eta \subseteq v \subseteq \delta$ and t and $t+1$ are in different states on T_v. Hence for some $\xi \subseteq v$, T_δ triggers T_ξ at stage $t+1$.

(vii) Suppose that $T_{\lambda,t}$ is not cancelled at stage $t+1$ and $T_{\lambda,t+1} \neq T_{\lambda,t}$. Fix the longest $\xi \subseteq \lambda$ such that $T_{\xi,t}$ is designated as an empty differentiating tree, a splitting tree, or the initial tree. By 4.2(iii), $T_{\xi,t+1} \neq T_{\xi,t}$. We note that by Step 0 of 3.5, if $T_{\xi,t} \neq \emptyset$ and r is the greatest stage $\leqslant t$ such that $T_{\xi,r}$ is newly designated as a splitting tree, then $T_{\xi,r} \neq \emptyset$ (else $T_{\xi,t} = T_{\xi,t+1} = \emptyset$). If $T_{\xi,t}$ is not designated as an empty differentiating tree, then by 3.5(ix) and 2.8(i), $T_{\xi,t}$ must receive and prefer some $\langle \alpha, i \rangle \in S_{\xi,t+1}$ with $i \leqslant 1$. By (i), there is a highest priority $\eta \supseteq \xi$ and a transmission sequence $\{\langle \sigma_\beta, i_\beta \rangle : \xi \subseteq \beta \subset \eta\}$ at stage $t+1$. Since $T_{\xi,t+1} \neq T_{\xi,t}$, it follows from 2.14(xii) and 3.5(ix) that $T_{\xi,t}$ has no transmission, and t and $t+1$ are in different states on T_ξ. Hence T_η triggers T_ξ at stage $t+1$. $T_{\eta,t}$ cannot have been cancelled at stage $t+1$, else $\langle \alpha, i \rangle \notin S_{\xi,t+1}$. Hence T_η is the unique trigger at stage $t+1$, and Case 1, Subcase 3 or Subcase 4 of 4.3 is followed at stage $t+1$. But then if v has lower priority than η, then $T_{v,t}$ is cancelled at stage $t+1$, and $T_{v,t+1}$ cannot be newly designated at stage $t+1$. Hence $T_{v,t+1}$ cannot be designated if v has lower priority than η. ∎

The proof of (3) will require that $T_\gamma(\emptyset)$ be a focal point of T_δ for all $\delta \subseteq \gamma$. A crucial step in the verification of this fact is that reception of pairs satisfies 2.3(i)–(v). 2.3(i)–(iv) will follow from the lemmas of Sects. 2 and 3. We turn our attention to the verification of 2.3(v). We will have to consider the relationship between $T_{\xi,t}(\emptyset)$, the pair transmitted by $T_{\xi,t}$, and $T_{\lambda,t}(\emptyset)$ whenever $\xi^- = \lambda^-$ and ξ has higher priority than λ. We specify this relationship along with some other useful facts about the construction of 4.3 in the next lemma.

4.5 Lemma. Fix $\lambda \in \mathscr{S}_h$, $\sigma \in \mathscr{S}_f$, $t \in N$ *and* $i \leqslant 3$ *such that* $\lambda \neq \emptyset$ *and* $T_{\lambda,t}$ *is designated. Then*:

(i) *If* $T_{\lambda,t}$ *transmits* $\langle \sigma, i \rangle$, *then* $\sigma \subset T_{\lambda^-,t}$. *If, in addition,* $T_{\lambda,t}(\emptyset)\downarrow$, *then* $\sigma \supseteq T_{\lambda,t}(\emptyset)$.

(ii) *If* $\lambda \neq \lambda^- * 0$ *then* $T_{p(\lambda),t}$ *transmits a unique pair* $\langle \sigma, i \rangle$. *For this pair,* $i \in \{2,3\}$, $\mathrm{lh}(\sigma) > \mathrm{ht}(T_{p(\lambda),t})$ *and* $T_{\lambda,t}(\emptyset)\downarrow = \sigma$.

(iii) *If* $\lambda \neq \lambda^- * 0$ *and* $T_{p(\lambda),t}(\emptyset)\downarrow$, *then* $T_{\lambda,t}(\emptyset) \supset T_{p(\lambda),t}(\emptyset)$ *and* $\mathrm{lh}(T_{\lambda,t}(\emptyset)) > \mathrm{ht}(T_{p(\lambda),t})$.

(iv) *For all* $\xi \in \mathscr{S}_h$, *if* $\xi^- = \lambda^-$, ξ *has higher priority than* λ, *and* $T_{\xi,t}$ *transmits* $\langle \sigma, i \rangle$, *then* $T_{\lambda,t}(\emptyset) \supseteq \sigma$ *and* $i \in \{2,3\}$.

(v) *If* $\langle \beta, j \rangle$, $\langle \gamma, k \rangle \in S_{\lambda,t+1}$ *and* $\langle \beta, j \rangle \neq \langle \gamma, k \rangle$, *then* $\beta \subseteq \gamma$ *or* $\gamma \subseteq \beta$. *If, in addition,* $k \in \{0,1\}$, *then* $\beta \subseteq \gamma$ *and* $j \in \{2,3\}$.

(vi) *If* $T_{\lambda,t}$ *receives* $\langle \sigma, 0 \rangle$, *then* $\mathrm{lh}(\sigma) < \mathrm{ht}(T_{\lambda,t})$.

(vii) *If* $T_{\lambda,t}$ *receives* $\langle \sigma, 0 \rangle$, $\beta \subseteq \sigma$ *and* β *is a potential focal point of* $T_{\lambda,t}$, *then* β *is a focal point of* $T_{\lambda,t}$.

(viii) *If* $T_{\lambda,t}$ *is newly designated, then* $\mathrm{dom}(T_{\lambda,t}) \subseteq \{\emptyset\}$; *and if* $T_{\lambda,t} = \emptyset$, *then either* $T_{\lambda,t}$ *is designated as a differentiating tree or* $T_{\lambda^-,t} = \emptyset$.

Proof. We proceed by induction on t, and then by induction on those λ such that $T_{\lambda,t}$ is designated, with lower priority λ considered first. Fix t, λ, σ and i as in the hypothesis of the lemma.

(i) Assume that $T_{\lambda,t}$ transmits $\langle \sigma, i \rangle$. If $T_{\lambda,t}$ is designated as an empty differentiating tree, then (i) follows from Lemma 2.14(iv). If $T_{\lambda,t}$ is designated as an extension tree or as a non-empty differentiating tree, then by 2.12(ii) and 2.14(iii) respectively, there is an $\eta \in \mathscr{S}_h$ such that $\eta^- = \lambda$ and $T_{\eta,t}$ transmits $\langle \sigma, i \rangle$. Applying (i) by induction to η, we see that $\sigma \subset T_{\lambda,t} \subseteq T_{\lambda^-,t}$ so $\sigma \supseteq T_{\lambda,t}(\emptyset)$. Finally, if $T_{\lambda,t}$ is designated as a splitting tree, then (i) follows from Lemma 3.5(iii) since $T_{\lambda,t}(\emptyset)\downarrow \subseteq \alpha^*(t)$.

(ii) Fix $\lambda \neq \lambda^- * 0$ such that $T_{\lambda,t}$ is designated. First consider the case where $T_{\lambda,t}$ is newly designated. Then Case 1, Subcase 1 or Subcase 2 of the construction of 4.3 is followed at stage t, and by (i) and 3.5(vi), $T_{p(\lambda),t-1}$ transmits a unique pair $\langle \sigma, i \rangle$, $i \in \{2,3\}$, $\mathrm{lh}(\sigma) > \mathrm{ht}(T_{p(\lambda),t-1})$, $T_{\lambda,t}(\emptyset)\downarrow = \sigma$, and there is a $\delta \geqslant p(\lambda)$ such that T_δ triggers $T_{p(\lambda)}$ at stage t and T_δ is not cancelled at stage t. Let S be the transmission sequence from T_δ to $T_{p(\lambda)}$ at stage t. By 4.4(v), 4.4(vii), 3.5(viii) and 2.14(ix), t and $t+1$ are in the same state on T_η and $T_{\eta,t} = T_{\eta,t-1}$ for all η such that $p(\lambda) \subseteq \eta \subseteq \delta$. Hence by 3.5(xi) and 2.14(x), $T_{p(\lambda),t}$ also transmits $\langle \sigma, i \rangle$. Hence (ii) holds.

Now assume that $T_{\lambda,t}$ is not newly designated. Since $T_{\lambda,t}$ is designated, $T_{\lambda,t-1}$ is designated and not cancelled at stage t. We assume by induction that S is still a transmission sequence at stage $t-1$. By 4.4(v), 4.4(vii), 3.5(viii) and 2.14(ix), t and $t-1$ are in the same state on $T_{p(\lambda)}$; and by 2.14(x), 3.5(xi) and 4.4(iv), S is a transmission sequence at stage t, and $T_{p(\lambda),t}$ transmits $\langle \sigma, i \rangle$.

(iii) Assume that $\lambda \neq \lambda^- * 0$ and that $T_{p(\lambda),t}(\emptyset)\downarrow$. By (ii), $T_{p(\lambda),t}$ has a unique transmission $\langle \sigma, 2 \rangle$ or $\langle \sigma, 3 \rangle$, $T_{\lambda,t}(\emptyset) = \sigma$, and $\mathrm{lh}(\sigma) > \mathrm{ht}(T_{p(\lambda),t})$. By (i), $\sigma \supset T_{p(\lambda),t}(\emptyset)$.

(iv) Fix ξ such that $\xi^- = \lambda^-$, ξ has higher priority than λ, and $T_{\xi,t}$ transmits $\langle \sigma, i \rangle$. Define $s^1(\xi) = s(\xi)$ and $s^{k+1}(\xi) = s(s^k(\xi))$. There must be a k such that $\lambda = s^k(\xi)$, and for all $j \leqslant k$, $T_{s^j(\xi),t}$ must be designated. By (ii), $T_{s(\xi),t}(\emptyset) = \sigma$ and $i \in \{2,3\}$. Iterating (iii), we see that $T_{\lambda,t}(\emptyset) = T_{s^k(\xi),t}(\emptyset) \supseteq \cdots \supseteq T_{s(\xi),t}(\emptyset) = \sigma$.

(v) Assume that $\langle \beta, j \rangle \neq \langle \gamma, k \rangle \in S_{\lambda,t+1}$. Then $T_{\lambda,t+1}$ is not newly designated, else $S_{\lambda,t+1} = \emptyset$. Hence $T_{\lambda,t}$ is not cancelled at stage $t+1$. Thus there are $\xi, \eta \in \mathscr{S}_h$ such that $\xi^- = \eta^- = \lambda$, $T_{\xi,t}$ transmits $\langle \beta, j \rangle$, and $T_{\eta,t}$ transmits $\langle \gamma, k \rangle$. If

$k \in \{0, 1\}$, then by (iv), either ξ has higher priority than η and $j \in \{2, 3\}$, or $\xi = \eta$. Assume first that ξ has higher priority than η. (A similar proof will work if η has higher priority than ξ.) Then $\eta \neq \eta^- * 0$. By (i) and (iv), $\beta \subseteq T_{\eta,t}(\emptyset) \subseteq \gamma$. Finally, assume that $\xi = \eta$. Then by 3.5(iii) and 2.14(iv), $T_{\xi,t}$ cannot be designated as a splitting tree or as an empty differentiating tree. Hence $T_{\xi,t}$ is designated either as an extension tree or as a non-empty differentiating tree. By 2.12(ii) and 2.14(iii), $T_{\xi,t}$ receives both $\langle \beta, j \rangle$ and $\langle \gamma, k \rangle$. Hence (v) follows by induction.

(vi) Let $T_{\lambda,t}$ receive $\langle \sigma, 0 \rangle$. Then there is a $\xi \in \mathscr{S}_h$ such that $\xi^- = \lambda$ and $T_{\xi,t}$ transmits $\langle \sigma, 0 \rangle$. If $T_{\xi,t}$ is designated as an extension tree or as a non-empty differentiating tree, then by 2.12(ii), 2.14(iii) and induction, $\mathrm{lh}(\sigma) < \mathrm{ht}(T_{\xi,t}) \leqslant \mathrm{ht}(T_{\lambda,t})$. If $T_{\xi,t}$ is designated as an empty differentiating tree or as a splitting tree, then by 2.14(v) and 3.5(iv), $\mathrm{lh}(\sigma) < \mathrm{ht}(T_{\lambda,t})$.

(vii) A potential focal point β of $T_{\lambda,t}$ will be a focal point of $T_{\lambda,t}$ if and only if $\mathrm{lh}(\beta) < \mathrm{ht}(T_{\lambda,t})$. Let $T_{\lambda,t}$ receive $\langle \sigma, 0 \rangle$, and let $\beta \subseteq \sigma$ be a potential focal point of $T_{\lambda,t}$. By (vi), $\mathrm{lh}(\sigma) < \mathrm{ht}(T_{\lambda,t})$. Since $\beta \subseteq \sigma$, $\mathrm{lh}(\beta) < \mathrm{ht}(T_{\lambda,t})$. Hence β is a focal point of $T_{\lambda,t}$.

(viii) Let $T_{\lambda,t}$ be newly designated, with $T_{\lambda,t} \neq \emptyset$. First assume that Case 1, Subcase 1 or Subcase 2 of the construction of 4.3 is followed at stage t. Then $T_{p(\lambda),t-1}$ is designated either as an empty differentiating tree or as a splitting tree and is not cancelled at stage t; and there is a $\delta \supseteq p(\lambda)$ such that T_δ triggers $T_{p(\lambda)}$ at stage t and T_δ is not cancelled at stage t. By 4.2(iv), 4.4(v), 2.14(xii) and 3.5(ix), t and $t - 1$ must be in different states on $T_{p(\lambda)}$. Hence by (i), there is a pair $\langle \sigma, i \rangle$ transmitted by $T_{p(\lambda),t-1}$ such that $\sigma \subset T_{\lambda^-,t-1}$, so $T_{\lambda,t}(\emptyset)\downarrow = \sigma$. By 2.14(xii), 3.5(ix) and 4.4(vii), $\mathrm{lh}(\sigma) = \mathrm{ht}(T_{\lambda^-,t-1}) = \mathrm{ht}(T_{\lambda^-,t})$. Hence $\mathrm{dom}(T_{\lambda,t}) = \{\emptyset\}$.

If Case 2, Subcase 2 of the construction of 4.3 is followed at stage t, then $T_{\lambda,t}$ is designated as a differentiating tree. Otherwise, Case 2, Subcase 1 of the construction of 4.3 is followed at stage t, and $\gamma_{t-1} = \lambda^-$. By 4.4(vi), either $T_{\lambda^-,t-1}$ is newly designated or $T_{\lambda^-,t-1} \neq T_{\lambda^-,t-2} = \emptyset$, and $T_{\lambda^-,t-1}$ is designated as a differentiating tree. Hence by 4.4(vii), 2.14(xii) and induction, $\mathrm{dom}(T_{\lambda^-,t-1}) \subseteq \{\emptyset\}$ and $T_{\lambda,t} \subseteq T_{\lambda^-,t-1}$. Hence $\mathrm{dom}(T_{\lambda,t}) \subseteq \{\emptyset\}$. ∎

We now turn our attention to proving that the construction generates special sequences of trees in which reception of strings satisfies 2.3(i)–(v). The proof will involve conditions about the preservation of focal points, as well as conditions about the mechanics of transmission and reception of pairs.

4.6 Lemma. *Let $\lambda \in \mathscr{S}_h$ and $t \in N$ be given such that $T_{\lambda,t}$ is designated. Then*:

(i) *If δ has lower priority than λ, $T_{\delta,t}$ is designated but not cancelled at stage $t + 1$, $\{\langle \sigma_\beta, i_\beta \rangle : \xi \subseteq \beta \subset \delta\}$ is a transmission sequence at stage $t + 1$, $T_{\xi,t}$ transmits $\langle \sigma, i \rangle$, and $T_{\lambda,t+1}(\emptyset)\downarrow$, then $T_{\lambda,t+1}(\emptyset) \subseteq \sigma$.*

(ii) *If $T_{\xi,t+1}$ is designated, $T_{\xi,t+1}(\emptyset)\downarrow$, ξ has lower priority than λ, and $T_{\lambda,t+1}(\emptyset)\downarrow$, then $T_{\lambda,t+1}(\emptyset) \subseteq T_{\xi,t+1}(\emptyset)$.*

(iii) *Reception of pairs by $T_{\lambda,t}$ satisfies* 2.3(i)–(v).

(iv) *If for all $\xi \subseteq \lambda$, β is a potential focal point of $T_{\xi,t}$, and for all $\langle \alpha, i \rangle \in S_{\lambda,t+1}$, either $i \in \{2, 3\}$ or $\beta \subseteq \alpha$, then β is a potential focal point of $T_{\lambda,t+1}$.*

(v) *$\{T_{\xi,t+1} : \xi \subseteq \lambda \;\&\; t \leqslant s \leqslant t + 1\}$ is special.*

(vi) *If $T_{\lambda,t+1} \neq \emptyset$, then for all $\xi \subseteq \lambda$, $T_{\lambda,t+1}(\emptyset)$ is a potential focal point of $T_{\xi,t+1}$.*

Proof. We proceed by induction on t. The lemma follows easily for $t = -1$.

(i) Fix δ of lower priority than λ such that $T_{\delta,t}$ is designated but not cancelled at stage $t+1$. Fix a maximal transmission sequence $S = \{\langle \sigma_\beta, i_\beta \rangle: \xi \subseteq \beta \subset \delta\}$ at stage $t+1$. Since S is a transmission sequence, $T_{\delta,t}$ is designated as an empty differentiating tree, hence $\delta = \delta^- * 0$. Thus if λ has higher priority than δ, then either $\lambda = \delta^-$ or λ has higher priority than δ^-. By 2.14(iv), $T_{\delta^-,t}(\emptyset)\downarrow$, else we have nothing to show. We note that $T_{\lambda,t}(\emptyset)\downarrow$ and $T_{\lambda,t}$ is not cancelled at stage $t+1$, else $T_{\delta,t}$ would be cancelled at stage $t+1$. It thus follows inductively from (ii) that $T_{\lambda,t+1}(\emptyset) = T_{\lambda,t}(\emptyset) \subseteq T_{\delta^-,t}(\emptyset)$. Let $\langle \sigma_{\xi^-}, i_{\xi^-} \rangle$ be the transmission of $T_{\xi,t}$ if such a transmission exists. It suffices to show that for all β such that $\xi^- \subset \beta \subseteq \delta$, $T_{\delta^-,t}(\emptyset) \subseteq \sigma_{\beta^-}$. We proceed by induction on $\{\beta: \xi^- \subset \beta \subseteq \delta\}$, longer strings first. If T_β is designated as an empty differentiating tree, then $\beta = \delta$, so by 2.14(iv), $T_{\delta^-,t}(\emptyset) \subseteq \sigma_{\delta^-}$. If T_β is designated as an extension tree or as a non-empty differentiating tree, then by induction, $T_{\delta^-,t}(\emptyset) \subseteq \sigma_\beta = \sigma_{\beta^-}$. And if T_β is designated as a splitting tree, then by 3.5(iii), $\alpha^*(t+1) \subseteq \sigma_{\beta^-}$. By (vi) and 4.5(vii), $T_{\delta^-,t}(\emptyset)$ is a potential focal point of $T_{\beta,t}$ and if $i_\beta = 0$, then $T_{\delta^-,t}(\emptyset)$ is a focal point of $T_{\beta,t}$. Hence $T_{\delta^-,t}(\emptyset) \subseteq \alpha^*(t+1) \subseteq \sigma_{\beta^-}$.

(ii) If $\xi \neq \gamma_{t+1}$, then $T_{\xi,t+1}$ is designated if and only if $T_{\xi,t}$ is designated and not cancelled at stage $t+1$. Hence if ξ has higher priority than λ, then applying (ii) by induction, we see that $T_{\lambda,t+1}(\emptyset) = T_{\lambda,t}(\emptyset) \subseteq T_{\xi,t}(\emptyset) = T_{\xi,t+1}(\emptyset)$. Assume that $\xi = \gamma_{t+1}$. If Case 2 of the construction of 4.3 is followed at stage $t+1$, then $\xi = \xi^- * 0$, so either $\xi^- = \lambda$ or λ has higher priority than ξ^-. By induction and since $T_{\xi,t+1} \subseteq T_{\xi^-,t}$, $T_{\lambda,t+1}(\emptyset) = T_{\lambda,t}(\emptyset) \subseteq T_{\xi^-,t}(\emptyset) \subseteq T_{\xi,t+1}(\emptyset)$. Otherwise, Case 1 of the construction of 4.3 is followed at stage $t+1$, and there is a $\delta \supset \xi^-$ and a transmission sequence $\{\langle \sigma_\beta, i_\beta \rangle: p(\xi) \subseteq \beta \subset \delta\}$ such that $T_{\xi,t+1}(\emptyset) = \sigma_\xi$. By the cancellation procedure and since $T_{\delta,t} = \emptyset$, λ has higher priority than δ. Hence by (i), $T_{\lambda,t+1}(\emptyset) \subseteq \sigma_\xi = T_{\xi,t+1}(\emptyset)$.

(iii) 2.3(i), (ii), and (v) for $T_{\lambda,t}$ follow from 4.5(i), 4.5(vi) and 4.5(v) respectively. Let $T_{\lambda,t}$ receive $\langle \alpha, i \rangle$ with $i \in \{1,2,3\}$. Fix the shortest $\xi \supset \lambda$ such that $T_{\xi,t}$ transmits $\langle \alpha, i \rangle$ and $T_{\xi,t}$ is designated either as a splitting tree or as an empty differentiating tree. If either $T_{\xi,t}$ is newly designated or if t and $t+1$ are in different states on T_ξ, then by 3.5(v) and 2.14(vi), for all $\eta \subseteq \lambda$ α is a potential focal point of $T_{\eta,t}$ which is not a focal point of $T_{\eta,t}$, so 2.3(iii) and 2.3(iv) hold for $T_{\lambda,t}$. We must now verify 2.3(iii) for subsequent stages. 2.3(iii) will follow for $T_{\lambda,t}$ by induction unless $T_{\lambda,t} \neq T_{\lambda,t-1}$. Thus assume that $T_{\lambda,t} \neq T_{\lambda,t-1}$. Since $T_{\xi,t-1}$ transmits $\langle \alpha, i \rangle$ and $T_{\xi,t-1}$ is not cancelled at stage t, it follows from 4.4(iii) that $T_{\lambda,t-1}$ prefers $\langle \alpha, i \rangle$. Applying (v) inductively, we see that $T_{\lambda,t}$ is a type i extension of $T_{\lambda,t-1}$ for α such that $\mathrm{ht}(T_{\lambda,t}) = \mathrm{ht}(T_{\emptyset,t})$. Since $T_{\lambda,t} \subseteq T_{\emptyset,t-1}$, $\mathrm{ht}(T_{\lambda,t}) = \mathrm{ht}(T_{\emptyset,t-1})$. Hence by 2.14(ixa) and 3.5(viiia), t and $t+1$ are in different states on T_ξ, a case which has already been considered.

(iv) We proceed by induction on $\mathrm{lh}(\lambda)$, shorter strings first. Assume that for all $\xi \subseteq \lambda$, β is a potential focal point of $T_{\xi,t}$, and that for all $\langle \alpha, i \rangle \in S_{\lambda,t+1}$, either $i \in \{2,3\}$ or $\beta \subseteq \alpha$. If $T_{\lambda,t+1} = T_{\lambda,t}$ then β is a potential focal point of $T_{\lambda,t+1}$. Assume that $T_{\lambda,t+1} \neq T_{\lambda,t}$. If $T_{\lambda,t}$ is designated as an initial tree or as a splitting tree, then by 2.8(i) and 3.5(ix), $T_{\lambda,t}$ receives and prefers some $\langle \alpha, i \rangle$ and $T_{\lambda,t+1}$ is a type i extension of $T_{\lambda,t}$ for α. By 4.5(vii), for all $\rho \subset T_{\lambda,t+1} - T_{\lambda,t}$, $\beta \subseteq \rho$. Hence β is a potential focal point of $T_{\lambda,t+1}$. $T_{\lambda,t} \neq \emptyset$, else β would not be defined. Suppose that $T_{\lambda,t}$ is designated as an extension tree or as a non-empty differentiating tree. Then $T_{\lambda^-,t}$ is designated and receives all pairs which $T_{\lambda,t}$ receives. Let $T_{\lambda^-,t}$ receive $\langle\alpha, i\rangle$ with $i \leqslant 1$. Then there is a ξ such that $\xi^- = \lambda^-$ and $T_{\xi,t}$ transmits $\langle\alpha, i\rangle$. By the construction of 4.3, λ cannot have higher priority than ξ. It now follows from 4.5(iv) that for all $\langle\alpha, i\rangle$ received by $T_{\lambda^-,t}$, either $\beta \subseteq \alpha$ or $i \in \{2, 3\}$. Hence (iv) follows by induction.

(v) If $\lambda = \emptyset$, then it follows from Remark 2.10 that $\{T_{\eta,s}\colon \eta \subseteq \lambda \,\&\, t \leqslant s \leqslant t + 1\}$ is special. Assume that $\lambda \neq \emptyset$. 2.9(i) for $\{T_{\eta,s}\colon \eta \subseteq \lambda \,\&\, t \leqslant s \leqslant t + 1\}$ follows from 2.12(i), 2.14(i) and 3.5(i). 2.9(ii) for $\{T_{\eta,s}\colon \eta \subseteq \lambda \,\&\, t \leqslant s \leqslant t + 1\}$ follows from (iv). We now verify 2.9(iii) for $\{T_{\eta,s}\colon \eta \subseteq \lambda \,\&\, t \leqslant s \leqslant t + 1\}$.

We proceed by induction on $\mathrm{lh}(\lambda)$, shorter strings first. Thus we may assume that $\{T_{\eta,s}\colon \eta \subset \lambda \,\&\, t \leqslant s \leqslant t + 1\}$ is special. We assume that $T_{\lambda,t+1} \neq T_{\lambda,t}$ or that $T_{\lambda,t}$ is newly designated. If $T_{\lambda,t+1}$ is designated as an initial tree, a splitting tree or an empty differentiating tree, then it follows from 2.8(i), 3.5(ix), and 2.14(xii) that $\mathrm{ht}(T_{\lambda,t+1}) = \mathrm{ht}(T_{\emptyset,t})$, $T_{\lambda,t}$ has no transmission, and if $T_{\lambda,t} = \emptyset$ or if $T_{\lambda,t+1}$ is newly designated, then it follows from 4.5(viii) that $T_{\lambda,t}$ receives and prefers some $\langle\alpha, i\rangle$ and $T_{\lambda,t+1}$ is a type i extension of $T_{\lambda,t}$ for α. By 4.4(vii), we cannot have $T_{\eta,t+1} \neq T_{\eta,t}$ for any $\eta \subset \lambda$. Hence $\mathrm{ht}(T_{\lambda,t+1}) = \mathrm{ht}(T_{\eta,t+1})$ for all $\eta \subseteq \lambda$, and 2.9(iii) holds in this case.

Suppose that $T_{\lambda,t+1}$ is designated as an extension tree or as a non-empty differentiating tree. First suppose that $T_{\lambda,t+1}$ is newly designated. Then Case 1, Subcase 1 or Subcase 2 of the construction of 4.3 is followed at stage $t + 1$, and by 4.5(viii), $\mathrm{dom}(T_{\lambda,t+1}) = \{\emptyset\}$. By 4.5(ii), $T_{p(\lambda),t}$ must newly transmit some $\langle\tau, j\rangle$ with $j \in \{2, 3\}$, hence by the proof of (iii), τ is a potential focal point of $T_{\eta,t}$ which is not a focal point of $T_{\eta,t}$ for all $\eta \subseteq \lambda^-$, and $T_{\lambda,t}$ has no transmission. Hence $\mathrm{ht}(T_{\lambda,t+1}) = \mathrm{ht}(T_{\emptyset,t})$. Again it follows from 4.4(vii) that $\mathrm{ht}(T_{\lambda,t+1}) = \mathrm{ht}(T_{\eta,t+1})$ for all $\eta \subseteq \lambda$. Next suppose that $T_{\lambda,t+1}$ is not newly designated. Fix the longest $\xi \subset \lambda$ such that $T_{\xi,t}$ is designated as the initial tree or as a splitting tree. Then $T_{\xi,t+1} \neq T_{\xi,t}$ so by induction, $\mathrm{ht}(T_{\xi,t+1}) = \mathrm{ht}(T_{\emptyset,t+1})$, $T_{\xi,t}$ receives and prefers some $\langle\alpha, i\rangle$ and $T_{\xi,t+1}$ is a type i extension of $T_{\xi,t}$ for α. By 2.12(ii) and 2.14(iii), it suffices to show that $T_{\lambda,t}$ receives $\langle\alpha, i\rangle$. But this follows from 4.5(iv) and induction, as if $\lambda^- = \xi$, then by the construction of 4.3, there can be no η such that $T_{\eta,t}$ is designated, $\eta^- = \xi$, and λ has higher priority than η.

(vi) Fix λ such that $T_{\lambda,t+1}(\emptyset)\!\downarrow$. First suppose that $T_{\lambda,t+1}$ is newly designated or that $T_{\lambda,t} = \emptyset$. By 4.5(viii), $\mathrm{dom}(T_{\lambda,t+1}) = \{\emptyset\}$. By (v) and 2.9(iii), $T_{\lambda,t+1}(\emptyset)$ is a potential focal point of $T_{\xi,t+1}$ for all $\xi \subseteq \lambda$. Otherwise, $T_{\lambda,t+1}(\emptyset) = T_{\lambda,t}(\emptyset)$ which, by induction, is a potential focal point of $T_{\xi,t}$ for all $\xi \subseteq \lambda$. By (iv), it suffices to show that for all $\xi \subseteq \lambda$ and $\langle\alpha, i\rangle \in S_{\xi,t+1}$, if $i \leqslant 1$ then $T_{\lambda,t+1}(\emptyset) \subseteq \alpha$. Let $\langle\alpha, i\rangle \in S_{\xi,t+1}$ be given with $i \leqslant 1$. By 4.4(i) and 4.4(iii), there is a transmission sequence $\{\langle\sigma_\beta, i_\beta\rangle\colon \xi \subseteq \beta \subset \delta\}$ for some $\delta \supset \xi$ at stage $t + 1$, with $\langle\sigma_\xi, i_\xi\rangle = \langle\alpha, i\rangle$ and $T_{\delta,t} = \emptyset$. If λ has higher priority than δ, then by (i), $T_{\lambda,t+1}(\emptyset) \subseteq \sigma_\xi = \alpha$. We cannot have $\lambda = \delta$, as $T_{\delta,t} = \emptyset$, so by 4.5(i), $S_{\delta,t+1} = \emptyset$. We complete the proof by assuming that δ has higher priority than λ and obtaining a contradiction. Fix the longest η such that $\eta \subseteq \delta$ and $\eta \subseteq \lambda$. Then $\xi \subseteq \eta$. Hence $i_\eta \leqslant 1$, contradicting 4.4(ii). ▯

We are now almost ready to prove that $\mathrm{lh}(\Gamma) = \infty$. This fact will follow from the next lemma about triggering sequences and their effect on the definition of γ_s.

4.7 Lemma. *Let $\delta \in \mathcal{S}_h$ and $t \in N$ be given such that $T_{\delta,t}$ is designated. Then*:

(i) *If $T_{\delta,t} = \emptyset$ then $T_{\delta,t}$ is designated as a differentiating tree. If, in addition, $\eta \in \mathcal{S}_h$ is given such that $\eta^- = \delta$, then $T_{\eta,t}$ is not designated.*

(ii) *If $T_{\delta,s}$ is not cancelled at stage $s + 1$ for all $s \geqslant t$, then $\{s: \gamma_s = \delta\}$ is finite.*

Proof. (i) We proceed by induction on t. (i) clearly holds for $t = 0$. We assume by induction that (i) holds for $t - 1$ in place of t.

Since at most one tree $T_{\gamma_t,t}$ is newly designated at stage t, we need only verify (i) for $\delta = \gamma_t$ when $T_{\gamma_t,t}$ is newly designated. We assume the following as an additional induction hypothesis:

(4) If $T_{\gamma_t,t} = \emptyset$, then either there is a $\xi \subset \gamma_t$ such that $T_{\xi,t} \neq T_{\xi,t-1}$ and T_{γ_t} triggers T_ξ at stage t, or $T_{\gamma_t,t}$ is newly designated and transmits some pair.

We analyze the construction at stage t. First assume that Case 1, Subcase 1 or 2 is followed at stage t. Then there is a δ such that T_δ triggers $T_{\gamma_t^-}$ at stage t and $T_{\gamma_t^-,t-1}$ transmits $\langle \tau, i \rangle$ with $i \in \{2, 3\}$. Furthermore, $\tau = T_{\gamma_t,t}(\emptyset) \subset T_{\gamma_t^-,t}$, so $T_{\gamma_t^-,t} \neq \emptyset$ and (i) holds. Since $T_{\gamma_t,t} \neq \emptyset$, (4) must also hold.

Next assume that Case 1, Subcase 3 or 4 is followed at stage t. Then no trees are newly designated at stage t, so (i) follows by induction. These subcases can be followed only if T_{γ_t} triggers some T_ξ at stage t and $T_{\xi,t-1}$ has no transmission. If Subcase 3 is followed, then $\xi = \gamma_t$, hence by 2.14(xiii), 4.6(v) and 2.9(iii), $T_{\gamma_t,t} \neq \emptyset$, so (4) holds. And if Subcase 4 is followed, then (4) follows from 2.8(i) if $\xi = \emptyset$ and from 4.6(iii) and 3.5(xiii) if $\xi \neq \emptyset$.

Suppose that Case 2 is followed at stage t. Then $T_{\gamma_t,t}$ is newly designated. If Subcase 1 is followed and $T_{\gamma_t^-,t} \neq \emptyset$ then $T_{\gamma_t,t} \neq \emptyset$. If Subcase 2 is followed, then by 4.6(v), 2.9(iii) and 2.14(xiii), $T_{\gamma_t,t}$ will have a transmission if $T_{\gamma_t,t} = \emptyset$ and $T_{\gamma_t^-,t} = T_{\gamma_t^-,t-1} \neq T_{\gamma_t^-,t-2}$. Note that $\gamma_t^- = \gamma_{t-1}$. Hence by 4.3, both (4) and (i) will follow once we show that $T_{\gamma_t^-,t-1} \neq \emptyset$. If $\gamma_{t-1} \neq \gamma_{t-1}^- * 0$, then by 4.5(ii), $T_{\gamma_{t-1},t-1} \neq \emptyset$. Suppose that $\gamma_{t-1} = \gamma_t^- * 0$. We apply (4) by induction. There are two cases to consider.

First suppose that $T_{\gamma_{t-1},t-1}$ is newly designated. Then by (4), if $T_{\gamma_{t-1},t-1} = \emptyset$, there is a longest transmission sequence S at stage t from $T_{\gamma_{t-1}}$ to T_ξ. If $\xi = \emptyset$, then S is a triggering sequence, so Case 2 would not be followed at stage t. If $T_{\xi,t-1}$ transmits $\langle \tau, i \rangle$, then $i \in \{2, 3\}$ by the maximality of S, and as the transmission sequence did not exist at stage $t - 1$, $t - 1$ and $t - 2$ are in different states on T_ξ. Again, S is seen to be a triggering sequence, so Case 2 could not have been followed at stage t. Otherwise, $\emptyset \subset \xi \subset \gamma_{t-1}$ and $T_{\xi,t-1}$ has no transmission. By 2.14(xiii), 4.6(iii) and (v), and 3.5(xiii), $T_{\xi,t-1} \neq T_{\xi,t-2}$. Hence by 4.4(vi) there is an η such that $T_{\gamma_{t-1}}$ triggers T_η at stage t. Again we see that Case 2 could not have been followed at stage t. We must therefore conclude, in this case, that $T_{\gamma_{t-1},t-1} \neq \emptyset$.

Finally, suppose that $T_{\gamma_{t-1},t-1}$ is not newly designated at stage $t - 1$. If $T_{\gamma_{t-1},t-1} = \emptyset$, then it follows from (4) and 4.4(vi) that $T_{\gamma_{t-1}}$ triggers some T_η at stage t, so Case 2 is not followed at stage t. Hence again we conclude that $T_{\gamma_{t-1},t-1} \neq \emptyset$.

(ii) Suppose that $T_{\delta,t}$ is designated and $T_{\delta,s}$ is not cancelled at any stage $s + 1 > t$. If $s > t$ and $\gamma_s = \delta$, then $T_\delta = \emptyset$ must be a trigger at stage s. We proceed by induction on $\{\beta: \beta \subseteq \delta\}$, longer strings first. For each such β, we show that

$\{s: T_\delta$ triggers T_β at stage $s\}$ is finite, thus proving (ii). First suppose that $\beta = \delta$. If $r > s > t$ and T_δ triggers T_δ at stages r and s, then by 2.14(xi) and 2.14(xiii), the state of s on T_δ lexicographically precedes the state of r on T_δ. Since there are only finitely many possible states, by 2.14(x), there must be a stage $t(\delta) > t$ and a pair $\langle \sigma_{\delta^-}, i_{\delta^-} \rangle$ such that for all $s \geqslant t(\delta)$, s and $t(\delta)$ are in the same state on T_δ and $T_{\delta,s}$ transmits $\langle \sigma_{\delta^-}, i_{\delta^-} \rangle$. Thus T_δ cannot trigger T_δ at any stage $s > t(\delta)$. If $i_{\delta^-} \in \{2, 3\}$, then T_δ cannot be a trigger after stage $t(\delta)$, so the proof of (ii) is complete.

Assume by induction that $\{s: T_\delta$ triggers T_β at stage $s\}$ is finite. We will have the following induction hypothesis:

(5) There is a stage $t(\beta) > t$ and a sequence $\{\langle \sigma_\lambda, i_\lambda \rangle: \beta^- \subseteq \lambda \subset \delta\}$ which is a transmission sequence for all $s \geqslant t(\beta)$.

(5) has been verified in the preceding paragraph for $\beta = \delta$. Note that $T_{\beta^-,s}$ is not cancelled at any stage $s + 1 > t$. Hence we may define $T_{\beta^-} = \bigcup\{T_{\beta^-,s}: s > t\}$ and designate it in the same way in which $T_{\beta^-,t}$ is designated. If T_{β^-} is designated as an extension tree or as a non-empty differentiating tree, then by 2.12(ii), 2.14(iii) and induction, $T_{\beta^-,s}$ transmits $\langle \sigma_{\beta^-}, i_{\beta^-} \rangle$ for all $s \geqslant t(\beta)$. Furthermore, T_δ cannot trigger T_{β^-}. Hence $t(\beta^-) = t(\beta)$ and (5) holds. If T_{β^-} is designated as an initial tree, then $\beta^- = \emptyset$ and T_δ triggers $T_\emptyset$ at stage $t(\beta)$. By (5) and 2.8(i), T_δ triggers some T_η at stage $t(\beta) + 1$ with $\emptyset \subset \eta \subseteq \delta$, contrary to the choice of $t(\beta)$. Hence if $\beta^- = \emptyset$, we have completed the verification of (ii).

The remaining case is when T_{β^-} is designated as a splitting tree. By 4.4(iii), $T_{\beta^-,s}$ prefers $\langle \sigma_{\beta^-}, i_{\beta^-} \rangle$ at all stages $s \geqslant t(\beta)$. If $r > s \geqslant t(\beta)$ and $T_{\beta^-,r}$ and $T_{\beta^-,s}$ transmit different pairs, then by 3.5(xi), r and s must be in different states on T_{β^-}; by 3.5(xii), the state of s on T_{β^-} lexicographically precedes the state of r on T_{β^-}, unless there is a stage u such that $s < u \leqslant r$ and $T_{\beta^-,u} \neq T_{\beta^-,u-1}$. If no such stage u exists, then all sufficiently large stages must be in the same state on T_{β^-}, so $t(\beta^-)$ must exist as specified in (5). But if u exists, then by (i), T_δ would trigger some T_η at stage $u + 1$ with $\beta^- \subset \eta \subseteq \delta$, contrary to the choice of $t(\beta)$. Hence (ii) must hold. ▯

The lemmas just proved enable us to verify the success of the construction. We first show that Γ is an infinite path through $\mathscr{S}_h$.

4.8 Proposition. $\mathrm{lh}(\Gamma) = \infty$. *Furthermore, there are $\gamma(m) \in \mathscr{S}_h$ and $t(m) \in N$ such that*:

(i) $\mathrm{lh}(\gamma(m)) = m$.
(ii) *For all $t \geqslant t(m)$, either $\gamma_t \supseteq \gamma(m)$ or γ_t has lower priority than $\gamma(m)$.*
(iii) $\{r: \gamma_r \supset \gamma(m)\}$ *is infinite.*

Proof. We proceed by induction on m. (i)–(iii) are easily verified for $m = 0$. Assume that (i)–(iii) hold for $m = k - 1$. Since $\{\gamma \in \mathscr{S}_h: \mathrm{lh}(\gamma) = k\}$ is finite, it follows from (iii) that there is a $\gamma(k) \supset \gamma(k-1)$ of highest priority such that $\mathrm{lh}(\gamma(k)) = k$ and $\gamma_s \supseteq \gamma(k)$ for infinitely many s. Fix a stage $t(k)$ such that $T_{\gamma(k),s}$ is not cancelled at any stage $s \geqslant t(k)$. By (ii) for $m = k - 1$, it follows that $T_{\gamma(k-1),s}$ is not cancelled at any stage $s \geqslant t(k-1)$, hence by 4.7(ii), $t(k)$ must exist. (i) and (ii) are now easily verified for $m = k$. And (iii) is immediate from the choice of $\gamma(k)$ and 4.7(ii).

We now note that $\Gamma = \bigcup\{\gamma(m): m \in N\}$, so by (i), $\mathrm{lh}(\Gamma) = \infty$. ▯

For each $\beta \in \mathscr{S}_h$ such that β has higher priority than Γ and $T_{\beta,t}$ is designated at infinitely many stages t, it follows from 4.8 that there is a least stage $t(\beta)$ such that $T_{\beta,t}$ is not cancelled at any stage $t > t(\beta)$, $T_{\beta,t(\beta)}$ is designated, and if there is a $t \geqslant t(\beta)$ such that $T_{\beta,t}(\emptyset)\downarrow$ then $T_{\beta,t(\beta)}(\emptyset)\downarrow$. For such β, let $T_\beta = \cup\{T_{\beta,t}: t > t(\beta)\}$, and let T_β have the same designation as $T_{\beta,t(\beta)}$.

Let $g(x) = \lim_s \alpha_s(x)$ for all x for which this limit is defined. We now show that $\mathrm{lh}(g) = \infty$.

4.9 Proposition. $\mathrm{lh}(g) = \infty$ *and* $\mathbf{g} \leqslant \mathbf{0}'$.

Proof. Since $\{\alpha_s\colon s \in N\}$ is recursive, if $\mathrm{lh}(g) = \infty$ then $\mathbf{g} \leqslant \mathbf{0}'$. We show that $\mathrm{lh}(g) = \infty$ in a two part proof. Let $g^* = \cup\{T_\gamma(\emptyset)\colon \gamma \subset \Gamma\}$. We first show that $\mathrm{lh}(g^*) = \infty$ and then show that $g = g^*$.

Let $\gamma \subset \Gamma$ be given such that $\gamma \neq \emptyset$ and $\mathrm{lh}(\gamma)$ is even. If $\gamma = \gamma^- * 0$, then by 2.14(viii), $\mathrm{lh}(T_\gamma(\emptyset)) > \mathrm{lh}(T_{\gamma^-}(\emptyset))$. And if $\gamma = \gamma^- * 1$ then there is a pair $\langle \alpha, i\rangle$ such that $T_{\gamma^- * 0,t}$ transmits $\langle \alpha, i\rangle$ at all sufficiently large stages, $i \in \{2, 3\}$, and $T_{\gamma,t}(\emptyset) = \alpha$. By 2.14(vii), $\mathrm{lh}(T_\gamma(\emptyset)) > \mathrm{lh}(T_{\gamma^-}(\emptyset))$. Hence $\lim_s\{\mathrm{lh}(T_\gamma(\emptyset))\colon \mathrm{lh}(\gamma) = s \;\&\; \gamma \subset \Gamma\} = \infty$. Since $T_\gamma \subseteq T_{\gamma^-}$ for all $\gamma \subset \Gamma$ such that $\gamma \neq \emptyset$, $\mathrm{lh}(g^*) = \infty$.

We will show that $g = g^*$ by proving that for all $\gamma \subset \Gamma$ such that $\mathrm{lh}(\gamma)$ is even and all $t \geqslant t(\gamma)$, $T_\gamma(\emptyset) \subseteq \alpha_t$. Fix such a γ. By 4.8(ii), for all $t \geqslant t(\gamma)$, either $\alpha_t = T_{\delta,t}(\emptyset)$ for some δ of lower priority than γ, or there is a δ of lower priority than γ such that $T_{\delta,t-1}$ is designated but not cancelled at stage t, $\{\langle \sigma_\beta, i_\beta\rangle\colon \xi \subseteq \beta \subset \delta\}$ is a transmission sequence at stage t, $T_{\xi,t-1}$ transmits $\langle \sigma, i\rangle$ and $\alpha_t = \sigma$. Hence by 4.6(i) and 4.6(ii), $T_\gamma(\emptyset) \subseteq \alpha_t$. ▯

We are now ready to prove the main theorem of this chapter.

4.10 Theorem. *Let* $\mathscr{L}$ *be a finite lattice. Then there is a function* g *of degree* $\leqslant \mathbf{0}'$ *such that* $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$.

Proof. By 4.9, it suffices to verify 1.15(i) and 1.15(ii). Fix $m \in N$ and $i, j \leqslant n$ such that $u_i \nleqslant u_j$. Let $\langle m, i, j\rangle$ be the eth triple in the ordering described earlier in this chapter. Fix $\gamma \subset \Gamma$ such that $\mathrm{lh}(\gamma) = 2e + 2$, and let $\gamma^* = \gamma^- * 0$. Then T_{γ^*} is designated as an $\langle m, i, j\rangle$-differentiating tree. If $\gamma = \gamma^*$, then since $\mathrm{lh}(g^*) = \infty$, $T_\gamma(\emptyset)\downarrow$, so by 2.14(i), T_γ is $\langle m, i, j\rangle$-differentiating. Otherwise, $\gamma = \gamma^- * 1$ and $\gamma^- \subset \Gamma$, so since $\mathrm{lh}(g^*) = \infty$, $T_{\gamma^-}(\emptyset)\downarrow$. By the construction, $T_{\gamma^*,t}$ will transmit $\langle T_\gamma(\emptyset), 2\rangle$ at all sufficiently large stages and $T_{\gamma^- * 0} = \emptyset$. Since $g \subset T_{\gamma^-}$, T_{γ^-} is infinite, and if $T_{\gamma^-}(\eta) = T_\gamma(\emptyset)$, then $\mathrm{PExt}_f(T_{\gamma^-}, \eta)$ is infinite since $T_\gamma(\emptyset) \subset g$. Hence by 2.14(xiii) and 2.15, T_γ is $\langle e, j\rangle$-divergent. Thus 1.15(i) holds.

Fix $e \in N$ and $\gamma \subset \Gamma$ such that $\mathrm{lh}(\gamma) = 2e + 1$. Let $\gamma = \gamma^- * m$. We verify 1.15(ii) by induction on k. We also assume the following induction hypothesis:

(6) There is a sequence $u_n = u_{n_0} > u_{n_1} > \cdots > u_{n_k}$ of elements of $\mathscr{L}$ such that $T_{\gamma^- * k}$ has no e-splittings mod n_k, and $T_{\gamma^- * k}$ is designated as an e-splitting tree for n_k.

The induction hypothesis is easily verified for $k = 0$, as no tree has e-splittings mod u_n. By 3.5(i), $T_{\gamma^- * k}$ is an e-splitting tree for n_k. Hence if $T_{\gamma^- * k}$ is infinite, then $k = m$, and 1.15(ii) will hold. Otherwise, $T_{\gamma^- * k}$ is finite, so $T_{\gamma^- * (k+1)}$ must be designated. Hence $T_{\gamma^- * k}$ must transmit some $\langle \alpha, i\rangle$ at all sufficiently large stages with $i \in \{2, 3\}$, so by 3.5(x), $T_{\gamma^- * k}$ must prefer some pair. If $T_{\gamma^-}(\eta) = \alpha$, then since $\alpha \subset g \subset T_{\gamma^-}$, $\mathrm{PExt}_f(T_{\gamma^-}, \eta)$ is infinite. Hence by 4.6(iii), 4.6(v), 3.6 and the construction, either $\gamma = \gamma^- * (k+1)$ and T_γ is $\langle e, n\rangle$-divergent, or there are no e-splittings mod n_{k+1} on $T_{\gamma^- * (k+1)}$ for some n_{k+1} such that $u_{n_{k+1}} < u_{n_k}$. If $n_{k+1} \neq 0$, then (6) holds for $k+1$ since $T_{\gamma^- * (k+1)}$ is designated as an e-splitting tree for n_{k+1}. And if $n_{k+1} = 0$, then $T_{\gamma^- * (k+1)}$ has no e-splittings mod 0, so $T_{\gamma^- * (k+1)}$ is an e-splitting tree for 0. Since $\mathscr{L}$ is finite, the induction must terminate with T_γ satisfying 1.15(ii). ▯

The methods introduced in this chapter can be combined with the methods of previous chapters to prove generalizations of Theorem 4.10. Some generalizations of this sort are discussed in the next section.

5. Generalizations and Applications

The methods introduced in this chapter are compatible with methods used to prove many of the theorems about minimal degrees. Thus similar theorems can be proved for other initial segments of the degrees. The proof of Theorem 4.10 can also be extended to embed other usls as ideals of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. We discuss such results in this section. We will not give any complete proofs. Rather, we will sketch the changes which need to be made in the proof of Theorem 4.10 in order to prove the more general results. We will also discuss applications of these results.

The first category of generalizations which we consider deals with embedding infinite usls which have a least element as initial segments of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. Not all countable usls with least elements can be embedded in this manner. In fact, by Theorem VIII.2.2, if $\mathbf{g} \leqslant \mathbf{0}'$ then $\mathscr{D}[\mathbf{0}, \mathbf{g}]$ is $\mathbf{g}^{(3)} \leqslant \mathbf{0}^{(4)}$ presentable. Furthermore, by IV.3.11, if $\mathscr{D}[\mathbf{0}, \mathbf{g}]$ is a lattice, then $\mathbf{g} \in \mathbf{L}_2$ so $\mathscr{D}[\mathbf{0}, \mathbf{g}]$ is $\mathbf{g}^{(3)} = \mathbf{0}^{(3)}$ presentable. Shore [1981] has shown that $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ has no presentation of degree $< \mathbf{0}^{(4)}$, so there is no nice characterization of the initial segments of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ in terms of the jumps of degrees of presentations. We will show that if $\mathscr{L}$ is a $\mathbf{0}^{(2)}$ presentable usl with least element, then there is a $\mathbf{g} \leqslant \mathbf{0}'$ such that $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$. We will lead up to this result, beginning with recursively presentable usls.

5.1 Theorem. *Let $\mathscr{L}$ be a recursively presentable usl with least element. Then there is a $\mathbf{g} \leqslant \mathbf{0}'$ such that $\mathscr{L} \simeq \mathscr{D}[\mathbf{0}, \mathbf{g}]$.*

Sketch of Proof. The proof of Theorem 4.10 readily combines with that of Theorem VIII.1.8. Set up a recursive list of all requirements, and a recursive increasing sequence $\{\mathscr{L}_i : i \in N\}$ of finite lattices such that the embedding $\mathscr{L}_i \hookrightarrow \mathscr{L}_{i+1}$ preserves the ordering and least upper bounds of $\mathscr{L}_i$, with $\mathscr{L} = \bigcup\{\mathscr{L}_i : i \in N\}$ having universe N. The sequence $\{\mathscr{L}_i : i \in N\}$ gives rise to a uniform sequential lattice table for $\mathscr{L}$. If we allow our trees to increase in width as we progress from tree to tree to make use of the lattice table as in Theorem VIII.1.8, then this modification to the proof of Theorem 4.10 and B.3.29 will yield a proof of the above theorem. ▯

The proof of Theorem VI.4.6 can be adjusted to use Theorem 4.10 to obtain the following result due to Epstein [1979] and Lerman.

5.2 Corollary. Th($\mathscr{D}[\mathbf{0},\mathbf{0}']$) *is undecidable.*

The characterization of the degree of Th($\mathscr{D}[\mathbf{0},\mathbf{0}']$) was obtained by Shore [1981], and relies on Theorem 5.1.

5.3 Theorem. Th($\mathscr{D}[\mathbf{0},\mathbf{0}']$) *has degree $\mathbf{0}^{(\omega)}$, the degree of the first order theory of arithmetic.*

The outline of the proof of Theorem 5.3 is the same as that of Theorem VIII.3.5. We need a way to code arithmetic through a lattice, and embed that lattice below a degree $\mathbf{d}$ as an initial segment. The lattice must be given in a sufficiently effective way so that exact pairs below $\mathbf{0}'$ will be available for the ideals which need to be picked out. Shore defines distributive lattices with this property, and proves an Exact Pair Theorem (see Theorem III.8.6) which produces exact pairs for the necessary ideals. Theorem 5.1 allows us to embed such lattices as initial segments below $\mathbf{0}'$ and thus prove Theorem 5.3.

Shore's Exact Pair Theorem can be used to pin down the sets which are coded by exact pairs in certain intervals of degrees. Using the translation of arithmetic provided by Theorem VIII.3.5 into the theory of various intervals of degrees, Shore [1981] obtains the following results.

5.4 Theorem. (i) $\mathscr{D}[\mathbf{0},\mathbf{0}'] \not\equiv \mathscr{D}[\mathbf{0},\mathbf{0}^{(2)}]$.

(ii) $\mathscr{D}[\mathbf{0},\mathbf{0}'] \not\equiv \mathscr{D}[\mathbf{0}',\mathbf{0}^{(2)}]$.

(iii) *If $\mathbf{a} \geqslant \mathbf{0}'$, then every presentation of the usl $\mathscr{D}[\mathbf{0},\mathbf{a}]$ has degree $\geqslant \mathbf{a}^{(3)}$.*

(iv) *If $\mathbf{a} \geqslant \mathbf{0}'$ then every presentation of the usl $\mathscr{D}[\mathbf{a},\mathbf{a}']$ has degree $\geqslant \mathbf{a}^{(4)}$.*

Theorem 5.1 can also be used to obtain the following improvement on Theorem VIII.4.1 (Shore [1981]).

5.5 Theorem. *If $\mathscr{D}' \equiv \mathscr{D}'[\mathbf{b},\infty)$ then $\mathbf{b}^{(3)} = \mathbf{0}^{(3)}$.*

The ideas mentioned in the sketch of proof for Theorem 5.1 can be extended to embed $\mathbf{0}'$ presentable usls with least elements as ideals of $\mathscr{D}[\mathbf{0},\mathbf{0}']$.

5.6 Theorem. *Let $\mathscr{L}$ be a $\mathbf{0}'$ presentable usl with least element. Then there is a $\mathbf{g} \leqslant \mathbf{0}'$ such that $\mathscr{L} \simeq \mathscr{D}[\mathbf{0},\mathbf{g}]$.*

Sketch of Proof. Changes must be made to the sketch of proof for Theorem 5.1 to take the non-recursiveness of $\mathscr{L}$, and hence the non-recursiveness of the set of requirements into account. By the Limit Lemma, there are recursive approximations to each of these sets. Whenever such an approximation changes its value for a given requirement or finite lattice in the approximation to $\mathscr{L}$, we cancel the part of the construction which was performed using the information which was just changed, and pick up from the last stage at which everything performed during the construction through that stage still seems to be correct based on current information. This cancellation agrees well with the construction carried out in Sect. 4, and is performed as the first step at every stage of the construction. The proof of the theorem now follows very closely the proof of Theorem 4.10 with this cancellation taken into account. Note that Appendix B.3.28 allows us to extend the sequential lattice table for $\mathscr{L}_i$ to one for $\mathscr{L}_{i+1}$, so this revised construction can be carried out without changing any trees based on correct information about $\mathscr{L}$. ▯

We now discuss the modifications to the above proof which will allow us to embed $\mathbf{0}^{(2)}$ presentable usls with least element as ideals of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$.

Let $\mathscr{L}$ be a $\mathbf{0}^{(2)}$ presentable usl. Then there is a sequence $\mathscr{L}_0 \subseteq \mathscr{L}_1 \subseteq \cdots$ of finite usls which is recursive in $\mathbf{0}^{(2)}$ such that $\mathscr{L} = \cup\{\mathscr{L}_i : i \in N\}$. The fact that $\{\mathscr{L}_i : i \in N\}$ is recursive in $\mathbf{0}^{(2)}$ implies, by two applications of the Limit Lemma, that there is a recursive array $\{\mathscr{L}_{i,j,k} : i, j, k \in N\}$ such that for each $i \in N$, $\mathscr{L}_i = \lim_j \lim_k \mathscr{L}_{i,j,k}$. Fixing $i \in N$, we thus have an array as in Fig. 5.1 such that if we look at the limit $\mathscr{L}^*_{i,j}$ along column j, then $\mathscr{L}^*_{i,j} = \mathscr{L}_i$ for all but finitely many $j \in N$. And if $\mathscr{L}^*_{i,j} = \mathscr{L}_i$, then $\mathscr{L}_{i,j,k} = \mathscr{L}_i$ for all but finitely many $k \in N$.

$$\begin{array}{cccc}
\mathscr{L}^*_{i,0} & \mathscr{L}^*_{i,1} & \mathscr{L}^*_{i,2} & \cdots \\
\vdots & \vdots & \vdots & \\
\mathscr{L}_{i,0,1} & \mathscr{L}_{i,1,1} & \mathscr{L}_{i,2,1} & \cdots \\
\mathscr{L}_{i,0,0} & \mathscr{L}_{i,1,0} & \mathscr{L}_{i,2,0} & \cdots
\end{array}$$

Fig. 5.1

We list all potential requirements (differentiating requirements of the form $\Phi_e^{g^{\langle j \rangle}} \neq g^{\langle i \rangle}$ are listed for all i and j), in a recursive list $\{R_i : i \in N\}$. We try to satisfy requirement R_i with respect to the sequence of trees generated by the tables for a sequence of lattices $\{\mathscr{L}^*_{j,k} : j \leqslant i\}$ for various choices of $k > i$. We will try to satisfy R_i with $\{\mathscr{L}^*_{j,k} : j \leqslant i\}$ at stage k only if either $k = i$ or $\{\mathscr{L}^*_{j,k-1} : j \leqslant i\} \neq \{\mathscr{L}^*_{j,k} : j \leqslant i\}$. Since, for all j and all sufficiently large m and k, $\mathscr{L}^*_{j,k} = \mathscr{L}^*_{j,m}$, only finitely many such attempts will be made.

Since the sequences $\{\mathscr{L}^*_{j,k} : j \leqslant i\}$ are recursive only in $\mathbf{0}'$, we will approximate to them recursively, using $\{\mathscr{L}_{j,k,r} : j \leqslant i\}$ at stage r. If such a sequence changes for fixed k between stages r and s, we cancel what we have done since stage r, and begin with a new attempt to satisfy R_i with the new sequence. Since, for all j and k, there is an r such that for all $t \geqslant r$, $\mathscr{L}_{j,k,t} = \mathscr{L}^*_{j,k}$, again only finitely many cancellations will be required.

The first few steps of the construction will proceed as follows. We will keep trying to satisfy R_0 on $\mathscr{L}_{0,0,t}$ using a tree $T^*_{0,0}$, cancelling what we have done whenever $\mathscr{L}_{0,0,t+1} \neq \mathscr{L}_{0,0,t}$. Eventually, $\mathscr{L}_{0,0,t} = \mathscr{L}^*_{0,0}$ for all $t \geqslant r$, so we will make a final attempt to satisfy R_0 through $\mathscr{L}^*_{0,0}$. We now try to use $\mathscr{L}^*_{0,1}$ and $\mathscr{L}^*_{1,1}$ to satisfy both R_0 and R_1. We approximate to $\mathscr{L}^*_{0,1}$ and $\mathscr{L}^*_{1,1}$ using $\mathscr{L}_{0,1,t}$ and $\mathscr{L}_{1,1,t}$ as before, arriving at final lattices. If $\mathscr{L}^*_{0,1} \neq \mathscr{L}^*_{0,0}$, we define $\alpha_0 = T^*_{0,0}(\emptyset)$ and try to satisfy R_0 using a tree $T^*_{0,1}$ with respect to the table for $\mathscr{L}^*_{0,1}$, with $\alpha_0 \subset T^*_{0,1}$; and we attempt to satisfy R_1 using a subtree $T^*_{1,1} \subseteq T^*_{0,1}$ defined in terms of the tables for $\mathscr{L}^*_{0,1}$ and $\mathscr{L}^*_{1,1}$. (Without loss of generality, we can arrange that $\mathscr{L}^*_{1,1}$ extends $\mathscr{L}^*_{0,1}$, and that all elements mentioned in any R_i are in $\mathscr{L}^*_{i,1}$.) If $\mathscr{L}^*_{0,1} = \mathscr{L}^*_{0,0}$, then we make no new attempt to satisfy R_0. By Appendix B.3.28, we can extend the table for $\mathscr{L}^*_{0,0}$ to one for $\mathscr{L}^*_{1,1}$, and so satisfy R_1 on a subtree of $T^*_{0,0}$. Thus we will eventually satisfy R_i for $\mathscr{L}_i$ if its conditions are consistent with the ordering of $\mathscr{L}_i$, and construct a sequence of subtrees satisfying all requirements for $\{\mathscr{L}_i : i \in N\}$. We have thus sketched a proof of the following theorem.

5.7 Theorem. *Let $\mathscr{L}$ be a $\mathbf{0}^{(2)}$ presentable usl with least element. Then there is a* $\mathbf{g} \leqslant \mathbf{0}'$ *such that* $\mathscr{L} \simeq \mathscr{D}[\mathbf{0}, \mathbf{g}]$.

Suppose that we start with a usl $\mathscr{L}$, and produce $\mathbf{g} \leqslant \mathbf{0}'$ such that $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$. We would like to locate $\mathbf{g}$ in the high/low hierarchy. If $\mathscr{L}$ is $\mathbf{0}'$ presentable, it will follow from the Jump Theorem or the existence of such a $\mathbf{g}$ below an arbitrary non-zero recursively enumerable degree that it is possible to find $\mathbf{g} \in \mathbf{L}_1$ in this case. Since narrow subtrees and e-total subtrees can be introduced into the construction and produce no new complications, their use as in Chap. V.3 enables us to produce such a $\mathbf{g} \in \mathbf{L}_2 - \mathbf{L}_1$. Hence:

5.8 Theorem. *Let $\mathscr{L}$ be a $\mathbf{0}^{(2)}$ presentable usl with least element. Then there is a* $\mathbf{g} \in \mathbf{L}_2 - \mathbf{L}_1$ *such that* $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$.

Note that if the $\mathscr{L}$ of Theorem 5.8 is a lattice, then by IV.3.11, the corresponding $\mathbf{g}$ must lie in $\mathbf{L}_2$.

The proof which we have presented for the Cooper Jump Inversion Theorem (Theorem X.2.1) makes use of an oracle of degree $\mathbf{0}'$, and so cannot be combined with the recursive approximation proof of Theorem 4.10. However, Cooper's [1973] original proof of this theorem proceeds by recursive approximation, and can be combined with the construction of Sect. 4 by approximating to the trees used in Chap. X instead of those used in this section. The resulting proof is similar in nature to producing $\mathbf{g}$ below a non-zero recursively enumerable degree, a construction which we will sketch. We will need $\mathscr{L}$ to be a $\mathbf{0}'$-presentable lattice, essentially since the proof requires that we determine whether we are looking at a sequence of trees (and hence a table) for the true approximation to $\mathscr{L}$. We state the jump theorem here without proof.

5.9 Theorem. *Let $\mathscr{L}$ be a $\mathbf{0}'$ presentable usl with least element, and let* $\mathbf{d} \geqslant \mathbf{0}'$ *be given. Then there is a degree* $\mathbf{g}$ *such that* $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$ *and* $\mathbf{g}' = \mathbf{d}$.

5.10 Corollary. *Let $\mathscr{L}$ be a $\mathbf{0}'$ presentable usl with least element. Then there is a* $\mathbf{g} \in \mathbf{L}_1$ *such that* $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$.

We now turn our attention to finding specified initial segments below fixed degrees. Two such theorems were proved for minimal degrees. If $\mathbf{d}$ is a degree, then either of the following conditions guarantee the existence of a minimal degree below $\mathbf{d}$:

(1) $\mathbf{d} \in \mathbf{GH}_1$.

(2) $\mathbf{d} \neq \mathbf{0} \,\&\, \mathbf{d}$ is recursively enumerable.

The proof of (1) used an oracle construction which cannot be combined with the proof of Theorem 4.10. We do not know if such a result holds for arbitrary finite lattices. However, if $\mathbf{d} \in \mathbf{H}_1$ then Cooper [1973] has produced a minimal degree $\leqslant \mathbf{d}$ through a proof which proceeds by recursive approximation. Posner [1980] presents an easier proof of this kind. Either of these proofs can be combined with the proof of Theorem 4.10 to yield:

5.11 Theorem. *Let $\mathscr{L}$ be a $\mathbf{0}'$ presentable usl with least element, and let* $\mathbf{d} \in \mathbf{H}_1$ *be given. Then there is a* $\mathbf{g} \leqslant \mathbf{d}$ *such that* $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$.

The proof of (2) in Theorem XI.2.2 uses a construction which proceeds by recursive approximation and is compatible with the proof of Theorem 4.10. Hence we can prove a generalization of (2) which will imply Corollary 5.6.

5.12 Theorem. *Let $\mathscr{L}$ be a $\mathbf{0}'$ presentable usl with least element and let $\mathbf{a} \neq \mathbf{0}$ be a recursively enumerable degree. Then there is a $\mathbf{g} \leqslant \mathbf{a}$ such that $\mathscr{D}[\mathbf{0}, \mathbf{g}] \simeq \mathscr{L}$.*

Sketch of Proof. Let A be a recursively enumerable set of degree $\mathbf{a}$, and let $\{a_s\colon s \in N\}$ be a recursive enumeration of A. A will *permit α_s to change to* α_{s+1} if $a_s \leqslant x$, where x is the least y such that $\alpha_s(y) \neq \alpha_{s+1}(y)$. If no such y exists, then A permits the change.

It is best now to view the construction of the trees as proceeding simultaneously with the construction of $\{\alpha_s\colon s \in N\}$. If there is no trigger at stage $s + 1$, then it can easily be verified that $\alpha_{s+1} \supseteq \alpha_s$ so no problems arise. So let us consider trees involved in triggering sequences. If such a tree is designated as an extension tree or is non-empty and is designated as a differentiating tree, then no changes need to be made.

We first consider T_δ designated as a differentiating tree with $T_\delta = \emptyset$. We require that A permit all transmissions of T_δ in the following sense. Let T_δ transmit $\langle \alpha, i\rangle$ at stage s. If $i \in \{0, 1\}$, then this is only allowed if $a_s \leqslant T_\delta\text{-}(\emptyset)$. And if $i = 2$, then A must permit α_s to change to α. If transmissions are disallowed, begin constructing a new attempt at a differentiating tree above α_s, proceeding as before but replacing $T_\delta\text{-}(\emptyset)$ above with α_s. This process continues as long as the obstacles to all trees are due to permitting. New states are inserted to reflect the wait for permitting, and earlier attempts have higher priority. Cancellation follows the priority ordering, but if a lower priority attempt reaches a later state than a higher priority attempt, then the higher priority attempt is cancelled. Since only finitely many states exist, if we look at the greatest state in which infinitely many attempts terminate (assuming that permitting is always the obstacle to the attempt), we see that attempts in this state are never cancelled. Hence if we wait for A to permit on larger and larger strings, then we can compute A recursively, and so obtain a contradiction. Hence differentiating trees will have the right properties, i.e., a last successful attempt will be made, and this tree will be used at all sufficiently large uncancelled stages.

Consider T_δ designated as a splitting tree. We assure the construction of an appropriate tree as in the preceding paragraph once several comments are made. If T_δ wants to transmit $\langle \alpha, 3\rangle$ at stage s, then this transmission is allowed if α_s permits α. $T_{\delta,t}$ may receive many pairs $\langle \alpha, i\rangle$ for $i \in \{0, 1\}$ at stage t. It prefers the pair of highest priority (in terms of its transmission sequence) for which A permits the tree to change state. If A will not permit a certain transmission and T_η is the corresponding trigger, then the new attempt begun is an attempt to replace $T_{\eta,t}$ with another differentiating tree. Thus a notion of *characteristic* must be defined for a transmission sequence, listing the state of each pair along the way (state 0 is specified for trees with no states such as extension trees). The argument about states in the previous paragraph becomes an argument using the ordering of the finitely many possible characteristics.

If T_δ is designated as an initial tree, then preference must be redefined as in the preceding paragraph. Otherwise, the construction of this tree is unchanged.

As we stated, each time a new attempt is started, it must be above α_s. We will always be able to choose some $\beta \supseteq \alpha_s$ which is a potential focal point of T_θ but not a focal point of $T_{\theta,t}$ at which to begin this attempt. Thus the sequences of trees will still be special. No other changes are required in the construction. ▯

Shore [1981] uses Theorems 5.11 and 5.12 combined with theorems about exact pairs to characterize the degree of Th($\mathscr{D}[\mathbf{0}, \mathbf{a}]$) for $\mathbf{a} \in \mathbf{H}_1$ and for $\mathbf{a} \neq \mathbf{0}$ and recursively enumerable. The proof is along the lines sketched for the proof of Theorem 5.3. Epstein [1979], [1981] had previously obtained the undecidability of Th($\mathscr{D}[\mathbf{0}, \mathbf{a}]$) for $\mathbf{a} \in \mathbf{H}_1$.

5.13 Theorem. Th($\mathscr{D}[\mathbf{0}, \mathbf{a}]$) *has degree* $\mathbf{0}^{(\omega)}$ *if either*:

(i) $\mathbf{a} \in \mathbf{H}_1$.

(ii) $\mathbf{a} \neq \mathbf{0}$ & $\mathbf{a}$ *is recursively enumerable.*

All the results of this section can be relativized. Care must be taken, when talking about arithmetic, to include the definability of certain degrees in the hypothesis of some of the relativizations.

We have just touched on some of the applications which can be made using initial segments results for the degrees below $\mathbf{0}'$ and other classes of degrees. We refer the reader to Nerode and Shore [1980], Shore [1981] and [1981a], and Epstein [1979] and [1981] for proofs of the applications mentioned in this section and some further results.