Chapter III
The Jump Operator

The jump operator is a naturally defined function taking each degree to a larger degree. It is also very closely related to the arithmetical hierarchy. We will study this relationship, as well as some algebraic structures whose universe is the set of degrees and on which the jump operator acts as a function.

The jump operator also allows us to pick out certain natural degrees other than **0**. Thus we are presented with certain natural intervals of degrees for which we can ask questions similar to those answered in Chap. II. We begin our study of *local degree theory*, i.e., the study of bounded intervals of degrees in this chapter. Many of the results proved throughout this book are local results which allow us to prove global theorems about the degrees.

1. The Arithmetical Hierarchy

The arithmetical hierarchy coincides with the $\exists_n/\forall_n$ hierarchy of sentences for the language of recursion theory specified below. It is introduced in this section, and characterizations of certain levels of this hierarchy are given.

There is one basic relation and one basic function for which we want symbols in our language for recursion theory. The first is the graph of the function φ of the Enumeration Theorem. The other is any one-one recursive correspondence $\pi: N^2 \to N$. π is called a *pairing function*. We will sometimes denote π by π_2, and note that for $n \geqslant 2$, we can recursively define $\pi_{n+1}: N^{n+1} \to N$ by

$$\pi_{n+1}(a_0, \ldots, a_n) = \pi(\pi_n(a_0, \ldots, a_{n-1}), a_n).$$

The underlying language for recursion theory is the language of the pure predicate calculus with equality, together with:

(1) For each $n \in N$, a constant symbol $\underset{\sim}{n}$ whose interpretation is n.

(2) A five place relation symbol $\underset{\sim}{\varphi}$ whose interpretation is the graph of the function of the Enumeration Theorem, i.e., $\{\langle \sigma, e, x, s, y\rangle : \varphi(\sigma, e, x, s)\downarrow = y\}$.

(3) A function symbol $\underset{\sim}{\pi}$ of two places whose interpretation is the pairing function π.

Henceforth, we will identify each symbol in this language with its interpretation.

Note that the above language is recursive. φ maps $\mathscr{S} \times N^3 \to N$, but we have a recursive one-one correspondence of $\mathscr{S}$ with N, so can treat φ as a map $N^4 \to N$ under this correspondence.

Given any recursive relation R, we recall that χ_R is the characteristic function of R. If R is a k-place relation, then we can effectively pass from R to the one-place relation R^* defined by

$$N \models R(n_1, \ldots, n_k) \Leftrightarrow N \models R^*(\pi_k(n_1, \ldots, n_k)) \Leftrightarrow \chi_R(\pi_k(n_1, \ldots, n_k)) = 1.$$

Furthermore, $\chi_R = \varphi_e$ for some $e \in N$ and χ_R is total. Hence

$$\chi_R(n) = 1 \Leftrightarrow \exists s(\varphi(\emptyset, e, n, s)\downarrow = 1) \Leftrightarrow \forall t \forall y(\varphi(\emptyset, e, n, t)\downarrow = y \to y = 1).$$

Since the domain of φ is recursive, we now see that every recursive relation has both an $\forall_1$ and $\exists_1$ expression in this language. It is easily verified that any relation expressible in both $\forall_1$ and $\exists_1$ form is recursive.

The language for recursion mentioned above is closely connected with the local structure theory of the degrees. We thus introduce notation to distinguish the $\exists_n$ and $\forall_n$ sentences of this language from those of other languages.

1.1 Definition. Let S be a formula in the above language. We say that S is Σ_n^0 if S is $\exists_n$, S is Π_n^0 if S is $\forall_n$, and S is Δ_n^0 if S is both Σ_n^0 and Π_n^0. Given $R \subseteq N^k$, we say that R is Σ_n^0 (Π_n^0, Δ_n^0 resp.) if there is a Σ_n^0 (Π_n^0, Δ_n^0 resp.) formula S of the language such that for all $a_1, \ldots, a_n \in N$

$$R(a_1, \ldots, a_n) \Leftrightarrow S(a_1, \ldots, a_n).$$

Similarly, given $f: N^k \to N$, we say that f is Σ_n^0 (Π_n^0, Δ_n^0 resp.) if the relation

$$R(x_1, \ldots, x_{n+1}) \Leftrightarrow f(x_1, \ldots, x_n) = x_{n+1}$$

is Σ_n^0 (Π_n^0, Δ_n^0 resp.). The classification $\{\Sigma_n^0, \Pi_n^0, \Delta_n^0 : n \geqslant 0\}$ is called the *arithmetical hierarchy*.

The arithmetical hierarchy has the following nice closure properties.

1.2 Remark. Let R, $S \subseteq N^k$ be Σ_n^0 relations. Then $R \cup S$ and $R \cap S$ are Σ_n^0 relations, and $N^k - R$ is a Π_n^0 relation.

1.3 Remark. Let $R \subseteq N^n$ be a recursive relation, and let $m \in N$ be given. Define the relation $S \subseteq N^{n-1}$ by

$$S(x_1, \ldots, x_{n-1}) \Leftrightarrow \exists x \leqslant m(R(x, x_1, \ldots, x_{n-1})).$$

Then S is a recursive relation. Hence the class Σ_0^0 is closed under bounded existential quantification. Since Σ_0^0 is closed under negation, it is also closed under bounded universal quantification.

1.4 Remark. For all $n \in N$ and all Σ_n^0 relations $S \subseteq N^k$, there is a Π_{n-1}^0 relation $R \subseteq N^{k+1}$ such that

$$\forall x_1, \ldots, x_k(S(x_1, \ldots, x_k) \Leftrightarrow \exists x(R(x, x_1, \ldots, x_k))).$$

We leave it to the reader to prove this remark (Exercise 1.16).

We now define a relativized version of the arithmetical hierarchy.

1.5 Remark. Given any function $f: N^k \to N$, we can expand our language by adding a symbol f to our language to be interpreted as the function f. If we follow the procedure of Definition 1.1 to define Σ_n^f, Π_n^f, and Δ_n^f, we obtain the *arithmetical hierarchy relativized to* f.

The recursively enumerable sets have been studied extensively by recursion theorists. We will, however, touch on these sets only peripherally in this book. Soare [1978] contains a survey of results in this area, and Soare [1984] is an excellent source from which to learn about recursively enumerable sets.

1.6 Definition. Let $f: N^k \to N$ and $A \subseteq N$ be given. A is said to be *recursively enumerable in* f if $A = \emptyset$ or A is the range of a function recursive in f. A is said to be *recursively enumerable* if A is recursively enumerable in some recursive function. (Note that A is recursively enumerable if and only if A is recursively enumerable in every recursive function.) A degree $\mathbf{d}$ is *recursively enumerable* (*in* f) if $\mathbf{d}$ is the degree of a set which is recursively enumerable (in f).

The next two propositions present alternate definitions of the recursively enumerable and recursive sets.

1.7 Proposition. *Let $f: N^k \to N$ and $A \subseteq N$ be given. Then the following are equivalent:*

(i) *A is recursively enumerable in f.*
(ii) *A is the domain of a partial function θ which is computable from f.*
(iii) *A is the range of a partial function θ which is computable from f.*

Proof. Exercise 1.17. ▯

1.8 Proposition. *Let $f: N^k \to N$ and $A \subseteq N$ be given. Then the following are equivalent:*

(i) *A is recursive in f.*
(ii) *Both A and $N - A$ are recursively enumerable in f.*

Proof. Exercise 1.18. ▯

The next theorem relates recursive enumerability to the arithmetical hierarchy.

1.9 Theorem. *Let $f: N^k \to N$ and $A \subseteq N$ be given. Then the following are equivalent:*

(i) *A is recursively enumerable in f.*
(ii) *$A \in \Sigma_1^f$.*

Proof. (i) $\Rightarrow$ (ii): Let A be recursively enumerable in f. If $A = \emptyset$ then $x \in A \Leftrightarrow \exists x(x \neq x)$ so $A \in \Sigma_1^f$. If $A \neq \emptyset$, let g be a function recursive in f with range A.

Then

$$y \in A \Leftrightarrow \exists x(g(x) = y).$$

As g is in our language, $A \in \Sigma_1^f$.

(ii) $\Rightarrow$ (i): If $A = \emptyset$ then the result is immediate. Assume that $A \neq \emptyset$, and fix $a \in A$. Since $A \in \Sigma_1^f$, there is a relation S recursive in f such that

$$x \in A \leftrightarrow \exists z(S(x, z)).$$

Let $\{\langle u_i, v_i \rangle : i \in N\}$ be a recursive enumeration of N^2. Define $g : N \to N$ by

$$g(i) = \begin{cases} u_i & \text{if } \langle u_i, v_i \rangle \in S, \\ a & \text{otherwise.} \end{cases}$$

g is clearly recursive in f and has range A. ▯

1.10 Corollary. *Let $f: N^k \to N$ and $A \subseteq N$ be given. Then the following are equivalent:*

(i) *A is recursive in f.*

(ii) $A \in \Delta_1^f$.

Proof. Recall that for all $B \subseteq N$, $B \in \Sigma_1^f \Leftrightarrow N - B \in \Pi_1^f$. Hence by Proposition 1.8 and Theorem 1.9,

> A is recursive in $f \Leftrightarrow$ both A and $N - A$ are recursively enumerable in $f \Leftrightarrow A,\ N - A \in \Sigma_1^f \Leftrightarrow A \in \Sigma_1^f \cap \Pi_1^f \Leftrightarrow A \in \Delta_1^f$. ▯

The final result of this section, the Limit Lemma, characterizes the class of Δ_2^0 functions. It is stated in terms of the following definition.

1.11 Definition. Let $f: N^{k+1} \to N$ and $g: N^k \to N$ be given. We say that $g = \lim_s f$ if

(i) $$\forall x_1, \ldots, x_k \exists s \forall t \geqslant s(f(t, x_1, \ldots, x_k) = g(x_1, \ldots, x_k)).$$

We say that $\lim_s f$ exists if there is a function $h: N^k \to N$ such that (i) holds with h in place of g.

1.12 Limit Lemma. *Let $f: N \to N$ be given. Then the following are equivalent:*

(i) $f \in \Delta_2^0$.

(ii) *There is a recursive function $g: N^2 \to N$ such that $f = \lim_s g$.*

Proof. (i) $\Rightarrow$ (ii): Let $\{\langle y_i, s_i \rangle : i \in N\}$ be a one-one recursive enumeration of N^2. By Remark 1.4, fix a recursive relation $S \subseteq N^4$ such that

$$f(x) = y \leftrightarrow \exists s \forall t(S(x, y, s, t)).$$

For each $u, x \in N$, define $i(u, x) = \mu i[\forall t \leqslant u(S(x, y_i, s_i, t))]$. Since f is total, we see that i is a total recursive function. Define $g(u, x) = y_{i(u,x)}$ for all $u, x \in N$. It is easily verified that $f = \lim_u g$.

(ii) $\Rightarrow$ (i): Let $f = \lim_s g$, where g is recursive. Then

$$f(x) = y \Leftrightarrow \exists s \forall t \geqslant s(g(t,x) = y) \Leftrightarrow \forall s \exists t \geqslant s(g(t,x) = y).$$

Hence $f \in \Delta_2^0$. ▯

1.13 Corollary. *Let $A \subseteq N$. Then $A \in \Delta_2^0$ if and only if there is a recursive function $f: N^2 \to \{0, 1\}$ such that for all $x \in N$*

$$x \in A \Leftrightarrow \lim_s f(s, x) = 1.$$

Proof. Fix $A \subseteq N$. Then $A \in \Delta_2^0$ if and only if the characteristic function of A is a Δ_2^0 function. Apply the Limit Lemma. ▯

There is a version of the Limit Lemma for functions $f: N^k \to N$ for all $k \geqslant 1$ which is proved by making the obvious modifications to the proof of the Limit Lemma. The Limit Lemma can also be relativized as follows.

1.14 Lemma. *Let $f: N \to N$ and $h: N^k \to N$ be given. Then the following are equivalent:*

(i) $f \in \Delta_2^h$.

(ii) *There is a function $g: N^2 \to N$ recursive in h such that $f = \lim_s g$.*

Proof. Exercise 1.19. ▯

1.15 Remarks. The results of this section are contained in Post [1944] with the exception of the Limit Lemma and its variants which were proved by Shoenfield [1959].

1.16–1.22 Exercises

***1.16** Show that for all $n \in N$ and all Σ_n^0 relations $S \subseteq N^k$, there is a Π_{n-1}^0 relation $R \subseteq N^{k+1}$ such that

$$\forall x_1, \ldots, x_k(S(x_1, \ldots, x_k) \Leftrightarrow \exists x(R(x, x_1, \ldots, x_k))).$$

(*Hint*: Use a recursive one-one correspondence between N and N^k.)

***1.17** Let $f: N^k \to N$ and $A \subseteq N$ be given. Prove the equivalence of the following conditions:

(i) A is recursively enumerable in f.

(ii) A is the domain of a partial function which is computable from f.

(iii) A is the range of a partial function which is computable from f.

***1.18** Let $f: N^k \to N$ and $A \subseteq N$ be given. Prove the equivalence of the following conditions:

(i) A is recursive in f.

(ii) Both A and $N - A$ are recursively enumerable in f.

***1.19** Let $f: N \to N$ and $h: N^k \to N$ be given. Prove the equivalence of the following conditions:

(i) $f \in \Delta_2^h$.

(ii) There is a function $g: N^2 \to N$ which is recursive in h such that $f = \lim_s g$.

1.20 Let $f: N \to N$ be a total Σ_n^0 function for $n \geqslant 1$. Show that f is Π_n^0 and hence that f is Δ_n^0.

1.21 Show that if A is an infinite recursively enumerable set, then there is a one-one recursive function with range A.

1.22 Show that if A is an infinite recursive set, then there is a one-one function f enumerating A in order of magnitude, i.e., f has range A and $\forall x, y(x < y \to f(x) < f(y))$.

2. The Jump Operator

The jump operator is a strictly increasing function from **D** to **D**. We will define the jump operator in this section, and examine its relationship to the arithmetical hierarchy.

2.1 Definition. Given $f: N \to N$, define $f' = \{e : \Phi_e^f(e)\downarrow\}$. f' is called the *completion* of f.

2.2 Definition. Given $f: N \to N$, define $f^* = \{\langle e, x\rangle : \Phi_e^f(x)\downarrow\}$.

The following theorem describes relationships between f, f' and f^*.

2.3 Theorem. *Let $A \subseteq N$ and $f, g: N \to N$ be given. Then:*

(i) *f' is recursively enumerable in f.*

(ii) $f' \nleqslant_T f$.

(iii) $f' \equiv_T f^*$.

(iv) *If A is recursively enumerable in f, then A is recursive in f'. Hence $f \leqslant_T f'$.*

(v) *If $f \leqslant_T g$ then $f' \leqslant_T g'$.*

Proof. (i) $e \in f' \Leftrightarrow \exists y \in N \exists \sigma \in \mathscr{S}(\sigma \subset f \,\&\, \Phi_e^\sigma(e)\downarrow = y)$. Hence by Theorem 1.9, f' is recursively enumerable in f.

(ii) Assume that $f' \leqslant_T f$ for the sake of obtaining a contradiction. Then by Proposition 1.8, $N - f' = \{e : \Phi_e^f(e)\uparrow\}$ is recursively enumerable in f, hence by Proposition 1.7, $N - f' = \mathrm{dom}(\Phi_p^f)$ for some $p \in N$. We now note that

$$\Phi_p^f(p)\uparrow \Leftrightarrow p \in N - f' \Leftrightarrow p \in \mathrm{dom}(\Phi_p^f) \Leftrightarrow \Phi_p^f(p)\downarrow,$$

a contradiction.

(iii) Clearly $f' \leqslant_T f^*$. By clause (v) of the Enumeration Theorem, there is a recursive function $h: N^2 \to N$ such that

$$\Phi_{h(x,y)}^f(z) = \begin{cases} 1 & \text{if } \Phi_x^f(y)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Now

$$\langle x, y\rangle \in f^* \Leftrightarrow \Phi_x^f(y)\downarrow \Leftrightarrow h(x,y) \in \mathrm{dom}(\Phi_{h(x,y)}^f) \Leftrightarrow h(x,y) \in f'.$$

Hence $f^* \leqslant_T f'$.

(iv) Let A be recursively enumerable in f. By Proposition 1.7, $A = \mathrm{dom}(\Phi_e^f)$ for some $e \in N$. Hence

$$x \in A \Leftrightarrow \Phi_e^f(x)\downarrow \Leftrightarrow \langle e, x\rangle \in f^*$$

so $A \leqslant_T f^* \equiv_T f'$. Letting $f = A$, we see that $f \leqslant_T f'$.

(v) Suppose that $f \leqslant_T g$. By (i) and Proposition 1.7, $f' = \mathrm{dom}(\Phi_e^f)$ for some $e \in N$. Since $f \leqslant_T g$, there is a $p \in N$ such that $\Phi_e^f = \Phi_p^g$, so $f' = \mathrm{dom}(\Phi_p^g)$. Now

$$x \in f' \Leftrightarrow \Phi_p^g(x)\downarrow \Leftrightarrow \langle p, x\rangle \in g^*$$

so $f' \leqslant_T g^* \equiv_T g'$. ▯

From Theorem 2.3(v), we note that if $f \equiv_T g$ then $f' \equiv_T g'$. Thus the completion operator induces the following well-defined operator on $\mathbf{D}$.

2.4 Definition. Let $\mathbf{a} \in \mathbf{D}$ be given. Then $\mathbf{a}'$, the *jump* of $\mathbf{a}$, is the degree of A' for any $A \in \mathbf{a}$.

The completion and jump operators can be iterated as follows.

2.5 Definition. Let $f: N \to N$ be given. Define $f^{(n)}$, the *nth completion* of f, inductively as follows: $f^{(0)} = f$, and for $n > 0$, $f^{(n+1)} = (f^{(n)})'$. Given $\mathbf{a} \in \mathbf{D}$, define $\mathbf{a}^{(\mathbf{n})}$, the *nth jump* of $\mathbf{a}$ as follows: Choose $A \in \mathbf{a}$. Then $\mathbf{a}^{(\mathbf{n})}$ is the degree of $A^{(n)}$.

The following theorem relates nth completions to the arithmetical hierarchy.

2.6 Theorem. *Let $f: N \to N$ and $A \subseteq N$ be given. Then*
(i) $A \in \Sigma_{n+1}^f \Leftrightarrow A$ *is recursively enumerable in* $f^{(n)}$.
(ii) (*Post's Theorem*) $A \in \Delta_{n+1}^f \Leftrightarrow A \leqslant_T f^{(n)}$.

Proof. (i) We proceed by induction. If $n = 0$, then (i) follows from Theorem 1.9. Suppose by induction that (i) holds for n. Assume first that A is recursively enumerable in $f^{(n)}$. Then $A = \mathrm{dom}(\Phi_e^{f^{(n)}})$ for some $e \in N$. Hence

$$(1) \qquad x \in A \Leftrightarrow \exists y \in N\, \exists \sigma \in \mathscr{S}(\Phi_e^\sigma(x)\downarrow = y \;\&\; \sigma \subset f^{(n)}).$$

Furthermore,

$$(2) \qquad \sigma \subset f^{(n)} \Leftrightarrow \forall i < \mathrm{lh}(\sigma)((\sigma(i) = 1 \to i \in f^{(n)}) \;\&\; (\sigma(i) = 0 \to i \notin f^{(n)})).$$

By Theorem 2.3(i), $f^{(n)}$ is recursively enumerable in $f^{(n-1)}$, hence by induction, $f^{(n)} \in \Sigma_n^f$. It now follows from (1), (2) and Remarks 1.2 and 1.3 that $A \in \Sigma_{n+1}^f$.

Conversely, assume that $A \in \Sigma_{n+1}^f$. By Remark 1.4, there is a Π_n^f relation $S \subseteq N^2$ such that

$$(3) \qquad x \in A \leftrightarrow \exists y(S(x, y)).$$

Hence $N^2 - S \in \Sigma_n^f$, so by induction, $N^2 - S$ is recursively enumerable in $f^{(n-1)}$. By Theorem 2.3(iv), $N^2 - S$ is recursive in $f^{(n)}$; so S must also be recursive in $f^{(n)}$. By (3), $A \in \Sigma_1^{f^{(n)}}$, hence by Theorem 1.9, A is recursively enumerable in $f^{(n)}$.

(ii) Immediate from (i) and Proposition 1.8. ▯

Post's Theorem will usually be applied, implicitly, in the following way. The only non-effective steps in the construction of a set A will be the need to answer certain Σ_1^0 questions, posed in a uniform way. Post's Theorem will then imply that A has degree less than or equal to $\mathbf{0}'$.

The degree $\mathbf{0}'$ is uniquely situated within the degrees as being simultaneously the degree of a recursively enumerable set and the jump of another degree. Some properties of the degrees below $\mathbf{0}'$ will be discussed in the next section.

3. Embeddings and Exact Pairs Below $\mathbf{0}'$

Some of the theorems proved in Chap. II for the degrees are also true for the degrees below $\mathbf{0}'$. The theorems which are reexamined in this section are those whose proofs for the degrees below $\mathbf{0}'$ depend on little more than combining the proof given in Chap. II with a bounding principle for forcing.

3.1 Notation. Let $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ be given such that $\mathbf{a} \leqslant \mathbf{b}$. $\mathbf{D}[\mathbf{a}, \mathbf{b}]$ will denote $\{\mathbf{d} \in \mathbf{D} : \mathbf{a} \leqslant \mathbf{d} \leqslant \mathbf{b}\}$, and $\mathbf{D}[\mathbf{a}, \infty)$ will denote $\{\mathbf{d} \in \mathbf{D} : \mathbf{d} \geqslant \mathbf{a}\}$. Notation for other intervals of degrees such as $\mathbf{D}[\mathbf{a}, \mathbf{b})$, $\mathbf{D}(\mathbf{a}, \mathbf{b}]$, $\mathbf{D}(\mathbf{a}, \mathbf{b})$, and $\mathbf{D}(\mathbf{a}, \infty)$ is interpreted in the obvious way. This notation is also carried over to structures, so, for example, $\mathscr{D}[\mathbf{a}, \mathbf{b}]$ will denote $\langle \mathbf{D}[\mathbf{a}, \mathbf{b}], \leqslant \rangle$.

We now present a bounding principle which transforms global structure theorems into local structure theorems.

3.2 Bounding Principle. *Let* $\mathbf{a} \in \mathbf{D}$, *a set* $\mathscr{T}$ *of requirements, a notion of forcing* $\langle F, \leqslant \rangle$, $p \in F$ *and a function* $f: F \times \mathscr{T} \to F$ *be given. Assume that*:

(i) *The sets* $\mathscr{T}$ *and* F *have degree* $\leqslant \mathbf{a}$, *as does the relation* $\leqslant$ *and the function* f.
(ii) $\forall q \in F \forall T \in \mathscr{T} (f(q, T) \leqslant q \,\&\, f(q, T) \Vdash T)$.

For each $T \in \mathscr{T}$, *let* $C_T = \{q \in F : q \Vdash T\}$, *and let* $\mathscr{C} = \{C_T : T \in \mathscr{T}\}$. *Then there is a sequence* $p_0 \geqslant p_1 \geqslant \cdots$ *of elements of* F *such that*

(iii) $p_0 = p$.
(iv) $\{\langle i, p \rangle : p = p_i\}$ *has degree* $\leqslant \mathbf{a}$.
(v) $G = \{q \in F : q \geqslant p_i \text{ for some } i\}$ *is a* $\mathscr{C}$*-generic set.*

Thus if h *is a function which can be computed from* $\{p_i : i \in N\}$ *through the use of an oracle of degree* $\mathbf{a}$, *then* h *has degree* $\leqslant \mathbf{a}$. *In particular, if* F *is a set of partial functions uniformly of degree* $\leqslant \mathbf{a}$, $P = \bigcup\{\mathrm{dom}(p) : p \in F\}$ *has degree* $\leqslant \mathbf{a}$, *and* $g: F \times P \to F$ *is a function of degree* $\leqslant \mathbf{a}$ *such that for all* $q \in F$ *and* $x \in P$, $g(q, x) \leqslant q$ *and* $x \in \mathrm{dom}(g(q, x))$, *then we can choose* $\{p_i : i \in N\}$ *so that* G *has degree* $\leqslant \mathbf{a}$.

Proof. For the first part of the proof, we follow the proof of the Existence Theorem (II.2.8) for $\mathscr{C}$-generic sets, except that p_{n+1} is specified by f instead of being picked arbitrarily. (ii) allows us to prove a density lemma in this way. (iii) and (v) are easily seen to be satisfied, and (iv) now follows from (i).

h defined as in the hypothesis of the principle will automatically have degree $\leqslant \mathbf{a}$. If P has degree $\leqslant \mathbf{a}$, then we can choose $\{p_i : i \in N\}$ as in the proof of Theorem II.2.9 using f to determine p_{2s+1} and g to determine p_{2s+2}. Then $\bigcup G(x) = y \Leftrightarrow p_{2x+2}(x) = y$, so by (iv), $\bigcup G$ has degree $\leqslant \mathbf{a}$. ∎

The application of the Bounding Principle to some of the results in Chapter II is straightforward. When the proof of the Density Lemma produces a function f of degree $\leqslant \mathbf{0}'$, then 3.2(i) is easily verified, and 3.2(ii) is just the Density Lemma. Hence it is important to determine the effectiveness of the passage from a condition p and a requirement R to a condition $q \leqslant p$ which forces R.

We first classify the finite posets which can be embedded into $\mathscr{D}[\mathbf{0},\mathbf{0}']$. The crucial step in this classification is the existence of a countable set of independent degrees in $\mathbf{D}[\mathbf{0},\mathbf{0}']$.

3.3 Theorem. *There is a countable set $\{a_i : i \in N\}$ of independent degrees such that $\mathbf{a}_i \leqslant \mathbf{0}'$ for all $i \in N$.*

Proof. Proceed essentially as in the proof of Theorem II.3.6. We construct a set $A \subseteq N^2$ such that for all $e, i \in N$, the requirement $R_{e,i}: \Phi_e^{A^{[\hat{i}]}} \neq A_i$ is satisfied. We note that we can recursively extend a condition to one converging on a given argument, so by the Bounding Principle, it suffices to show that there is a function f of degree $\leqslant \mathbf{0}'$ such that for all $\psi \in F$ and $e, i \in N$, $\psi \subseteq f(\psi, R_{e,i}) = \theta \in F$ and $f(\psi, R_{e,i}) \Vdash R_{e,i}$. We note that in the proof of Theorem II.3.6, F is a space, so we can recursively order $F \times \mathscr{S}_2$ as $\{\langle \theta_i, \sigma_i \rangle : i \in N\}$. The definition of θ in the proof of Theorem II.3.6 depends on whether

$$(1) \qquad \exists \theta \in F\, \exists \sigma \in \mathscr{S}_2 (\theta \supseteq \psi \;\&\; \sigma \subseteq \theta^{[\hat{i}]} \;\&\; \Phi_e^{\sigma}(x)\downarrow),$$

for a specified x depending recursively on ψ. By Theorem 1.9, $S = \{\langle \psi, e, i \rangle : (1)$ holds for ψ, e and $i\}$ is recursively enumerable, hence by 2.3(iv), S has degree $\leqslant \mathbf{0}'$. If $\langle \psi, e, i \rangle \notin S$, then we set $f(\psi, R_{e,i}) = \psi$; and if $\langle \psi, e, i \rangle \in S$, then we search for the least j such that $\theta_j \supseteq \psi$, $\sigma_j \subseteq \theta^{[i]}$ and $\Phi_e^{\sigma_j}(x)\downarrow$ (noting that such a j will be found through a recursive search) and set $F(\psi, R_{e,i}) = \theta_j$. f is easily seen to have degree $\leqslant \mathbf{0}'$. ∎

As in Section II.3, the following corollaries can now be drawn.

3.4 Corollary. *Let $\mathscr{U} = \langle U, \leqslant \rangle$ be a finite poset. Then $\mathscr{U} \hookrightarrow \mathscr{D}$.*

3.5 Corollary. $\mathrm{Th}(\mathscr{D}[\mathbf{0},\mathbf{0}']) \cap \exists_1$ *is decidable.*

Having discussed the localization of the embedding theorems of Sect. II.3, we turn our attention to the extension theorems of Sect. II.4. Some of the extension theorems of that section are false in $\mathscr{D}[\mathbf{0},\mathbf{0}']$, while others require new proofs. The new proofs will be presented in subsequent sections. For the remainder of this section, we concentrate primarily on a local version of Theorem II.4.8, the Exact Pair Theorem for Countable Ideals. The Exact Pair Theorem is false in $\mathscr{D}[\mathbf{0},\mathbf{0}']$, but a weak version is true and can be used to show that $\mathscr{D}[\mathbf{0},\mathbf{0}']$ is not a lattice. We first demonstrate the falsity of the Exact Pair Theorem in $\mathscr{D}[\mathbf{0},\mathbf{0}']$.

3.6 Corollary. *There are $2^{\aleph_0}$ many ideals $\mathbf{I}$ of $\mathscr{D}[\mathbf{0},\mathbf{0}']$ such that for all $\mathbf{a},\mathbf{b} \leqslant \mathbf{0}'$, $\mathbf{I} \neq \{\mathbf{d} : \mathbf{d} \leqslant \mathbf{a} \;\&\; \mathbf{d} \leqslant \mathbf{b}\}$.*

Proof. Choose a countable set $\{\mathbf{a_i}: i \in N\}$ of independent degrees $\leqslant \mathbf{0}'$ as in Theorem 3.3. For each $S \subseteq N$, let

$$\mathbf{I}_S = \{\mathbf{d} \in \mathbf{D}: \exists F \subseteq S(F \text{ is finite } \& \, \mathbf{d} \leqslant \cup\{\mathbf{a_i}: i \in F\})\}.$$

It is easily verified that for all $S \subseteq N$ $\mathbf{I}_S$ is an ideal, and that

(2) $\qquad \forall S, T \subseteq N(S \neq T \rightarrow \mathbf{I}_S \neq \mathbf{I}_T).$

A counting argument will now complete the proof of the corollary. $|\{S: S \subseteq N\}| = 2^{\aleph_0} > \aleph_0$. $|\mathbf{D[0, 0']}| = \aleph_0$, so $|\mathbf{D[0, 0']} \times \mathbf{D[0, 0']}| = \aleph_0$. Thus by (2), there are $2^{\aleph_0}$ many ideals of $\mathbf{D[0, 0']}$ and only countably many possible exact pairs for such ideals. Hence $2^{\aleph_0}$ many ideals of $\mathbf{D[0, 0']}$ cannot have exact pairs below $\mathbf{0}'$. ▯

The major problem we face in trying to use the Bounding Principle to localize extension theorems to $\mathbf{D[0, 0']}$ is that it may be impossible to define $f(\theta, R) \supseteq \theta$ so that $f(\theta, R) \mid\vdash R$ and $f \in \Delta_2^0$. In all the theorems considered in Section II.4, if we start with a poset $\mathscr{T} = \langle \mathbf{T}, \leqslant \rangle$ of degrees and representatives $B \in \mathbf{b}$ for all $\mathbf{b} \in \mathbf{T}$, the definition of $f(\theta, R)$ depends on the truth value of a Σ_1^B sentence for some such B rather than on the truth value of a Σ_1^0 sentence. Hence if there is a set $A \subseteq N^2$ of degree $\mathbf{a}$ which effectively provides a set of representatives for $\mathbf{T}$, i.e., if there is a function $g: N \rightarrow N$ of degree $\leqslant \mathbf{a}$ such that, letting $\mathbf{T} = \{\mathbf{b_i}: i \in N\}$, then for all $i, j \in N$, if $g(i) = j$ then $A^{[j]}$ has degree $\mathbf{b_i}$, then the Bounding Principle will tell us that the particular extension theorem is true in $\mathscr{D}[\mathbf{0}, \mathbf{a}']$. In particular, the following local version of the Exact Pair Theorem is seen to be true for the reasons mentioned above.

3.7 Theorem. *Let $A \subseteq N^2$ have degree $\mathbf{a}$ with $\mathbf{a}' = \mathbf{0}'$. Let*

$$\mathbf{I} = \{\mathbf{d} \leqslant \mathbf{0}': \exists F \subseteq N(F \text{ is finite } \& \, \mathbf{d} \leqslant \mathbf{A}^{[\mathbf{F}]})\}.$$

Then there are degrees $\mathbf{b}, \mathbf{c} \leqslant \mathbf{0}'$ such that $\mathbf{I} = \{\mathbf{d} \in \mathbf{D}: \mathbf{d} \leqslant \mathbf{b} \, \& \, \mathbf{d} \leqslant \mathbf{c}\}$.

In order to show that $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ is not a lattice, we proceed as in Sect. II.4. Thus we must produce an ideal $\mathbf{I}$ of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ with no greatest element to which the Exact Pair Theorem applies. The ideal used in Sect. II.4 was generated by an infinite independent set of degrees. In order to apply Theorem 3.7, however, we need a set $A \subseteq N^2$ such that A has degree $\mathbf{a}$, $\mathbf{a}' = \mathbf{0}'$, and $\{\mathbf{A}^{[\mathbf{i}]}: i \in N\}$ is an infinite set of independent degrees. Additional requirements must therefore be incorporated into the proof of Theorem 3.3 in order to insure that A' has degree $\mathbf{0}'$. One way in which this can be accomplished is to require A to force its jump.

3.8 Definition. Let $A \subseteq N$ be given. We say that *A forces its jump* if for all $e \in N$, there is a $\sigma \subset \mathscr{S}_2$ such that $\sigma \subset A$ and either

(i) $\qquad \Phi_e^\sigma(e)\downarrow$

or

(ii) $\qquad \forall \tau \supseteq \sigma(\Phi_e^\tau(e)\uparrow).$

3.9 Lemma. *Let $A \subseteq N$ be given such that A forces its jump. Let A have degree* $\mathbf{a}$. *Then* $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}'$. *Hence if* $\mathbf{a} \leqslant \mathbf{0}'$, *then* $\mathbf{a}' = \mathbf{0}'$.

Proof. By Theorem 2.3(iv) and (v), $\mathbf{a} \cup \mathbf{0}' \leqslant \mathbf{a}'$. To see that $\mathbf{a}' \leqslant \mathbf{a} \cup \mathbf{0}'$, we enumerate $\{\sigma : \sigma \subset A\}$, shorter strings first, and for each such σ, ask whether 3.8(i) or 3.8(ii) holds. Since A forces its jump, we eventually find a shortest σ such that 3.8(i) or 3.8(ii) holds, using A and $\emptyset'$ oracles. Now $e \in A' \Leftrightarrow \Phi_e^\sigma(e)\downarrow$. Hence $\mathbf{a}' \leqslant \mathbf{a} \cup \mathbf{0}'$. ▯

We now construct an infinite set of independent degrees which can be used to show that $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ is not a lattice.

3.10 Theorem. *There is a set $A \subseteq N^2$ such that:*
(i) *A has degree $\leqslant \mathbf{0}'$.*
(ii) *A forces its jump.*
(iii) *$\{A^{[i]} : i \in N\}$ is an infinite set of independent degrees.*

Proof. We modify the proof of Theorem 3.3 by adding requirements $\{T_e : e \in N\}$ whose satisfaction will guarantee that A forces its jump. Our notion of forcing is again $\langle F, \supseteq \rangle$ where $F = \{\sigma \subseteq N^2 : \mathrm{dom}(\sigma) \text{ is finite}\}$. Since N^2 is a space, we can treat $\sigma \in F$ as a subset of N. The new requirements are defined as follows:

(3) $\quad T_e : \exists \sigma \in \mathscr{S}_2(\sigma \subset A \;\&\; \sigma$ and e satisfy either 3.8(i) or 3.8(ii)).

For $\xi \in F$, we define $\xi \Vdash T_e$ as in (3) but with ξ in place of A. By the proof of Theorem 3.3 and the Bounding Principle, it suffices to show that there is a function $f : F \times N \to F$ of degree $\leqslant \mathbf{0}'$ such that for all $\xi \in F$ and $e \in N$, $\xi \subseteq f(\xi, e)$ and $f(\xi, e) \Vdash T_e$. Fix such ξ and e. We ask if there is a $\sigma \in \mathscr{S}_2$ which is compatible with ξ such that $\Phi_e^\sigma(e)\downarrow$. Such a question is a Σ_1^0 question, so can be answered by a $\emptyset'$ oracle. If the answer to this question is no, then we let $f(\xi, e) = \xi$, and note that 3.8(ii) is satisfied for $f(\xi, e)$ in place of σ. If the answer to this question is yes, we search for the least such σ under some fixed recursive one-one correspondence of N with $\mathscr{S}_2$, noting that a $\emptyset'$ oracle can identify such a σ. We let $f(\xi, e)$ be a common refinement of σ and ξ, and note that 3.8(i) is satisfied for $f(\xi, e)$ in place of σ. f is now seen to have all the desired properties. ▯

The following corollary is now proved in the same way as Corollary II.4.10, using Theorem 3.10 to provide the countable independent set of degrees, and Theorem 3.7 instead of the Exact Pair Theorem. Note that by Lemma 3.9, we can apply Theorem 3.7 in this situation.

3.11 Corollary. *$\mathscr{D}[\mathbf{0}, \mathbf{0}']$ is not a lattice.*

All the theorems and corollaries of this section have relativized versions. We leave these to the reader to formulate and prove, the proofs being straightforward.

Exact pairs for ideals of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ can be obtained under less restrictive assumptions on the set of representatives for generators of such an ideal than were placed in the hypothesis of Theorem 3.7. The proof requires more powerful techniques, and will be given in Sect. 8. In the next section, we will characterize the range of the jump operator on certain classes of degrees.

3.12 Remarks. The theorems proved in this section were proved at the same time as their global counterparts. Theorem 3.3 was proved by Kleene and Post [1954], and Theorem 3.7 and Corollary 3.11 were proved by Spector [1956].

3.13 Exercise. Let $B \in \mathbf{b} \in \mathbf{D}$, $f: N \to N$ and a countable ideal $\mathbf{I}$ of $\mathscr{D}[\mathbf{0}, \mathbf{b}]$ be given such that for all $C \in \mathbf{c} \in \mathbf{D}$,

$$\mathbf{c} \in \mathbf{I} \leftrightarrow \exists e \in N(C \leqslant_T \Phi^B_{f(e)}).$$

Show that there is an exact pair $\langle \mathbf{a}, \mathbf{d} \rangle$ for $\mathbf{I}$ such that $\mathbf{a}, \mathbf{d} \leqslant \mathbf{b}' \cup \mathbf{f}$.

4. Jump Inversion

The methods which have been used to this point can be combined with new coding techniques to characterize the range of the jump operator both on $\mathbf{D}$ and on $\mathbf{D}[\mathbf{0}, \mathbf{0}']$. We first characterize the degrees which are jumps of other degrees.

The sets constructed in this section will force their jumps. The description of the constructions is nicely given in terms of the following sets. Let $P = \{\langle \sigma, e \rangle \in \mathscr{S}_2 \times N: \sigma$ and e satisfy 3.8(i)$\}$ and let $Q = \{\langle \sigma, e \rangle \in \mathscr{S}_2 \times N: \sigma$ and e satisfy 3.8(ii)$\}$. Note that P is recursively enumerable as is $(\mathscr{S}_2 \times N) - Q$. Hence

(1) $\quad \mathbf{P} \leqslant \mathbf{0}' \,\&\, \mathbf{Q} \leqslant \mathbf{0}'$.

The first construction which we present constructs a set by alternately forcing its jump on an integer and coding another set into the string. This type of procedure will also be used to prove other theorems. The following principle replaces the Bounding Principle in such constructions.

4.1 Bounding Principle for Forcing and Coding. *Let $C \subseteq N$ and $f: \mathscr{S}_2 \times N \to \mathscr{S}_2$ be given. Define $\{\alpha_n \in \mathscr{S}_2: n \in N\}$ as follows: $\alpha_0 = \emptyset$, and $\alpha_{n+1} = \alpha_n * f(\alpha_n, n) * C(n)$. Let $A = \bigcup\{\alpha_n: n \in N\}$. Then*

(i) $\mathbf{A} \leqslant \mathbf{f} \cup \mathbf{C}$.

(ii) $\mathbf{C} \leqslant \mathbf{f} \cup \mathbf{A}$.

Proof. (i) is immediate from the definition of $\{\alpha_n: n \in N\}$. We verify (ii) by inductively computing $C(n)$ using an f oracle and an A oracle. Suppose by induction that α_n has been computed. Use the f oracle to compute $f(\alpha_n, n) = \sigma$. Next use the A oracle to find $\tau \in \mathscr{S}_2$ such that $\alpha_n * \sigma \subset \tau \subset A$ and $\mathrm{lh}(\tau) = \mathrm{lh}(\alpha_n) + \mathrm{lh}(\sigma) + 1$. Then $\alpha_{n+1} = \tau$ and $C(n) = \tau(\mathrm{lh}(\tau) - 1)$. ▯

We now characterize the range of the jump operator.

4.2 Friedberg Jump Inversion Theorem. *Let $\mathbf{c} \in \mathbf{D}$ be given. Then there is an $\mathbf{a} \in \mathbf{D}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{c} \cup \mathbf{0}'$.*

Proof. Fix a recursive one-one correspondence of N with $\mathscr{S}_2$. Define $f: \mathscr{S}_2 \times N \to \mathscr{S}_2$ by $f(\sigma, n) = \tau$ where τ is the least element of $\mathscr{S}_2$ (under the above correspondence) such that $\langle \sigma * \tau, n\rangle \in P$ if such a τ exists, and $\tau = \emptyset$ otherwise. Since P is recursively enumerable, $\mathbf{f} \leqslant \mathbf{0}'$. Fix $C \subseteq N$ such that $C \in \mathbf{c}$. Let $\alpha_0 = \emptyset$ and $\alpha_{n+1} = \alpha_n * f(\alpha_n, n) * C(n)$, and let $A = \bigcup\{\alpha_n : n \in N\}$. It is easily verified that A forces its jump, so by Lemma 3.9, $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}'$. Since $\mathbf{0}' \leqslant \mathbf{a} \cup \mathbf{0}'$ and $\mathbf{0}' \leqslant \mathbf{c} \cup \mathbf{0}'$, it follows from the Bounding Principle for Forcing and Coding that $\mathbf{c} \cup \mathbf{0}' = \mathbf{a} \cup \mathbf{0}'$. ▯

4.3 Corollary. *Let* $\mathbf{c} \in \mathbf{D}$ *be given. Then the following are equivalent:*
(i) $\exists \mathbf{a} \in \mathbf{D}(\mathbf{a}' = \mathbf{c})$.
(ii) $\mathbf{c} \geqslant \mathbf{0}'$.

Proof. (i) $\Rightarrow$ (ii): Immediate from Theorem 2.3(v).
(ii) $\Rightarrow$ (i): By the Friedberg Jump Inversion Theorem. ▯

The following is a relativization of Theorem 4.2. Its proof is straightforward, and is left to the reader (Exercise 4.15).

4.4 Corollary. *Let* $\mathbf{d} \in \mathbf{D}$ *and* $\mathbf{c} \in \mathbf{D}[\mathbf{d}, \infty]$ *be given. Then there is an* $\mathbf{a} \in \mathbf{D}[\mathbf{d}, \infty)$ *such that* $\mathbf{a}' = \mathbf{a} \cup \mathbf{d}' = \mathbf{c} \cup \mathbf{d}'$.

The next corollary follows easily from Corollary 4.4 and induction. We leave its proof to the reader (Exercise 4.16).

4.5 Corollary. *Let* $n \in N$ *and* $\mathbf{c} \in \mathbf{D}$ *be given. Then there is a degree* $\mathbf{a}$ *such that* $\mathbf{a}^{(n)} = \mathbf{a} \cup \mathbf{0}^{(n)} = \mathbf{c} \cup \mathbf{0}^{(n)}$.

The proof of Theorem 4.2 can be repeated for other recursively enumerable sets in place of P. For example, if $e \in N$ and we let $P_e = \{\langle \sigma, s\rangle : s \in W_e^\sigma\}$ and repeat the proof of Theorem 4.2 with P_e in place of P, then we obtain the following result.

4.6 Theorem. *Let* $e \in N$ *and* $C \leqslant N$ *be given such that the degree of* C *is* $\geqslant \mathbf{0}'$. *Then there is a set* A *such that* $A \oplus W_e^A \equiv_T C$.

Theorem 4.6 has the following consequence for $C = \emptyset'$. W_e^A is viewed as the construction of a set recursively enumerable in A which possesses a certain property for all A. Suppose that for all $A \subseteq N$, $A \leqslant_T W_e^A$. Then if the property is degree invariant, Theorem 4.6 tells us that $\emptyset'$ has this property with respect to some set of degree $\leqslant \mathbf{0}'$. A sample application of this result is now given.

4.7 Corollary. *There is a degree* $\mathbf{a} < \mathbf{0}'$ *such that* $\mathbf{a}' = \mathbf{0}^{(2)}$.

Proof. We prove in Sect. 7 that there is an $e \in N$ such that for all $A \subseteq N$, $A <_T W_e^A$ and $A' \equiv_T (W_e^A)'$. If we choose $C = \emptyset'$ in Theorem 4.6, then there is a set $A \subseteq N$ of degree $\mathbf{a}$ such that $A <_T W_e^A \equiv_T \emptyset'$ for which $\mathbf{a}' = (\mathbf{0}')' = \mathbf{0}^{(2)}$. ▯

In Chap. IV, we will investigate a hierarchy defined by jumps of degrees $\leqslant \mathbf{0}'$. In order to show that this hierarchy does not degenerate, we will need a stronger version of Theorem 4.6 which allows us to choose A to be recursively enumerable when $C = \emptyset'$. We will prove this result in Sect. 7.

Let $\mathbf{J}[\mathbf{0}, \mathbf{0}']$ denote the range of the jump operator on $\mathbf{D}[\mathbf{0}, \mathbf{0}']$. We will prove a local version of the Friedberg Jump Inversion Theorem, and so obtain a characterization of $\mathbf{J}[\mathbf{0}, \mathbf{0}']$. We cannot use the proof of the Friedberg Jump Inversion Theorem, however, since that theorem produces degrees in $\mathbf{D}[\mathbf{0}, \mathbf{0}']$ only if those degrees have jump equal to $\mathbf{0}'$.

Certain restrictions can immediately be placed on the degrees in $\mathbf{J}[\mathbf{0},\mathbf{0}']$. It follows from Theorem 2.3(v) that for all $\mathbf{c}\in\mathbf{J}[\mathbf{0},\mathbf{0}']$,

(2) $\qquad \mathbf{0}' \leqslant \mathbf{c}.$

Furthermore, since any degree which is recursively enumerable in $\mathbf{b} \leqslant \mathbf{d}$ must also be recursively enumerable in $\mathbf{d}$, it follows from Theorem 2.3(i) that for all $\mathbf{c}\in\mathbf{J}[\mathbf{0},\mathbf{0}']$,

(3) $\qquad \mathbf{c}$ is recursively enumerable in $\mathbf{0}'$.

We will show that $\mathbf{J}[\mathbf{0},\mathbf{0}']$ is just the set of all degrees which satisfy (2) and (3).

A new coding strategy is needed to invert the jump operator on $\mathbf{D}[\mathbf{0},\mathbf{0}']$. We will construct a set $A \subseteq N^2$ of degree $\leqslant \mathbf{0}'$ whose definition depends on a set C which is recursively enumerable in $\emptyset'$ such that:

(4) $\qquad \forall n\in N(n\in C \rightarrow \{x : A^{[n]}(x) \neq 1\}$ is finite.

(5) $\qquad \forall n\in N(n\notin C \rightarrow \{x : A^{[n]}(x) \neq 0\}$ is finite.

The following lemma shows that, under these circumstances, $C \leqslant_T A'$.

4.8 Lemma. *Let $A \subseteq N^2$ and $C \subset N$ be given such that* (4) *and* (5) *are satisfied. Then* $\mathbf{C} \leqslant \mathbf{A}'$.

Proof. $n\in C \leftrightarrow \exists x\,\forall y \geqslant x(A^{[n]}(y) = 1) \leftrightarrow \forall x\,\exists y \geqslant x(A^{[n]}(y) = 0)$. Hence $C\in\Delta_2^A$. By the relativization of Post's Theorem, $\mathbf{C} \leqslant \mathbf{A}'$. ▯

Before we characterize $\mathbf{J}[\mathbf{0},\mathbf{0}']$, we give the following definition.

4.9 Definition. Let $\{x_s : x, s\in N\}$ be an array of numbers, and let $m\in N$ be given. We say that $\liminf_s x_s = m$ if m is the least element of N such that $\{s : x_s = m\}$ is infinite. We say that $\liminf_s x_s = \infty$ if for all $k\in N$, $\{s : x_s = k\}$ is finite.

We now characterize $\mathbf{J}[\mathbf{0},\mathbf{0}']$. The proof that this characterization is correct differs from previous proofs in that it depends on a *priority ordering* of requirements. Thus instead of designating a particular step of the construction as the step at which a given predetermined requirement is satisfied, we have a situation where each requirement, while unsatisfied, tries to manipulate the construction at all sufficiently large steps in order to satisfy itself. The requirement of highest priority which, at a given step, can make some progress towards satisfying itself is the one chosen to determine the action taken at that step. We will show that each requirement is so chosen only finitely often, so that we will have the opportunity to try to satisfy all requirements.

4.10 The Shoenfield Jump Inversion Theorem. *Let $\mathbf{c}\in\mathbf{D}$ be given such that $\mathbf{c}$ satisfies* (2) *and* (3). *Then there is an $\mathbf{a} \leqslant \mathbf{0}'$ such that $\mathbf{a}' = \mathbf{c}$.*

Proof. Fix $\mathbf{c}\in\mathbf{D}$ satisfying (2) and (3). We construct a sequence of partial functions $\{\alpha_s : s\in N\}$ such that for all $s\in N$, $\alpha_s : N^2 \rightarrow \{0,1\}$ and $\alpha_s \subseteq \alpha_{s+1}$. $A = \bigcup\{\alpha_s : s\in N\}$ will be the desired set of degree $\mathbf{a}$. By (3), $\mathbf{c}$ is recursively enumerable in $\mathbf{0}'$ so we can fix a set $C \in \mathbf{c}$ which is recursively enumerable in $\emptyset'$ and a one-one function f recursive in $\emptyset'$ whose range is C.

Since N^2 is a space, we may identify N^2 with N and so treat any partial function with domain N^2 as if it had domain N. For such a partial function θ with domain N^2, it will thus make sense to say $\sigma \subseteq \theta$ for $\sigma \in \mathscr{S}_2$.

$\{\alpha_s : s \in N\}$ will be defined so that (4) and (5) are satisfied by A and C. We will thus be able to conclude from Lemma 4.8 that $\mathbf{c} \leqslant \mathbf{a}'$. The strategy to make $\mathbf{a}' \leqslant \mathbf{c}$ will be to satisfy requirements which attempt to make A force its jump, but subject to constraints imposed in order to satisfy (5). Recall that $P = \{\langle \sigma, e \rangle : \Phi_e^\sigma(e)\downarrow\}$ and $Q = \{\langle \sigma, e \rangle : \forall \tau \supseteq \sigma(\Phi_e^\tau(e)\uparrow)\}$. We establish the following requirement for each $e \in N$:

(6) $\qquad R_e : \exists \sigma \subset A(\langle \sigma, e \rangle \in P \text{ or } \langle \sigma, e \rangle \in Q).$

We say that $\theta : N^2 \to \{0,1\}$ *satisfies* R_e if (6) holds with θ in place of A.

Along with $\{\alpha_s : s \in N\}$, we define a sequence $\{i_s : s \in N\}$, letting i_{s+1} be the requirement which we try to satisfy at stage $s+1$ of the construction. It will be the case that $\liminf_s i_s = \infty$, so we will be able to satisfy all requirements. We begin the construction by defining $\alpha_0 = \emptyset$ and $i_0 = 0$. The construction now proceeds as follows:

Stage $s+1$. Find the least $i \leqslant s$, if any, such that α_s does not satisfy R_i and such that there is a finite $\theta : N^2 \to \{0,1\}$ which satisfies the following conditions:

(7) $\qquad \operatorname{dom}(\theta) \cap \operatorname{dom}(\alpha_s) = \emptyset.$

(8) $\qquad \alpha_s \cup \theta$ satisfies R_i.

(9) $\qquad \forall j < i\ \forall x \in N(\theta(j,x)\downarrow \to \theta(j,x) = 0).$

If such i and θ exist, let $i_{s+1} = i$ and let θ_s be the least θ (under some fixed recursive one-one correspondence of N with $\{\psi : N^2 \to \{0,1\} : \operatorname{dom}(\psi) \text{ is finite}\}$) which satisfies (7)–(9). Otherwise, let $i_{s+1} = s+1$ and $\theta_s = \emptyset$. Let $\beta_{s+1} = \alpha_s \cup \theta_s$. It follows from (7) that β_{s+1} is well-defined.

We define α_{s+1} as follows:

$$\alpha_{s+1}(j,x) = \begin{cases} \beta_{s+1}(j,x) & \text{if } \beta_{s+1}(j,x)\downarrow \\ 1 & \text{if } \beta_{s+1}(j,x)\uparrow \,\&\, f(s) = j \\ 0 & \text{if } \beta_{s+1}(j,x)\uparrow \,\&\, f(s) \neq j \,\&\, j, x \leqslant s \\ \uparrow & \text{otherwise.} \end{cases}$$

This completes the construction. The imposition of (9) will allow us to show that (5) is satisfied. However, it also prevents us from immediately satisfying (6). For if $f(t) = j < i$ for some $t > s$, then (9) will no longer apply to j as we will be filling column j with 1s at stage t. Hence once such a constraint is removed, we may then be able to find θ satisfying (7)–(9), and also $\sigma \subset \theta$ such that $\langle \sigma, j \rangle \in P$. Since satisfaction of requirements cannot be finally determined until f produces enough information about C, priorities are used to choose i_s so as to allow an attempt at the satisfaction of each requirement after f has produced all needed information about C.

Fix $j \in N$. If $i_s = j < s$ for some s, then α_s satisfies R_j, hence for all $t \geqslant s$, α_t satisfies R_j. Thus $\{s: i_s = j\}$ is finite, so $\liminf_s i_s = \infty$. We now note that by (9) and the second line of the definition of α_{s+1}, (4) and (5) are satisfied. Hence $\mathbf{c} \leqslant \mathbf{a}'$.

The definition of $\{\alpha_s : s \in N\}$ is seen to be recursive in $\emptyset'$ since P, Q and f are recursive in $\emptyset'$. Hence $A = \bigcup\{\alpha_s : s \in N\}$ has degree $\leqslant \mathbf{0}'$. A is readily seen to be total from the second and third lines of the definition of α_{s+1}.

We complete the proof of the theorem by indicating how to compute A' recursively from C. Let $e \in N$ be given. Using a C oracle, we can find a stage s such that for all $t \geqslant s, f(t) \geqslant e$. Fix the least stage $r > s$ such that $i_r \geqslant e$. Then for all $j < e$ and $x \in N$, if $\theta_r(j, x)\uparrow$ then $A(j, x) = 0$. Hence if there is a $\sigma \subset A$ such that $\langle \sigma, e \rangle \in P$, then either α_{r-1} satisfies R_e or $i_r = e$ and α_r satisfies R_e. Since r can be found recursively from a C oracle, and since we can determine whether or not α_r satisfies R_e from a $\emptyset'$ oracle (note that $\mathrm{dom}(\alpha_r)$ is recursive), it follows from (2) that $\mathbf{a}' \leqslant \mathbf{c}$. ▯

The following corollary is now immediate.

4.11 Corollary. *Let* $\mathbf{c} \in \mathbf{D}$ *be given. Then there is an* $\mathbf{a} \leqslant \mathbf{0}'$ *such that* $\mathbf{a}' = \mathbf{c}$ *if and only if* $\mathbf{c} \geqslant \mathbf{0}'$ *and* $\mathbf{c}$ *is recursively enumerable in* $\mathbf{0}'$.

The degree $\mathbf{a}$ constructed in both the Friedberg and Shoenfield Jump Inversion Theorems can also be subjected to other restrictions. Some of these restrictions are discussed in the exercises. Other jump inversion theorems have been proved using more powerful techniques than those discussed so far. Sacks [1963a] has shown that the range of the jump operator on the recursively enumerable degrees is $\mathbf{J}[\mathbf{0}, \mathbf{0}']$, and Cooper [1973] has shown that the range of the jump operator on the set of minimal degrees (those degrees $\mathbf{d} > \mathbf{0}$ whose only predecessor is $\mathbf{0}$) is $\mathbf{D}[\mathbf{0}', \infty)$.

Suppose that $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ are given. The theorems which we have proved can be used to determine the extent to which the configuration of $\mathbf{a}$ and $\mathbf{b}$ specifies the configuration of $\mathbf{a}'$ and $\mathbf{b}'$. By Theorem 2.3(v), if $\mathbf{a} \leqslant \mathbf{b}$ then $\mathbf{a}' \leqslant \mathbf{b}'$. The next corollary shows that all the remaining possibilities can occur.

4.12 Corollary. *There are* $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ *such that*:

(i) $\mathbf{a} < \mathbf{b}$ *and* $\mathbf{a}' < \mathbf{b}'$.
(ii) $\mathbf{a} < \mathbf{b}$ *and* $\mathbf{a}' = \mathbf{b}'$.
(iii) $\mathbf{a} \mid \mathbf{b}$ *and* $\mathbf{a}' < \mathbf{b}'$.
(iv) $\mathbf{a} \mid \mathbf{b}$ *and* $\mathbf{a}' = \mathbf{b}'$.
(v) $\mathbf{a} \mid \mathbf{b}$ *and* $\mathbf{a}' \mid \mathbf{b}'$.

Proof. (i) Choose $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}'$. By Theorem 2.3(iv), $\mathbf{a}' = \mathbf{0}' = \mathbf{b} < \mathbf{b}'$.

(ii) Let $\mathbf{a} = \mathbf{0}$ and let $\mathbf{b}$ be the degree of the set A constructed in Theorem 3.10. Then $\mathbf{a} < \mathbf{b}$ and by Lemma 3.9, $\mathbf{a}' = \mathbf{b}' = \mathbf{0}'$.

(iii) By the Friedberg Jump Inversion Theorem applied to $\mathbf{c} = \mathbf{0}^{(3)}$, there is a degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{b} \cup \mathbf{0}' = \mathbf{0}^{(3)}$. Let $\mathbf{a} = \mathbf{0}'$. Then $\mathbf{a}' < \mathbf{b}'$. Now $\mathbf{a} \neq \mathbf{0}^{(3)}$ and by Theorem 2.3(ii), $\mathbf{b} \neq \mathbf{0}^{(3)}$. Hence $\mathbf{a} \mid \mathbf{b}$.

(iv) Let $\mathbf{a}$ and $\mathbf{b}$ be the degrees of the sets $A^{[0]}$ and $A^{[1]}$ constructed in Theorem 3.10. Then $\mathbf{a} \mid \mathbf{b}$ and $\mathbf{a}' = \mathbf{b}' = \mathbf{0}'$.

(v) By the relativized version of Theorem 3.3, there are incomparable degrees $\mathbf{c}, \mathbf{d} \geqslant \mathbf{0}'$. By the Friedberg Jump Inversion Theorem, there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}' = \mathbf{c}$ and $\mathbf{b}' = \mathbf{d}$. $\mathbf{a}$ and $\mathbf{b}$ must be incomparable, else by Theorem 2.3(v), $\mathbf{c}$ and $\mathbf{d}$ would be comparable. ▯

Given $\mathbf{a}, \mathbf{b} \in \mathbf{D}$, we now ask about the relationship between $\mathbf{a}' \cup \mathbf{b}'$ and $(\mathbf{a} \cup \mathbf{b})'$. Since $\mathbf{a} \leqslant \mathbf{a} \cup \mathbf{b}$ and $\mathbf{b} \leqslant \mathbf{a} \cup \mathbf{b}$, Theorem 2.3(v) tells us that $\mathbf{a}' \cup \mathbf{b}' \leqslant (\mathbf{a} \cup \mathbf{b})'$. And if $\mathbf{a} \leqslant \mathbf{b}$, then Theorem 2.3(v) tells us that $\mathbf{a}' \cup \mathbf{b}' = (\mathbf{a} \cup \mathbf{b})'$. The next corollary tells us that all remaining possibilities can occur.

4.13 Corollary. *There are* $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ *such that* $\mathbf{a} \mid \mathbf{b}$ *and*:

(i) $\mathbf{a}' \cup \mathbf{b}' = (\mathbf{a} \cup \mathbf{b})'$.

(ii) $\mathbf{a}' \cup \mathbf{b}' < (\mathbf{a} \cup \mathbf{b})'$.

Proof. (i) Choose A as in Theorem 3.10. Let $\mathbf{a}$ be the degree of $A^{[0]}$, let $\mathbf{b}$ be the degree of $A^{[1]}$, and let $\mathbf{c}$ be the degree of A. Then $\mathbf{a}, \mathbf{b} \leqslant \mathbf{c}$ and $\mathbf{a}' = \mathbf{b}' = \mathbf{0}'$. Hence $\mathbf{a}' \cup \mathbf{b}' = \mathbf{0}' = \mathbf{c}'$ and $\mathbf{0}' \leqslant (\mathbf{a} \cup \mathbf{b})' \leqslant \mathbf{c}'$.

(ii) Apply the Friedberg Jump Inversion Theorem to $\mathbf{c} = \mathbf{0}''$ to obtain a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{0}''$. Let $\mathbf{b} = \mathbf{0}'$. Then $\mathbf{a}' \cup \mathbf{b}' = \mathbf{0}''$ while $(\mathbf{a} \cup \mathbf{b})' = \mathbf{0}^{(3)}$. ▯

Corollary 4.12 and Corollary 4.13 tell us that certain $\exists_1$ sentences are true in the language for $\mathscr{D}\mathscr{U}' = \langle \mathbf{D}, \leqslant, \cup, ' \rangle$. However, it is not known whether $\mathrm{Th}(\mathscr{D}\mathscr{U}') \cap \exists_1$ is decidable. In fact, it is not known whether $\mathrm{Th}(\mathscr{D}') \cap \exists_1$ is decidable, where $\mathscr{D}' = \langle \mathbf{D}, \leqslant, ' \rangle$. The inability to determine the truth of the following $\forall_2$ sentence is a major stumbling block in deciding these classes of sentences:

$$\forall \mathbf{c}, \mathbf{d} \in \mathbf{D}(\mathbf{c}, \mathbf{d} \geqslant \mathbf{0}' \,\&\, \mathbf{c} \mid \mathbf{d} \,\&\, \mathbf{c}, \mathbf{d} \text{ recursively enumerable in } \mathbf{0}'$$

$$\rightarrow \exists \mathbf{a}, \mathbf{b} \in \mathbf{D}(\mathbf{a}, \mathbf{b} \leqslant \mathbf{0}' \,\&\, \mathbf{a}' = \mathbf{c} \,\&\, \mathbf{b}' = \mathbf{d} \,\&\, (\mathbf{a} \cup \mathbf{b})' = \mathbf{c} \cup \mathbf{d})).$$

(It follows from 4.11 that this sentence is $\forall_2$.) If we drop the relation $\leqslant$ however, then Jockusch and Soare have shown that the corresponding elementary theory is decidable. We leave the proof of this result to the reader (Exercise 4.21).

The methods of this section are extended in the next section to obtain results about maximal antichains of $\mathbf{D}[\mathbf{0}, \mathbf{0}']$. Those methods will allow us to prove even more general jump inversion theorems than are mentioned in the exercises below.

4.14 Remarks. Theorem 4.2 was proved by Friedberg [1957], and Theorem 4.10 was proved by Shoenfield [1959]. Corollary 4.5 was proved by Selman [1972], but an easier proof appears in Jockusch [1974]. Theorem 4.6 was proved by Jockusch and Shore [1983].

4.15–4.24 Exercises

***4.15** Let $\mathbf{d} \in \mathbf{D}$ and $\mathbf{c} \geqslant \mathbf{d}$ be given. Show that there is a degree $\mathbf{a} \geqslant \mathbf{d}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{d}' = \mathbf{c} \cup \mathbf{d}'$.

***4.16** Let $n \in N$ and $\mathbf{c}, \mathbf{d} \in \mathbf{D}$ be given. Show that there is a degree $\mathbf{a} \geqslant \mathbf{d}$ such that $\mathbf{a}^{(n)} = \mathbf{a} \cup \mathbf{d}^{(n)} = \mathbf{c} \cup \mathbf{d}^{(n)}$. (*Hint*: Apply 4.15 in an induction proof.)

4.17 Characterize the set $\mathbf{J}[\mathbf{0}, \mathbf{0}^{(n)}] = \{\mathbf{d} : \exists \mathbf{a} \in \mathbf{D}[\mathbf{0}, \mathbf{0}^{(n)}](\mathbf{a}' = \mathbf{d})\}$.

4.18 Let $n \in N$ and $\mathbf{b}, \mathbf{c} \in \mathbf{D}$ be given such that $\mathbf{b} > \mathbf{0}$. Show that there is a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{c} \cup \mathbf{0}'$ and $\mathbf{b} \nleqslant \mathbf{a}$. (*Hint*: Fix $B \in \mathbf{b}$ and construct A as in 4.2

so that A satisfies all requirements of the form $\Phi_e^A \neq B$. If $\mathbf{b} \nleqslant \mathbf{c}$, then the construction of 4.2 automatically satisfies these requirements. If $\mathbf{b} \leqslant \mathbf{c}$, then in order to satisfy such a requirement recursively in $\mathbf{c} \cup \mathbf{0}'$, search for $\sigma, \tau \supset \alpha_n$ and $x \in N$ such that $\Phi_e^\sigma(x)\downarrow \neq \Phi_e^\tau(x)\downarrow$. If no such σ, τ and x exist, show that for all $C \subseteq N$, if $\alpha_n \subset C$ then Φ_e^C is recursive.)

A binary *tree* is a function $T: \mathscr{S}_2 \to \mathscr{S}_2$ such that for all $\sigma, \tau \in \mathscr{S}_2$: (i) If $\sigma \subset \tau$ then $T(\sigma) \subset T(\tau)$ and; (ii) If $\sigma | \tau$ then $T(\sigma) | T(\tau)$. If T is a tree and $A \subseteq N$, then $T(A) = \bigcup\{T(\sigma): \sigma \subset A\}$ is a subset of A. (Note that by (i), $T(A)$ is well-defined.)

4.19 Construct a binary tree T such that for all $A, B \subseteq N$, if $A \neq B$ then $\mathbf{T(A)} \neq \mathbf{T(B)}$ and $\mathbf{T(A)}' = \mathbf{T(A)} \cup \mathbf{0}'$. (*Hint*: Proceed as in the proof of Theorem 4.2 except that a tree is defined instead of a sequence. Thus at the nth step, those $\sigma \in \mathscr{S}_2$ such that $\text{lh}(\sigma) = n$ are used to do the coding part for $T(\sigma)$ (instead of using C to do the coding). In addition, for each $e \in N$, establish a requirement which asserts that for all $\sigma, \tau \in \mathscr{S}_2$, if $\text{lh}(\sigma) = \text{lh}(\tau) = e$ then for all $A \supset T(\sigma)$ and $B \supset T(\tau)$, $\Phi_e^A \neq B$. Take care to satisfy this requirement when $\{T(\sigma): \text{lh}(\sigma) = e\}$ is defined.)

4.20 Show that for each $\mathbf{c} \geqslant \mathbf{0}'$, there are infinitely many $\mathbf{a} \in \mathbf{D}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{c}$. (*Hint*: Use Exercise 4.19.)

4.21 (Jockusch and Soare). Prove that $\text{Th}(\langle \mathbf{D}, ' \rangle)$ is decidable. (*Hint*: Start with the language of the pure predicate calculus with equality together with a one-place function symbol f to be interpreted as the jump operator. Add definable function symbols f^n for each $n \in N$ where $f^n(x) = x^{(n)}$ and definable relation symbols $R_n(x)$ for each $n \in N$ where $R_n(y) \equiv \exists x(f^n(x) = y)$. Recursively axiomatize the theory using a relativized version of Exercise 4.20 to generate some of the axioms and to show that $\text{Th}(\langle \mathbf{D}, ' \rangle)$ satisfies all the axioms. Show that the set of axioms is complete. Conclude that $\text{Th}(\langle \mathbf{D}, ' \rangle)$ is decidable.)

4.22 Let $\mathbf{c}, \mathbf{d} \in \mathbf{D}$ be given such that $\mathbf{d} \leqslant \mathbf{0}'$, $\mathbf{c} \geqslant \mathbf{d}'$ and $\mathbf{c}$ is recursively enumerable in $\mathbf{0}'$. Prove that there is a degree $\mathbf{a} \leqslant \mathbf{0}'$ such that $\mathbf{d} \leqslant \mathbf{a}$ and $\mathbf{a}' = \mathbf{c}$.

4.23 Let $\mathbf{b}, \mathbf{c} \in \mathbf{D}$ be given such that $\mathbf{0} < \mathbf{b} \leqslant \mathbf{0}'$, $\mathbf{c} \geqslant \mathbf{0}'$, and $\mathbf{c}$ is recursively enumerable in $\mathbf{0}'$. Show that there is a degree $\mathbf{a} \leqslant \mathbf{0}'$ such that $\mathbf{a}' = \mathbf{c}$ and $\mathbf{b} \nleqslant \mathbf{a}$. (See hint to Exercise 4.18.)

Exercises 4.22 and 4.23 mention possible restrictions which can be placed on the degree $\mathbf{a}$ constructed in the Shoenfield Jump Inversion Theorem. Other potential restrictions come to mind. If $\mathbf{e}, \mathbf{f} < \mathbf{0}'$, can we construct $\mathbf{a} \leqslant \mathbf{e}$? Can we construct $\mathbf{a} \nleqslant \mathbf{f}$? We will show in the next chapter that it is not always possible to construct $\mathbf{a} \leqslant \mathbf{e}$. Methods introduced in the next section will allow us to construct $\mathbf{a} \nleqslant \mathbf{f}$. The proofs of Exercises 4.22 and 4.23 can be combined and extended to prove the following result.

4.24 Let $B, E \subseteq N^2$ be given such that $\mathbf{B}, \mathbf{E} \leqslant \mathbf{0}'$. Assume that for all $i \in N$ and all finite $F \subseteq N$, $B^{[i]} \nleqslant_T E^{[F]}$. Let $\mathbf{c} \in \mathbf{D}$ be given such that $\mathbf{c} \geqslant \mathbf{0}'$, $\mathbf{c}$ is recursively enumerable in $\mathbf{0}'$ and $\mathbf{c} \geqslant (\mathbf{E}^{[\mathbf{F}]})'$ for every finite $F \subseteq N$. Show that there is a degree $\mathbf{a} \leqslant \mathbf{0}'$ such that: (i) $\mathbf{a}' = \mathbf{c}$; (ii) for all $i \in N$, $\mathbf{B}^{[\mathbf{i}]} \nleqslant \mathbf{a}$; (iii) for all $i \in N$, $\mathbf{E}^{[\mathbf{i}]} \leqslant \mathbf{a}$. (See Fig. 4.1.)

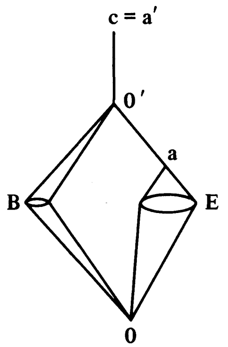

Fig. 4.1

5. *Maximal Antichains and Maximal Independent Sets Below* $\mathbf{0'}$

$\mathscr{D}[\mathbf{0},\mathbf{0'}]$ has two trivial maximal antichains, $\{\mathbf{0}\}$ and $\{\mathbf{0'}\}$. Since $\mathbf{D}[\mathbf{0},\mathbf{0'}]$ is countable, all antichains of $\mathscr{D}[\mathbf{0},\mathbf{0'}]$ are countable. We will show in this section that all non-trivial maximal antichains of $\mathscr{D}[\mathbf{0},\mathbf{0'}]$ are infinite. In contrast to this, we will show that $\mathscr{D}[\mathbf{0},\mathbf{0'}]$ has finite non-trivial maximal independent sets.

We present two proofs that every non-trivial maximal antichain of $\mathscr{D}[\mathbf{0},\mathbf{0'}]$ is infinite. The first proof uses the methods introduced in Sect. 4, but is non-uniform in nature. The second proof is uniform, and relies on two new ideas, one of which is used in the construction of a finite non-trivial maximal independent set of $\mathscr{D}[\mathbf{0},\mathbf{0'}]$.

5.1 Theorem. *Let* $\mathbf{a}_0,\ldots,\mathbf{a}_{n-1}\in\mathbf{D}(\mathbf{0},\mathbf{0'})$ *be given. Then there is a degree* $\mathbf{b}\in\mathbf{D}(\mathbf{0},\mathbf{0'})$ *such that for all* $i\leqslant n-1$, $\mathbf{b}\mid\mathbf{a}_i$.

Proof. Let $\mathbf{a}_0,\ldots,\mathbf{a}_{n-1}\in\mathbf{D}(\mathbf{0},\mathbf{0'})$ be given. For each $i\leqslant n-1$, fix $A_i\subseteq N$ such that $A_i\in\mathbf{a}_i$. We construct functions $f_0,\ldots,f_n\colon N\to[0,2n+1]$ such that:

(1) $\quad \forall i\leqslant n(\mathbf{f}_i\leqslant\mathbf{0'})$.

(2) $\quad \forall i,j\leqslant n(\mathbf{0'}\leqslant\mathbf{f}_i\cup\mathbf{f}_j)$.

(3) $\quad \forall i\leqslant n-1\,\forall j\leqslant n(\mathbf{a}_i\nleqslant\mathbf{f}_j)$.

Suppose that (1)–(3) hold. There cannot be $i,j\leqslant n$ and $k\leqslant n-1$ such that both $\mathbf{f}_i\leqslant\mathbf{a}_k$ and $\mathbf{f}_j\leqslant\mathbf{a}_k$ else $\mathbf{f}_i\cup\mathbf{f}_j\leqslant\mathbf{a}_k$, so by (2), $\mathbf{0'}\leqslant\mathbf{a}_k$ contradicting the choice of $\mathbf{a}_k\in\mathbf{D}(\mathbf{0},\mathbf{0'})$. Hence there must be at least one $i\leqslant n$ such that for all $k\leqslant n-1$, $\mathbf{f}_i\nleqslant\mathbf{a}_k$. By (1) and (3), we can then set $\mathbf{b}=\mathbf{f}_i$ to prove the theorem.

For each $e\in N$, $j\leqslant n$ and $i\leqslant n-1$, establish the requirement

$$R_{e,i,j}\colon \Phi_e^{f_j}\neq A_i.$$

Recursively order these requirements as $\{P_s : s \in N\}$. Let $s \in N$ and $\sigma \in \mathcal{S}_{2n+2}$ be given. Let $x = \mathrm{lh}(\sigma)$ and let $P_s = R_{e,i,j}$. If there are $\tau, \rho \in \mathcal{S}_{2n+2}$ such that $\Phi_e^{\sigma * \tau}(x)\downarrow \neq \Phi_e^{\sigma * \rho}(x)\downarrow$, fix the least such $\langle \tau, \rho \rangle$ (under some fixed recursive one-one correspondence of N with $\mathcal{S}^2_{2n+2}$) and let $g(\sigma, s)$ be the first $\eta \in \{\tau, \rho\}$ such that $\Phi_e^{\sigma * \eta}(x)\downarrow \neq A_i(x)$. Otherwise, let $g(\sigma, s) = \emptyset$; in this case, it follows that for all $f: N \to [0, 2n + 1]$ such that $f \supset \sigma$ if Φ_e^f is total, then Φ_e^f is recursive, hence $\neq A_i$. Note that g has degree $\leqslant \mathbf{0}'$. It follows from the Enumeration Theorem that

$$(4) \qquad \forall h \supset \sigma * g(\sigma, s)(\Phi_e^h \neq A_i).$$

For each $i \leqslant n$, we will define a sequence of strings $\{\alpha_s^i : s \in N\}$ such that $\bigcup\{\alpha_s^i : s \in N\} = f_i$. We begin by setting $\alpha_0^i = \emptyset$. Given $s \in N$, let $P_s = R_{e,i,j}$; define $\alpha_{s+1}^i = \alpha_s^i * g(\alpha_s^j) * k_s^i$ where

$$k_s^i = \begin{cases} 2i & \text{if } \ s \in \emptyset' \\ 2i + 1 & \text{if } \ s \notin \emptyset'. \end{cases}$$

By the proof of the Bounding Principle for Forcing and Coding, $\mathbf{g} \leqslant \mathbf{0}'$, so (1) holds. By (4), we see that (3) holds. We now verify (2). To decide whether $s \in \emptyset'$ from f_i and f_j oracles, we find the sth integer x in order of magnitude such that $f_i(x) \neq f_j(x)$. Then $s \in \emptyset' \leftrightarrow f_i(x) = 2i$. ∎

The following corollary is now immediate.

5.2 Corollary. *Every non-trivial maximal antichain of $\mathcal{D}[\mathbf{0}, \mathbf{0}']$ is infinite.*

The nonuniformity of the proof of Theorem 5.1 is due to the fact that we cannot identify a $j \leqslant n$ such that $\mathbf{f_j} \mid \mathbf{a_i}$ for all $i \leqslant n - 1$. The next proof of Theorem 5.1 attacks requirements more directly. The proof utilizes a slowdown procedure together with a domination lemma.

5.3 Definition. Let $f, g: N \to N$ be given. We say that *f dominates g* if $\{x : g(x) \geqslant f(x)\}$ is finite.

5.4 Upward Domination Lemma. *Let $\mathbf{d} \in \mathbf{D}$ and $\mathbf{a} \in \mathbf{D}(\mathbf{d}, \mathbf{d}']$ be given. Then there is a function g of degree $\mathbf{a}$ which is not dominated by any function of degree $\leqslant \mathbf{d}$.*

Proof. Fix $A \subseteq N$ such that $A \in \mathbf{a}$. By the relativized version of the Limit Lemma, there is a function $h: N^2 \to N$ which is recursive in $\mathbf{d}$ such that for all $x \in N$, $A(x) = \lim_s h(s, x)$. Define $g: N \to N$ by

$$(5) \qquad g(x) = \mu s \geqslant x[\forall y \leqslant x(h(s, y) = A(y)].$$

(g is called the *computation function* for A *relative to* h.) It is easily verified that $\mathbf{d} \cup \mathbf{g} = \mathbf{a}$.

Under the assumption that the lemma is false, we obtain a contradiction. Thus choose a function f of degree $\leqslant \mathbf{d}$ which dominates g. Without loss of generality, we may assume that $f(s) \geqslant s$ and $f(s) > g(s)$ for all $s \in N$. The contradiction is obtained by showing that $A \leqslant_T f$, so $\mathbf{a} \leqslant \mathbf{d}$. Fix $x \in N$. To compute $A(x)$, find an $s > x$ such that

$$(6) \qquad \forall t \in N(f(s) \leqslant t \leqslant f(f(s)) \to h(t, x) = h(f(s), x)).$$

Such $s \in N$ must exist since $\lim_t h(t, x)$ exists, and can be found recursively from an oracle of degree **d**. By choice of s and f,

(7) $\quad f(s) \leqslant g(f(s)) < f(f(s)).$

It now follows from (5)–(7) that $A(x) = h(g(f(s)), x) = h(f(s), x)$, the latter giving a computation of $A(x)$ recursively from an oracle of degree **d**. ▯

5.5 Definition. Let $\sigma, \tau, \rho \in \mathscr{S}$ and $e \in N$ be given. We call $\langle \tau, \rho \rangle$ an *e-splitting* of σ *on* x if

(i) $\Phi_e^\tau(x)\downarrow \neq \Phi_e^\rho(x)\downarrow$.
(ii) $\sigma \subset \tau$, $\sigma \subset \rho$.

5.6 Second Proof of Theorem 5.1. Let $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ be given. For each $i \leqslant n - 1$, fix $A_i \subseteq N$ of degree $\mathbf{a}_i$. Since $\mathbf{a}_i < \mathbf{0}' \leqslant \mathbf{a}_i'$ for all $i \leqslant n - 1$, the Upward Domination Lemma allows us to pick a function f_i of degree $\mathbf{0}'$ which is not dominated by any function of degree $\leqslant \mathbf{a}_i$ for each such i. Thus the function $f: N \to N$ defined by $f(x) = \max(\{f_i(x): i \leqslant n - 1\})$ is a function of degree $\mathbf{0}'$ which is not dominated by any function of degree $\mathbf{a}_i$ for any $i \leqslant n - 1$.

We define a sequence of strings $\{\beta_s: s \in N\}$ and let $B = \cup\{\beta_s: s \in N\}$. B will have degree $\leqslant \mathbf{0}'$, and $\mathbf{b} = \mathbf{B}$ will be the desired degree. It suffices to satisfy the following requirements for each $e, i \in N$:

(8) $\quad P_{e,i}: \Phi_e^{A_i} \neq B.$

(9) $\quad Q_{e,i}: \Phi_e^B \neq A_i.$

We order the set of all such requirements recursively as $\{P_s: s \in N\}$.

In order to use the Upward Domination Lemma, we must avoid making a commitment of the form $B(x) = y$ too early. Thus we *slow down* the definition of β_s, letting $\mathrm{lh}(\beta_s) = s$. When we try to force a requirement to be satisfied, we may be prevented from doing so immediately because the extension we need to make is too long. We therefore establish *targets* for requirements, namely strings which can be used to satisfy the requirements. Priorites determine which requirement heads towards its target at a given stage of the construction.

Given $\sigma \in \mathscr{S}_2$ and $k, s \in N$, we define $g(\sigma, k, s)$, the *k-target for* σ *at stage* s as follows:

Case 1. $R_k = P_{e,i}$. Search for $\tau \subset A_i$ such that $\mathrm{lh}(\tau) \leqslant f(s)$ and $\Phi_e^\tau(s)\downarrow$ if such a τ exists. If τ does not exist or if $\mathrm{lh}(\sigma) \neq s$, then $g(\sigma, k, s) = \sigma$. If τ exists and $\mathrm{lh}(\sigma) = s$, let $g(\sigma, k, s)$ be the least $\xi \in \mathscr{S}_2$ such that $\sigma \subset \xi$, $\mathrm{lh}(\xi) = \mathrm{lh}(\sigma) + 1$, and $\Phi_e^\tau(s) \neq \xi(s)$. Note that the definition of g in this case is given using only an oracle of degree $\leqslant \mathbf{0}'$.

Case 2. $R_k = Q_{e,i}$. Ask if there is an e-splitting of σ. Note that this question can be answered by an oracle of degree $\mathbf{0}'$. If no such e-splitting exists, set $g(\sigma, k, s) = \sigma$. If there is such an e-splitting, fix the least $\langle \tau, \rho, x \rangle \in \mathscr{S}_2^2 \times N$ (under some recursive one-one correspondence of N with $\mathscr{S}_2^2 \times N$) such that $\langle \tau, \rho \rangle$ e-splits σ on x. Let $g(\sigma, k, s)$ be the first $\xi \in \{\tau, \rho\}$ such that $\Phi_e^\xi(x) \neq A_i(x)$. Again note that the definition of g requires only the use of an oracle of degree $\mathbf{0}'$.

The construction proceeds as follows: Set $\beta_0 = \emptyset$ and $i_0 = 0$.

Stage $s+1$. Fix the least $k \in N$ such that R_k is not yet satisfied and $g(\beta_s, k, s) \supset \beta_s$. If no such k exists, let $\beta_{s+1} = \beta_s * 0$ and $i_{s+1} = s+1$. Otherwise, let $i_{s+1} = k$ and let β_{s+1} be the unique $\xi \in \mathscr{S}_2$ such that $\beta_s \subset \xi \subseteq g(\beta_s, k, s)$ and $\mathrm{lh}(\xi) = \mathrm{lh}(\beta_s) + 1$. If $\beta_{s+1} = g(\beta_s, k, s)$ then R_k *becomes satisfied at stage* $s+1$.

This completes the construction. We note that if $i_{s+1} = k$ and $R_k = P_{e,i}$ then R_k becomes satisfied at stage $s+1$. If $i_{s+1} = k$ and $R_k = Q_{e,i}$, then either R_k becomes satisfied at stage $s+1$ or $g(\beta_{s+1}, k, s+1) = g(\beta_s, k, s) \supset \beta_s$ and so $i_{s+2} \leqslant i_{s+1}$. Furthermore, in the latter case, if $i_t \geqslant i_{s+1}$ for all $t \geqslant s+1$, then R_k becomes satisfied at some stage $t > s$. It follows from an induction proof and the fact that if R_k becomes satisfied at stage t then $i_s \neq k$ for all $s > t$, that $\liminf_s i_s = \infty$.

We complete the proof of the theorem by verifying that (8) and (9) hold for all $e, i \in N$. Fix $e, i \in N$.

First consider $P_{e,i} = R_k$. If $\Phi_e^{A_i}$ is not total, then (8) holds. So assume that $\Phi_e^{A_i}$ is total. For each $x \in N$, define

$$f_e^i(x) = \mu t[\exists \tau \in \mathscr{S}_2(\tau \subset A_i \,\&\, \Phi_e^{\tau}(x)\downarrow \,\&\, \mathrm{lh}(\tau) = t)].$$

Note that f_e^i is recursive in A_i, so there are infinitely many $s \in N$ such that $f(s) \geqslant f_e^i(s)$. For each such s, $g(\beta_s, k, s) \supset \beta_s$. Since $\liminf_s i_s = \infty$, there is a $t \in N$ such that $i_{t+1} > k$ and $g(\beta_t, k, t) \supset \beta_t$. But then by the construction, R_k must be satisfied at some stage $s < t$, else $i_{t+1} \leqslant k$. Hence (8) holds.

Now consider $Q_{e,i} = R_k$. If R_k is satisfied, then (9) holds. If R_k is not satisfied, then since $\liminf_s i_s = \infty$, there must be $\beta_s \subset B$ which has no e-splittings. We show that either Φ_e^B is not total or Φ_e^B is recursive. Since A_i is not recursive, $\Phi_e^B \neq A_i$. Hence (9) will hold.

To compute $\Phi_e^B(x)$, search for $\sigma \in \mathscr{S}_2$ such that $\sigma \supseteq \beta_s$ and $\Phi_e^{\sigma}(x)\downarrow$. Since Φ_e^B is total, such σ must exist and can be found recursively. Since there are no e-splittings of β_s, $\Phi_e^B(x) = \Phi_e^{\sigma}(x)$. This procedure computes Φ_e^B recursively. ▯

The next theorem will produce a maximal independent set of $\mathscr{D}(\mathbf{0}, \mathbf{0}')$ having two elements. The proof uses the Upward Domination Lemma and a forcing and coding argument. The recovery of the coding depends on the following lemma.

5.7 Lemma. *There is a recursive sequence* $\{\lambda_i : i \in N\}$ *of elements of* $\mathscr{S}_2$ *such that for all* $i, j \in N$, *if* $i \neq j$ *then* $\lambda_i \mid \lambda_j$.

Proof. The sequence $\{\lambda_i : i \in N\}$ defined by

$$\lambda_i(x) = \begin{cases} 0 & \text{if} \quad x < i, \\ 1 & \text{if} \quad x = i, \\ \uparrow & \text{if} \quad x > i \end{cases}$$

for all $x \in N$ is easily seen to have the desired properties. ▯

5.8 Join Theorem for $\mathbf{0}'$. *Let* $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ *be given. Then there is a degree* $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ *such that* $\mathbf{a}' = \mathbf{0}' = \mathbf{a} \cup \mathbf{b}$.

Proof. Let $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ be given and fix a set B of degree $\mathbf{b}$. By the Upward Domination Lemma, we can fix a function g of degree $\mathbf{b}$ which is not dominated by any recursive function. Fix a set C of degree $\mathbf{0}'$ and let $\{\lambda_i : i \in N\}$ be a sequence of strings as in Lemma 5.7. Let

$$P = \{\langle \sigma, e\rangle \in \mathcal{S}_2 \times N : \Phi_e^\sigma(e)\downarrow\}$$

and let

$$Q = \{\langle \sigma, e\rangle \in \mathcal{S}_2 \times N : \forall \tau \in \mathcal{S}_2(\tau \supseteq \sigma \rightarrow \Phi_e^\tau(e)\uparrow)\}.$$

We have previously noted that P is recursively enumerable as is the complement of Q, so both P and Q have degree $\leqslant \mathbf{0}'$. Fix a recursive enumeration $\{\langle \sigma_i, e_i\rangle : i \in N\}$ of P and let $P^s = \{\langle \sigma_i, e_i\rangle : i \leqslant s\}$ for all $s \in N$.

In order to apply the Bounding Principle for Forcing and Coding, we must define the function f. For each $\sigma \in \mathcal{S}_2$ and $e \in N$, let

$$j(\sigma, e) = \mu k[\langle \sigma * \lambda_k, e\rangle \in Q \text{ or } \exists \tau \supseteq \sigma * \lambda_k(\langle \tau, e\rangle \in P^{g(k)})].$$

$j(\sigma, e)$ must be defined since g is not dominated by any recursive function, and the function $h: N \rightarrow N$ defined by

$$h(i) = \mu m[\exists \tau \supseteq \sigma * \lambda_i(\langle \tau, e\rangle \in P^m)]$$

is recursive whenever $\langle \sigma * \lambda_k, e\rangle \notin Q$ for all $k \in N$. Let

$$f(\sigma, e) = \begin{cases} \sigma * \lambda_{j(\sigma,e)} & \text{if } \langle \sigma * \lambda_{j(\sigma,e)}, e\rangle \in Q, \\ \sigma * \tau & \text{otherwise} \end{cases}$$

where $\langle \sigma * \tau, e\rangle$ is the first element enumerated in $P^{g(j(\sigma,e))}$ such that $\tau \supseteq \lambda_{j(\sigma,e)}$.

Define $\alpha_0 = \emptyset$ and $\alpha_{s+1} = \alpha_s * f(\alpha_s, s) * C(s)$ for all $s \in N$, and $A = \cup\{\alpha_s : s \in N\}$. Since j and g have degree $\leqslant \mathbf{0}'$, f has degree $\leqslant \mathbf{0}'$. Hence by the Bounding Principle for Forcing and Coding, $\mathbf{a} = \mathbf{A} \leqslant \mathbf{0}'$. It follows easily from the definition of A that A forces its jump. Hence by Lemma 3.9, $\mathbf{a}' = \mathbf{0}'$.

We compute C from A and B oracles as follows. Assume by induction that we have found α_x, and we wish to compute $C(x)$. As $\{\lambda_k : k \in N\}$ is a recursive set of pairwise incompatible strings, we can use the A oracle to find the unique $k \in N$ such that $\alpha_x * \lambda_k \subset A$. Fix this k. We now use the B and A oracles to determine whether there is a $\sigma \subset A$ such that $\sigma \supseteq \alpha_x * \lambda_k$ and $\sigma \in P^{g(k)}$. If the answer is yes, fix the least such σ; if the answer is no, let $\sigma = \alpha_x * \lambda_k$. In either case, the A oracle now gives us the unique $r \in N$ such that $\sigma * r \subset A$; $\alpha_{x+1} = \sigma * r$ and $x \in C \leftrightarrow r = 1$. Thus $C \leqslant_T A \oplus B$, so $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$. Since $\mathbf{b} \neq \mathbf{0}'$, $\mathbf{a} \neq \mathbf{0}$. Since $\mathbf{a}' = \mathbf{0}'$, $\mathbf{a} \neq \mathbf{0}'$. ∎

5.9 Corollary. *Every* $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ *is part of a maximal independent subset of* $\mathcal{D}[\mathbf{0}, \mathbf{0}']$ *consisting of two elements.*

Proof. Given $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ choose $\mathbf{a}$ as in the Join Theorem for $\mathbf{0}'$. Then $\mathbf{a} \mid \mathbf{b}$ and for all $\mathbf{c} \leqslant \mathbf{0}'$, $\mathbf{c} \leqslant \mathbf{a} \cup \mathbf{b}$. Hence $\{\mathbf{a}, \mathbf{b}\}$ must be a maximal independent subset of $\mathcal{D}[\mathbf{0}, \mathbf{0}']$. ∎

The Upward Domination Lemma can be generalized in such a way so as to allow the construction of $\mathbf{a}$ joining each of finitely many $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ to $\mathbf{0}'$. This generalization is discussed in the exercises.

Other related theorems such as meet theorems and complementation theorems have also been proved. We list some of these results:

Meet Theorem (Shoenfield [1966]). For all $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ there exists $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$.

Complementation Theorem (Posner and Robinson [1981], Posner [1981]). For all $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ there exists $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$.

A proof of the Meet Theorem combines Theorem 5.1 and the existence of a minimal degree below $\mathbf{0}'$ which is proved in Chap. IX. The proof of the Complementation Theorem involves a split into cases depending on the location of $\mathbf{b}$ in the high/low hierarchy. This hierarchy is introduced in the next chapter.

5.9 Remarks. Theorem 5.1 was proved by Shoenfield [1959]. The slowdown technique was introduced by Shoenfield [1966]. The Upward Domination Lemma was proved by Miller and Martin [1968]. The Join Theorem was proved by Posner and Robinson [1981].

5.10–5.22 Exercises

5.10 Let $\mathbf{a} \leqslant \mathbf{0}'$ and $A \subseteq N$ be given such that $A \in \mathbf{a}$. Let $h: N^2 \to N$ be given such that $A = \lim_s h$, and let f be the computation function for A relative to h.

(i) Show that $\mathbf{f} = \mathbf{a}$.

(ii) Show that f is not dominated by any function of degree $\mathbf{b} \leqslant \mathbf{a}$.

Let $\mathbf{C} \subseteq \mathbf{D}$ be given. We say that $\mathbf{C}$ is *uniformly of degree* $\leqslant \mathbf{b}$ if there is a set $B \subseteq N^2$ of degree $\leqslant \mathbf{b}$ such that $\mathbf{C} = \cup\{\mathbf{B}^{[i]}: i \in N\}$.

5.11 Let $\mathbf{C} \subseteq \mathbf{D}[\mathbf{0}, \mathbf{0}']$ be given such that $\mathbf{C}$ is uniformly of degree $\leqslant \mathbf{0}'$. Show that there is a degree $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ such that $\mathbf{a} \notin \mathbf{C}$.

5.12 (Shoenfield [1959]) Show that there is a degree $\leqslant \mathbf{0}'$ which is not the degree of a recursively enumerable set. (*Hint*: Use Exercise 5.11.)

5.13 (Shoenfield [1959]) Let $\mathbf{a} \leqslant \mathbf{0}'$ and $g: N \to N$ be given such that $\mathbf{g} = \mathbf{a}$. Let $h: N^2 \to N$ be given such that $g = \lim_s h$ and h is recursive.

The *modulus function f for g relative to h* is defined by

$$f(x) = \mu s[\forall t \geqslant s \, \forall y \leqslant x(h(t, y) = g(y))].$$

(i) Show that $\mathbf{g} \leqslant \mathbf{f}$.

(ii) Show that $\mathbf{a}$ is the degree of a recursively enumerable set if and only if there is a function g^* of degree $\mathbf{a}$ with modulus function f such that $\mathbf{f} \leqslant \mathbf{g}$.

(iii) Conclude that there is a degree $\mathbf{d} \leqslant \mathbf{0}'$ such that no function g of degree $\mathbf{d}$ has a modulus function of degree $\mathbf{d}$.

5.14 Let $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ be given. Show that there is a degree $\mathbf{a} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$.

5.15 Let $\mathbf{b} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ be given such that $\mathbf{b}' = \mathbf{0}'$. Show that there is a degree $\mathbf{a} \in \mathbf{D}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$.

The next two exercises use the techniques introduced in this section to extend the Shoenfield Jump Inversion Theorem.

5.16 Let $\mathbf{b}, \mathbf{c} \in \mathbf{D}$ be given such that $\mathbf{0} < \mathbf{b} < \mathbf{0}' \leqslant \mathbf{c}$ and $\mathbf{c}$ is recursively enumerable in $\mathbf{0}'$. Show that there is a degree $\mathbf{a} \leqslant \mathbf{0}'$ such that $\mathbf{a}' = \mathbf{c}$ and $\mathbf{a} \mid \mathbf{b}$.

5.17 Let $\mathbf{d} \in \mathbf{D}$ be given such that $\mathbf{0}' \leqslant \mathbf{d}$ and $\mathbf{d}$ is recursively enumerable in $\mathbf{0}'$. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \subseteq \mathbf{D}$ be given such that:

(i) $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are uniformly of degree $\leqslant \mathbf{0}'$.
(ii) $\mathbf{C}$ is an ideal of $\mathbf{D}$.
(iii) $\forall \mathbf{b} \in \mathbf{B}\, \forall \mathbf{c}_0, \ldots, \mathbf{c}_n \in \mathbf{C}(\mathbf{b} \nleqslant \cup\{\mathbf{c}_j : j \leqslant n\})$.
(iv) For every finite $F \subseteq N$, $\mathbf{d} \geqslant (\cup\{\mathbf{c}_i : i \in F\})'$.

Show that there is a degree $\mathbf{e} \leqslant \mathbf{0}'$ such that $\mathbf{e}' = \mathbf{d}$ and:

(v) $\forall \mathbf{c} \in \mathbf{C}(\mathbf{c} \leqslant \mathbf{e})$.
(vi) $\forall \mathbf{b} \in \mathbf{B} - \{\mathbf{0}\}(\mathbf{b} \nleqslant \mathbf{e})$.
(vii) $\forall \mathbf{a} \in \mathbf{A} - \{\mathbf{0}'\}(\mathbf{e} \nleqslant \mathbf{a})$.

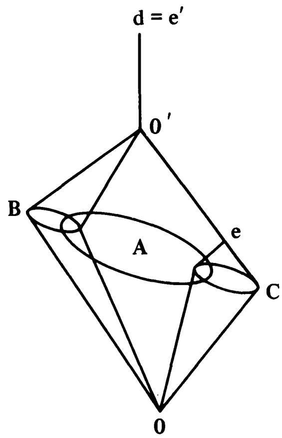

Fig. 5.1

5.18 (Yates [1967]) Show that there is a degree $\mathbf{d} \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ which is incomparable with every recursively enumerable degree except $\mathbf{0}$ and $\mathbf{0}'$.

5.19 Show that for all $n \geqslant 2$, there is a maximal independent subset of $\mathbf{D}(\mathbf{0}, \mathbf{0}')$ consisting of n degrees. (*Hint*: Start with a set of $n - 1$ independent degrees obtained from Theorem 3.10.)

5.20 (Posner and Robinson [1981]) Let $\mathbf{a}_0, \ldots, \mathbf{a}_n \leqslant \mathbf{0}'$ and $A_0, \ldots, A_n \subseteq N$ be given such that for all $i \leqslant n$, A_i has degree $\mathbf{a}_i$. For each $i \leqslant n$, let g_i be a computation function for A_i relative to some h_i, and define $g: N \to N$ by $g(x) = \min(\{g_i(x) : i \leqslant n\})$. Show that g is not dominated by any recursive function. (*Hint*: Proceed by induction on n, showing that if f is a recursive function which dominates g and f does not dominate g_n, then f dominates $\min(\{g_i : i < n\})$, thus obtaining a contradiction.)

5.21 (Posner and Robinson [1981]) Let $\mathbf{b}_0, \ldots, \mathbf{b}_n \in \mathbf{D}(\mathbf{0}, \mathbf{0}')$ be given. Show that there is a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{0}'$ and $\mathbf{a} \cup \mathbf{b_i} = \mathbf{0}'$ for all $i \leqslant n$. (*Hint*: Use Exercise 5.20.)

5.22 Let $\mathbf{A}, \mathbf{B} \subseteq \mathbf{D}$ be given such that $\mathbf{0} \notin \mathbf{B}$ and $\mathbf{A}$ and $\mathbf{B}$ are uniformly of degree $\leqslant \mathbf{0}'$ (see Fig. 4.1). Show that there is a degree $\mathbf{e}$ which satisfies:

(i) $\mathbf{e}' = \mathbf{0}'$.
(ii) $\forall \mathbf{a} \in \mathbf{A}(\mathbf{e} \cup \mathbf{a} = \mathbf{0}')$.
(iii) $\forall \mathbf{b} \in \mathbf{B} - \{\mathbf{0}\}(\mathbf{e} \not\geqslant \mathbf{b})$.

6. *Maximal Chains Below* $\mathbf{0}'$

Theorem 3.3 implies that $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ has infinite chains. For let $\{\mathbf{a_i}: i \in N\}$ be an infinite set of independent degrees in $\mathbf{D}[\mathbf{0}, \mathbf{0}']$ and for each $n \in N$, let $\mathbf{b_n} = \cup\{\mathbf{a_i}: i \leqslant n\}$. Then $\{\mathbf{b_n}: n \in N\}$ is an infinite chain of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. We will show, in this section, that all maximal chains of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ are infinite.

In order to characterize the size of maximal chains of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$, we need to show that for every $n \in N$ and every chain $\mathbf{C} = \{\mathbf{d_i}: i \leqslant n\}$ of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$, there is an element $\mathbf{d_{n+1}} \in \mathbf{D}[\mathbf{0}, \mathbf{0}'] - \mathbf{C}$ such that $\{\mathbf{d_i}: i \leqslant n + 1\}$ is a chain of degrees. We will show in Chap. XII that we cannot always hope to find such a $\mathbf{d_{n+1}}$ between $\mathbf{d_i} < \mathbf{d_j}$ unless $\mathbf{d_j} = \mathbf{0}'$. Thus our strategy will be to prove that for all $\mathbf{d} < \mathbf{0}'$, there is a $\mathbf{c} \in \mathbf{D}$ such that $\mathbf{d} < \mathbf{c} < \mathbf{0}'$. The proof of the existence of this $\mathbf{c}$ will depend on the fact that $\mathbf{0}'$ is recursively enumerable in $\mathbf{d}$. We will prove a theorem which implies that no recursively enumerable set can have minimal degree (i.e., it cannot be the case that $\mathbf{D}(\mathbf{0}, \mathbf{d}) = \emptyset$ if $\mathbf{d}$ is recursively enumerable), and relativize this result to obtain the existence of $\mathbf{c}$.

6.1 Theorem. *Let* $\mathbf{a} > \mathbf{0}$ *be a recursively enumerable degree. Then there are degrees* $\mathbf{b_0}, \mathbf{b_1} \leqslant \mathbf{a}$ *such that* $\mathbf{b_0} \mid \mathbf{b_1}$.

Proof. Let A be a recursively enumerable set of degree $\mathbf{a}$ and let $f: N \to N$ be a one-one recursive function which enumerates A. Let g be the computation function for A corresponding to f, i.e., for each $x \in N$, $g(x)$ is the least $s \in N$ such that for all $t \geqslant s$, $f(t) > x$. Then $x \in A \Leftrightarrow x \in A^{g(x)} = \{f(y): y \leqslant g(x)\}$, so $A \leqslant_T g$.

We construct sets $B_0, B_1 \subseteq N$ and let $\mathbf{b_0}$ and $\mathbf{b_1}$ be the respective degrees of B_0 and B_1. B_0 and B_1 are constructed to satisfy the following requirements for each $e \in N$ and $i \leqslant 1$:

$$P_{e,i}: \Phi_e^{B_i} \neq B_{1-i}.$$

We define a *priority ordering* of requirements, letting $P_{e,i}$ have *higher priority than* $P_{n,j}$ if either $e < n$ or both $e = n$ and $i < j$. We use elements of $\mathscr{S}_2^2$ to force the satisfaction of requirements, letting

$$\langle \sigma_0, \sigma_1 \rangle \Vdash P_{e,i} \leftrightarrow \exists x < \mathrm{lh}(\sigma_{1-i})(\Phi_e^{\sigma_i}(x)\downarrow \neq \sigma_{1-i}(x)).$$

We fix a recursive ordering $\{\langle \sigma_0^s, \sigma_1^s \rangle: s \in N\}$ of $\mathscr{S}_2^2$.

For $i \leqslant 1$, we construct the set $B_i = \cup\{\beta_i^s : s \in N\}$ as follows. We begin by setting $\beta_0^0 = \beta_1^0 = \emptyset$. At stage s, we search for $e \leqslant s$, $i \leqslant 1$ and $t \leqslant g(s)$ such that:

(1) $\forall i \leqslant 1(\sigma_i^t \supset \beta_i^s)$.

(2) $\langle \sigma_0^t, \sigma_1^t \rangle \Vdash P_{e,i}$.

(3) $\langle \beta_0^s, \beta_1^s \rangle \nVdash P_{e,i}$.

If no such σ, e and i are found, let $\beta_i^{s+1} = \beta_i^s * 0$ for $i \leqslant 1$. Otherwise, from all such e, i and t, we first fix e and i for which such a t exists for $P_{e,i}$ of highest priority, and then fix the least t for the e and i which have just been chosen. For $i \leqslant 1$, we set $\beta_i^{s+1} = \sigma_i^t$.

Since $\mathrm{lh}(\beta_i^{s+1}) > \mathrm{lh}(\beta_i^s)$ for all $s \in N$ and $i \leqslant 1$, $B_i \subseteq N$ for $i \leqslant 1$. The construction of B_0 and B_1 is recursive in $g \leqslant_T A$, so $B_0, B_1 \leqslant_T A$. We proceed by induction on the priority ordering of requirements, showing that all requirements are satisfied.

Fix $e \in N$ and $i \leqslant 1$ and assume that all requirements of higher priority than $P_{e,i}$ are satisfied. By (3) and the Enumeration Theorem, if $m \in N$ and $j \leqslant 1$ then $P_{m,j}$ can be selected to determine β_k^{s+1} at most at one stage. Hence we may fix $s \in N$ so that for all $t \geqslant s$, neither $P_{e,i}$ nor a requirement of higher priority than $P_{e,i}$ determines β_k^t. If $\langle \beta_0^s, \beta_1^s \rangle \Vdash P_{e,i}$, then $P_{e,i}$ is satisfied. Otherwise, for all $t \geqslant s$, $\langle \beta_0^t, \beta_1^t \rangle \nVdash P_{e,i}$. We obtain a contradiction under the assumption that $\Phi_e^{B_i}$ is total and equal to B_{1-i}. Under this assumption, for all $t \geqslant s$, there is a least $h(t) \in N$ such that $\langle \sigma_0^{h(t)}, \sigma_1^{h(t)} \rangle \Vdash P_{e,i}$ and $\sigma_k^{h(t)} \supset \beta_k^t$ for $k \leqslant 1$. Given $t \geqslant s$, we show how to compute β_k^{t+1}, $h(t)$ and $g(t)$ recursively from β_0^t and β_1^t for all $k \leqslant 1$. This will imply that g, and hence A, is recursive, contradicting the hypotheses of the theorem.

Given β_0^t and β_1^t, we search for the least $r \in N$ such that (1) and (2) hold with r in place of t. By the assumption that $P_{e,i}$ is not satisfied, such an r will be found, and $h(t) = r$. Since our assumption implies that (3) holds for t in place of s, it must be the case that $h(t) > g(t)$, so $g(t)$ is the least $u < h(t)$ such that $A^u \restriction t+1 = A^{h(t)} \restriction t+1$. Having determined $g(t)$, the construction can now recursively determine β_k^{t+1} for $k \leqslant 1$. ▯

Since the degrees $\mathbf{b_0}$ and $\mathbf{b_1}$ constructed in Theorem 6.1 are incomparable, they are nonrecursive and neither equals $\mathbf{a}$. We can thus conclude:

6.2 Corollary. *Let* $\mathbf{b} \neq \mathbf{0}$ *be a recursively enumerable degree. Then there is a degree* $\mathbf{c}$ such that $\mathbf{0} < \mathbf{c} < \mathbf{b}$.

Theorem 6.1 and Corollary 6.2 have straightforward relativizations. We state these relativizations, and leave the proofs to the reader.

6.3 Theorem. *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ *be given such that* $\mathbf{a} < \mathbf{b}$ *and* $\mathbf{b}$ *is recursively enumerable in* $\mathbf{a}$. *Then there is a* $\mathbf{c} \in \mathbf{D}$ *such that* $\mathbf{a} < \mathbf{c} < \mathbf{b}$.

The characterization of maximal chains of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ follows from Theorem 6.3.

6.4 Corollary. *Every maximal chain of* $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ *is infinite.*

Proof. Let $\mathbf{C} = \mathbf{a_0} < \mathbf{a_1} < \cdots < \mathbf{a_n}$ be a chain of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$. If $\mathbf{a_n} \neq \mathbf{0}'$, then $\mathbf{C}^* = \mathbf{C} \cup \{\mathbf{0}'\}$ properly extends this chain. If $\mathbf{a_n} = \mathbf{0}'$, then since $\mathbf{0}'$ is recursively enumerable, $\mathbf{0}'$ is recursively enumerable in $\mathbf{a}_{n-1}$, so by Theorem 6.3, there is a degree $\mathbf{a}_{n+1} \in \mathbf{D}[\mathbf{0}, \mathbf{0}'] - \mathbf{C}$ such that $\mathbf{a}_{n-1} < \mathbf{a}_{n+1} < \mathbf{0}'$. $\mathbf{C}^* = \mathbf{C} \cup \{\mathbf{a}_{n+1}\}$ is then a chain of $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ which properly extends $\mathbf{C}$. Hence $\mathscr{D}[\mathbf{0}, \mathbf{0}']$ has no finite maximal chains. ▯

6.5 Remarks. Corollary 6.4 has been noticed by many people. The proof given here for Theorem 6.1 uses a method introduced by Shore. Other theorems about the degrees below a recursively enumerable degree appear in Yates [1966].

6.6–6.8 Exercises

6.6 Let $\mathbf{a} > \mathbf{0}$ be a recursively enumerable degree.

(i) Prove that there is an infinite set $\{\mathbf{b}_i : i \in N\}$ of independent degrees such that $\mathbf{b}_i < \mathbf{a}$ for all $i \in N$.

(ii) Prove that every finite poset can be embedded into $\mathscr{D}[\mathbf{0}, \mathbf{a}]$.

(iii) Prove that $\mathrm{Th}(\mathscr{D}[\mathbf{0}, \mathbf{a}]) \cap \exists_1$ is decidable.

6.7 Let $\mathbf{a} > \mathbf{0}$ be a recursively enumerable degree.

(i) Simultaneously construct an ideal $\mathbf{I}$ of $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ with no greatest element and an exact pair $\langle \mathbf{b}, \mathbf{c} \rangle$ for $\mathbf{I}$ such that $\mathbf{b}, \mathbf{c} \in \mathbf{D}[\mathbf{0}, \mathbf{a}]$.

(ii) Prove that $\mathscr{D}[\mathbf{0}, \mathbf{a}]$ is not a lattice.

6.8 Let $\mathbf{a} > \mathbf{0}$ be a recursively enumerable degree. Show that there is a set B of degree $\mathbf{b} < \mathbf{a}$ such that B forces its jump. Conclude that $\mathbf{b}' = \mathbf{0}'$.

7. Classes of Degrees Determined by the Jump Operation

Hierarchies of classes of degrees determined by the jump operation will be discussed in Chap. IV. This section is devoted to proving results which imply that this hierarchy is nondegenerate. We prove an effective version of Theorem 4.6 in which the set constructed is recursively enumerable.

The construction of a recursively enumerable set A cannot make use of a nonrecursive oracle. Hence, in general, we cannot construct $A = \bigcup\{\alpha_s \in \mathscr{S}_2 : s \in N\}$. Rather, we must enumerate elements of N into A during the construction. We thus construct a recursive sequence $\{\alpha_s : s \in N\}$ of strings such that if $i \leqslant j$ then $\alpha_i^{-1}(1) \subseteq \alpha_j^{-1}(1)$ and $A = \bigcup\{\alpha_s^{-1}(1) : s \in N\}$, where $\sigma^{-1}(y) = \{x : \sigma(x) = y\}$. We will use the following notation.

7.1 Notation. Let $\sigma, \tau \in \mathscr{S}_2$ be given. We say that $\sigma \preccurlyeq \tau$ if $\mathrm{lh}(\sigma) \leqslant \mathrm{lh}(\tau)$ and $\sigma^{-1}(1) \subseteq \tau^{-1}(1)$.

7.2 Remark. Let $\{\alpha_s \in \mathscr{S}_2 : s \in N\}$ be a recursive sequence of strings such that for all $i, j \in N$, if $i < j$ then $\alpha_i \prec \alpha_j$. Then $\lim_s \alpha_s$ is a recursively enumerable set.

Proof. $x \in \lim_s \alpha_s \leftrightarrow \exists i \in N(\alpha_i(x) = 1)$. Apply Theorem 1.9. ▯

A sequence of strings, $\{\alpha_s : s \in N\}$ will be defined in an attempt to satisfy certain requirements. For each $e \in N$, we may try to satisfy the following requirements:

$P_e\colon A \neq \Phi_e.$

$Q_e\colon e \in W_n^A$ (i.e., $\Phi_n^A(e)\downarrow$).

Such requirements have been encountered before. However, in this setting, once α_s forces the satisfaction of a requirement, we may enumerate some $x < \mathrm{lh}(\alpha_s)$ into A via α_t, and so *injure* the satisfaction of the requirement. Each requirement may specify *restraints* in order to try to avoid injury. However, in order to satisfy all requirements, we must occasionally violate such restraints. A priority ordering of requirements will determine when such violations may occur, and insure the satisfaction of as many requirements as possible. We thus will try to satisfy the highest priority requirement whenever possible, ignoring all restraints, and so allow injuries to lower priority requirements. Once this is done, the next requirement in the priority ordering takes over. Since it is possible for a requirement never to require any action in order to become satisfied, we cannot recursively predict the set of injuries which will occur.

The requirement P_e will be satisfied as follows. During the course of the construction, numbers x will be appointed as *followers* of P_e. Such followers may be cancelled, and in no case may P_e have more than one follower at a given stage. The purpose of the follower x is to try to witness $\Phi_e(x)\downarrow \neq A(x)$. Thus P_e imposes restraint to keep x out of A as long as $\Phi_e^{\alpha_s}(x)\uparrow$, and if and when this computation converges, decides whether or not to place x into A. If a requirement of higher priority than P_e either restrains x or causes x to be placed into A, then P_e receives a new follower. Requirements of higher priority than P_e will act only finitely often, so P_e will have a final follower which will either cause it to be satisfied or witness the fact that Φ_e is not total.

The requirement Q_e is used to make sure that W_n^A has degree $\leqslant \mathbf{0}'$. Q_e is satisfied as follows. We seek to find α_s such that $e \in W_n^{\alpha_s}$. If $t < s$ or no such s exists, the restraint $u(e, t)$ is set equal to 0. Once such an α_s is found, the requirement imposes restraint $u(e, s) = \mathrm{lh}(\alpha_s)$, trying to guarantee that $\alpha_s \restriction \mathrm{lh}(\alpha_s) = \alpha_t \restriction \mathrm{lh}(\alpha_s)$ for all $t \geqslant s$, and so insure that the requirement is satisfied. The restraint remains in effect forever unless some higher priority requirement enumerates a new $x < \mathrm{lh}(\alpha_s)$ into A, at which point the restraint begins this definition process anew.

We will also code a set K of degree $\mathbf{0}'$ into A and show that K can be recovered from $A \oplus W_e^A$. This coding will also respect the restraint imposed by higher priority requirements, and will impose restraint to preserve the location for coding, protecting that location from being used by requirements of lower priority.

Before proceeding further, we define notation which aids in the description of the interaction of the restraint function with the definition of A.

7.3 Notation. Let $\sigma, \tau \in \mathscr{S}_2$ and $r \in N$ be given. We say that $\sigma \preccurlyeq_r \tau$ if $\sigma \preccurlyeq \tau$ and $\sigma \restriction r = \tau \restriction r$.

We now note that

(1) If α_s satisfies a requirement $R \in \{P_e, Q_e\colon e \in N\}$, $m = \mathrm{lh}(\alpha_s)$, and for all $t \geqslant s$, $\alpha_t \succcurlyeq_m \alpha_s$, then R is satisfied.

The jump hierarchy which we will discuss has classes

$$\mathbf{L_n} = \{\mathbf{a} \leqslant \mathbf{0}'\colon \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\}, \qquad \mathbf{H_n} = \{\mathbf{a} \leqslant \mathbf{0}'\colon \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\},$$

and

$$\mathbf{I} = \{\mathbf{a} \leqslant \mathbf{0}' : \forall n \in N(\mathbf{0}^{(n)} < \mathbf{a}^{(n)} < \mathbf{0}^{(n+1)})\}.$$

We wish to show that all classes of the form $\mathbf{L_{n+1}} - \mathbf{L_n}$, $\mathbf{H_{n+1}} - \mathbf{H_n}$ and $\mathbf{I}$ are non-empty. The proof of this fact relies on an effective version of Theorem 4.6.

7.4 Theorem. *Let $n \in N$ be given. Then there is a non-recursive recursively enumerable set A such that $A \oplus W_n^A \equiv_T \emptyset'$.*

Proof. We construct a recursive sequence $\{\alpha_s : s \in N\}$ of binary strings such that for all $i, j \in N$, if $i < j$ then $\alpha_i \prec \alpha_j$. By Remark 7.2, $A = \lim_s \alpha_s$ will be a recursively enumerable set. A will be constructed to satisfy the requirements in $\{P_e : e \in N\}$ and $Q_e : e \in N\}$ whenever possible, and we will also code a recursively enumerable set K of degree $\mathbf{0}'$ into A. Let f be a one-one recursive function which enumerates K, and let $K^s = \{f(x) : x \leqslant s\}$.

For each $e \in N$, let S_e be the requirement which codes e into A. We establish a priority ordering $\{R_i : i \in N\}$ of all requirements mentioned above, letting R_i have *higher priority than* R_j if $i < j$. Each requirement has a restraint function $r(i, s)$ associated with it at the end of stage s, protecting it from interference by lower priority requirements.

Following ideas introduced by Rogers [1967], we will use *movable markers* to describe the construction. If $R_i = P_e$, then Λ_i will be a marker whose location $\lambda(i, s)$ at stage s is the follower of P_e which is currently designated. If $R_i = S_e$ then Λ_i will be a marker whose position $\lambda(i, s)$ will designate the current location for coding into A whether or not $e \in K$.

We say that α_s *satisfies* P_e if $\Phi_{e,s}(\lambda(i, s))\downarrow \neq \alpha_s(\lambda(i, s))\downarrow$, where $P_e = R_i$ and $\Phi_{e,s}(x) = \Phi_e^{0^s}(x)$. ($0^s$ is the string consisting of s consecutive 0s.) We say that β *satisfies* $R_i = Q_e$ if $\Phi_n^\beta(e)\downarrow$.

We say that R_i *requires attention at stage* $s + 1$ if either:

(2) $\quad R_i = P_e$ or $R_i = S_e$ and Λ_i does not have a position at the end of stage s,

or α_s does not satisfy R_i and either:

(3) $\quad R_i = P_e \;\&\; \Phi_{e,s}(\lambda(i, s))\downarrow$; or

(4) $\quad R_i = Q_e \;\&\; \exists \beta \in \mathscr{S}_2(\mathrm{lh}(\beta) = \mathrm{lh}(\alpha_s) + 1 \;\&\; \beta$ satisfies Q_e $\;\&\; \forall m \leqslant i(\alpha_s \preccurlyeq_{r(m,s)} \beta)$.

For β as in (4), we say that R_i *requires attention through* β *at stage* $s + 1$.

The construction proceeds as follows. At stage 0, we set $\alpha_0 = \emptyset$ and $r(e, 0) = 0$ for all $e \in N$. No marker is assigned a position.

Stage $s + 1$. We proceed by cases.

Case 1. $s + 1 = 2k + 1$. We code K into A at this stage. Let $f(k) = e$ and let $S_e = R_i$. Let α_{s+1} be the string of length $\max(\{\mathrm{lh}(\alpha_s) + 1, \lambda(i, s) + 1\})$ (if $\lambda(i, s)\uparrow$ then $\mathrm{lh}(\alpha_{s+1}) = \mathrm{lh}(\alpha_s) + 1$) such that

$$\alpha_{s+1}(x) = \begin{cases} \alpha_s(x) & \text{if } x < \text{lh}(\alpha_s) \,\&\, x \neq \lambda(i,s), \\ 1 & \text{if } x = \lambda(i,s), \\ 0 & \text{otherwise.} \end{cases}$$

For all $j \in N$, marker Λ_j remains where it is unless $j > i$ in which case this marker is removed from its position, and we define

$$r(j, s+1) = \begin{cases} r(j,s) & \text{if } j \leqslant i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we are coding K into A and cancelling everything done for requirements of lower priority than R_i, since such requirements may just have been injured.

Case 2. $s + 1$ is even. Let R_i be the requirement of highest priority which requires attention at stage $s + 1$. (There will always be a requirement satisfying (2).) We again proceed by cases.

Case 2a. (2) holds for R_i. Let $\alpha_{s+1} = \alpha_s * 0$ and let $r(j, s+1) = r(j,s)$ and $\lambda(j, s+1) = \lambda(j,s)$ for all $j \in N$ such that $j \neq i$. Let $\lambda(i, s+1) = \text{lh}(\alpha_{s+1})$ and $r(i, s+1) = \lambda(i, s+1) + 1$. Thus we are placing the highest priority marker which has no current location on a position which is beyond all restraints and which has not yet entered A, and are setting restraints to prevent this marker position from being placed into A by a requirement of lower priority than R_i.

Case 2b. (3) holds for R_i. Let $R_i = P_e$. We proceed as in Case 1. Thus we satisfy P_e, and cancel everything done for requirements of lower priority than R_i, since such requirements may just have been injured.

Case 2c. (4) holds for R_i. Let $R_i = Q_e$. Fix the least β (under some fixed recursive one-one correspondence of N with $\mathscr{S}_2$) such that R_i requires attention through β at stage $s + 1$. Let $\alpha_{s+1} = \beta$. For all $j \in N$, marker Λ_j remains where it is unless $j > i$ in which case Λ_j is removed from its position. Define

$$r(j, s+1) = \begin{cases} r(j,s) & \text{if } j < i, \\ \text{lh}(\alpha_{s+1}) & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we satisfy Q_e, and cancel everything which has been done for lower priority requirements.

This completes the construction. We note that if R_i has higher priority than R_j, $\lambda(i,s)\downarrow$ and $\lambda(j,s)\downarrow$, then $\lambda(i,s) < \lambda(j,s)$. We now show that each requirement is injured only finitely often.

7.5 Lemma. *For all $i \in N$ there is a least stage s such that for all $t \geqslant s$ and $j \leqslant i$, R_j does not determine the action taken by the construction at stage t.*

Proof. We proceed by induction on i. Assume that the lemma holds for all $j < i$, and let $s(i-1) = 0$ if $i = 0$, and let $s(i+1)$ be the stage produced by the lemma for $i - 1$ otherwise.

Assume first that $R_i = P_e$. Then Λ_i does not have a position at the end of stage $s(i-1)$, so by (2) and the induction hypothesis, R_i determines the action taken by the construction at stage $s(i-1) + 1$, when Λ_i is given a position not yet in A which is protected by the restraint function $r(i, s(i-1)+1)$. This position and restraint remain unchanged at all later stages, and by the construction, no requirement of lower priority than R_i can place this marker position into A. Thus R_i can determine the action taken by the construction at at most one later stage, that occurring if (3) holds.

Assume next that $R_i = Q_e$. Then R_i can only require attention at stage $t > s(i-1)$ when (4) holds, at which stage α_t satisfies R_i and restraints are imposed to protect this satisfaction from injury by requirements of lower priority than R_i. Hence R_i never again requires attention.

Finally, assume that $R_i = S_e$. As in the first case, R_i determines the action taken by the construction at stage $s(i-1)+1$ at which stage Λ_i is assigned a final position. R_i can determine the action taken by the construction at most at one more stage, that occurring if $f(t) = e$. ▯

For all $i \in N$, let $s(i)$ be the stage given by Lemma 7.5 for i. We note that by the construction, neither the marker position for Λ_i nor the restraint function for R_i changes after stage $s(i)$, so for all $i \in N$ both $r(i) = \lim_s r(i, s)$ and $\lambda(i) = \lim_s \lambda(i, s)$ exist, the latter limit being defined only if $R_i = P_e$ or S_e for some $e \in N$. By the Limit Lemma, the functions r, λ, and s all have degree $\leqslant \mathbf{0}'$. We also note that

$$(5) \qquad \forall t \geqslant s(i)(\alpha_t \succcurlyeq_{r(i)} \alpha_{s(i)}).$$

Since the construction is recursive, it follows from (5) and 7.2 that A has degree $\leqslant \mathbf{0}'$.

We next show that A is not recursive. It suffices to show that for all $e \in N$, P_e is satisfied. Fix $e \in N$ and let $P_e = R_i$. If $\alpha_{s(i)}$ satisfies P_e, then by (5), A satisfies P_e. Otherwise, it must be the case that $\Phi_e(\lambda(i))\uparrow$, else if this computation converges and outputs the value k, then if $k \neq 0$, R_i never requires attention after stage $s(i)$ and $A(\lambda(i)) = 0$; and if $k = 0$, then we set $A(\lambda(i)) = 1$ at stage $s(i)$. In either case, R_i is satisfied.

We next show that W_n^A has degree $\leqslant \mathbf{0}'$. This fact follows easily, since (5) implies that $e \in W_n^A \leftrightarrow \alpha_{s(i)}$ satisfies $Q_e = R_i$. We thus conclude that $A \oplus W_n^A \leqslant_T K$.

We complete the proof of the theorem by showing that $K \leqslant_T A \oplus W_n^A$. We proceed by induction on i, simultaneously computing $\lambda(i)$, $r(i)$, and $s(i)$, and if $R_i = S_e$, we decide at step i whether or not $e \in K$. Consider step i of the induction. We proceed by cases.

Suppose first that $R_i = P_e$. Then $\lambda(i) = \lambda(i, s(i-1)+1)$ and $r(i) = r(i, s(i-1)+1)$. $s(i) = s(i-1)+1$ unless $\lambda(i) \in A$, in which case, $s(i)$ is the stage at which $\lambda(i)$ was placed into A.

Next suppose that $R_i = Q_e$. Then $\lambda(i)$ is undefined. If $e \notin W_n^A$, then $s(i) = s(i-1)$ and $r(i) = 0$. If $e \in W_n^A$, let s be the least stage $\geqslant s(i-1)$ such that α_s satisfies Q_e. Then $r(i) = r(i, s)$, and $s(i) = s$.

Finally, suppose that $R_i = S_e$. Then $\lambda(i) = \lambda(i, s(i-1)+1)$ and $r(i) = r(i, s(i-1)+1)$. $s(i) = s(i-1)+1$ unless $\lambda(i) \in A$, in which case, we find the unique k such that $f(k) = e$. If $2k+1 \leqslant s(i-1)$, then $s(i) = s(i-1)+1$ and $e \in K \Leftrightarrow e \in K^{s(i)}$. If $2k+1 > s(i-1)$, then $s(i) = 2k+1$ and $e \in K \Leftrightarrow \lambda(i) \in A$. ▯

Theorem 7.4 relativizes to any set X. The proof of this relativization is uniform in n and X. We summarize these facts.

7.6 Theorem. *There is a recursive function f such that for all $n \in N$ and $X \subseteq N$, the following conditions hold*:

(i) $X <_T W^X_{f(n)}$.

(ii) $W^X_{f(n)} \oplus W_n^{W^X_{f(n)}} \equiv_T X'$.

Furthermore, if for all $Y \subseteq N$, $Y <_T W^Y_n$, then

(iii) $W_n^{W^X_{f(n)}} \equiv_T X'$.

Proof. The proofs of (i) and (ii) are straightforward relativizations of the corresponding proofs in Theorem 7.4. The only modification which needs to be made in the construction is to build A so that $A(2m) = X(m)$ for all $m \in N$. We leave the verification of these conditions to the reader.

Let $Y = W^X_{f(n)}$. If $Y <_T W^Y_n$, then by (ii)

$$W_n^{W^X_{f(n)}} = W^Y_n \equiv_T Y \oplus W^Y_n \equiv_T X'. \quad \square$$

Theorem 7.6 is most useful under the assumption that for all $X \subseteq N$, $X <_T W^X_n$. If we let $X = \emptyset$, then the theorem implies that there is a correspondence as in Fig. 7.1 which preserves all uniform degree invariant properties. Thus the jumps of A are related to the jumps of $\emptyset$ in exactly the same way as the jumps of $\emptyset'$ are related to the jumps of B. The next two lemmas make this latter statement precise.

$$\begin{tikzcd}
\emptyset' \arrow[d, dash] & \emptyset' \equiv_T W^B_n \arrow[d, dash] \\
W_n = A \arrow[ur] \arrow[d, dash] & W_{f(n)} = B \arrow[d, dash] \\
\emptyset \arrow[ur] & \emptyset
\end{tikzcd}$$

Fig. 7.1

7.7 Lemma. *Let $m, n \in N$ be given such that the following conditions hold for all $X \subseteq N$*:

(i) $X <_T W^X_n$.

(ii) $(W^X_n)^{(m)} \equiv_T X^{(m+1)}$.

Let f be as in Theorem 7.6. Then the following conditions hold for all $Y \subseteq N$:

(iii) $Y <_T W^Y_{f(n)}$.

(iv) $(W^Y_{f(n)})^{(m+1)} \equiv_T Y^{(m+1)}$.

(v) *If $m > 0$ and $\forall X \subseteq N((W^X_n)^{(m-1)} \not\equiv_T X^{(m)})$ then $(W^Y_{f(n)})^{(m)} \not\equiv_T Y^{(m)}$.*

Proof. (iii) Immediate from Theorem 7.6(i).

(iv) By (i) and Theorem 7.6(iii),

(6) $W^Y_{f(n)} <_T W_n^{W^Y_{f(n)}} \equiv_T Y'$.

Hence by (ii) applied to $X = W^Y_{f(n)}$ and Theorem 7.6(iii),

$$(W^Y_{f(n)})^{(m+1)} \equiv_T (W_n^{W^Y_{f(n)}})^{(m)} \equiv_T (Y')^{(m)} \equiv_T Y^{(m+1)}.$$

(v) Assume that $m > 0$ and that

(7) $\quad \forall X \subseteq N((W^X_n)^{(m-1)} \not\equiv_T X^{(m)}).$

Let $X = W^Y_{f(n)}$ in (7). Then by (6),

$$Y^{(m)} \equiv_T (Y')^{(m-1)} \equiv_T (W_n^{W^Y_{f(n)}})^{(m-1)} \not\equiv_T (W^Y_{f(n)})^{(m)}. \quad \square$$

Lemma 7.7 is used to show that if $\mathbf{H_m} - \mathbf{H_{m-1}} \neq \emptyset$ then $\mathbf{L_{m+1}} - \mathbf{L_m} \neq \emptyset$. The next lemma is used to show that if $\mathbf{L_{m+1}} - \mathbf{L_m} \neq \emptyset$ then $\mathbf{H_{m+1}} - \mathbf{H_m} \neq \emptyset$. Its proof is similar to the proof of Lemma 7.7.

7.8 Lemma. *Let $m, n \in N$ be given such that the following conditions hold for all $X \subseteq N$:*

(i) $\quad X <_T W^X_n.$

(ii) $\quad (W^X_n)^{(m+1)} \equiv_T X^{(m+1)}.$

Let f be as in Theorem 7.6. Then the following conditions hold for all $Y \subseteq N$:

(iii) $\quad Y <_T W^Y_{f(n)}.$

(iv) $\quad (W^Y_{f(n)})^{(m+1)} \equiv_T Y^{(m+2)}.$

(v) $\quad$ *If $\forall X \subseteq N((W^X_n)^{(m)} \equiv_T X^{(m)})$ then $(W^Y_{f(n)})^{(m)} \not\equiv_T Y^{(m+1)}$.*

Proof. (iii) Immediate from Theorem 7.6.

(iv) We note that (6) again follows from (i) and Theorem 7.6(iii). Hence by (ii) applied to $X = W^Y_{f(n)}$ and Theorem 7.6(iii),

$$(W^Y_{f(n)})^{(m+1)} \equiv_T (W_n^{W^Y_{f(n)}})^{(m+1)} \equiv_T (Y')^{(m+1)} \equiv_T Y^{(m+2)}.$$

(v) Assume that

(8) $\quad \forall X \subseteq N((W^X_n)^{(m)} \not\equiv_T X^{(m)}).$

Let $X = W^Y_{f(n)}$ in (8). Then by (6),

$$Y^{(m+1)} \equiv_T (Y')^{(m)} \equiv_T (W_n^{W^Y_{f(n)}})^{(m)} \not\equiv_T (W^Y_{f(n)})^{(m)}. \quad \square$$

We are now ready to prove the nondegeneracy of the hierarchy.

7.9 Theorem. *For all $m \in N$, $\mathbf{H_{m+1}} - \mathbf{H_m} \neq \emptyset$ and $\mathbf{L_{m+1}} - \mathbf{L_m} \neq \emptyset$.*

Proof. Fix f given by Theorem 7.6. We note that $\mathbf{L_0} = \{\mathbf{0}\}$ and $\mathbf{H_0} = \{\mathbf{0'}\}$. Fix $e \in N$ such that for all $X \subseteq N$, $W^X_e = X'$. Then for all $X \subseteq N$, $X <_T W^X_e$. We proceed by induction on m. At the end of step m of the induction, we assume that we have $n \in N$ such that for all $X \subseteq N$:

(9) $X <_T W_n^X$.

(10) $(W_n^X)^{(m)} \equiv_T X^{(m+1)}$.

(11) $m > 0 \to (W_n^X)^{(m-1)} \not\equiv_T X^{(m)}$.

For $m = 0$, the induction hypotheses hold for $n = e$. (9), (10) and (11) allow us to apply Lemma 7.7, and conclude that for all $Y \subseteq N$, if $k = f(n)$ then:

(12) $Y <_T W_k^Y$.

(13) $(W_k^Y)^{(m+1)} \equiv_T Y^{(m+1)}$.

(14) $(W_k^Y)^{(m)} \not\equiv_T Y^{(m)}$.

Hence $W_{f(n)}^{\emptyset} \in \mathbf{L_{n+1}} - \mathbf{L_n}$. (12), (13) and (14) allow us to apply Lemma 7.8, and conclude that for all $X \subseteq N$, if $r = f(k)$ then (9), (10) and (11) hold for $m + 1$ in place of m and r in place of n. Thus $W_{f(k)}^{\emptyset} \in \mathbf{H_{m+1}} - \mathbf{H_m}$ and the induction hypotheses are verified. ▯

In order to show that $\mathbf{I} \neq \emptyset$, we must state a different form of the relativization of Theorem 7.4. We note that the set $W_{f(n)}^X$ defined in Theorem 7.6 was obtained as $X \oplus A$ for some set A, and we let this set A be obtained as $W_{g(n)}^X$ where g is also a recursive function. We then conclude:

7.10 Theorem. *There is a recursive function g such that for all $n \in N$* and $X \subseteq N$, *the following conditions hold*:

(i) $X <_T X \oplus W_{g(n)}^X$.

(ii) $X \oplus W_{g(n)}^X \oplus W_n^{X \oplus W_{g(n)}^X} \equiv_T X'$.

7.11 Theorem. $\mathbf{I} \neq \emptyset$.

Proof. We will apply the Recursion Theorem to show that $\mathbf{I} \neq \emptyset$. For each $e \in N$ and $X \subseteq N$, let $J_e(X) = X \oplus W_e^X$. We define iterates of the operator J_e by induction; $J_e^0(X) = X$ and $J_e^{n+1}(X) = J_e(J_e^n(X))$. We note that for all $e \in N$,

$$J_e J_{g(e)}(X) = J_e(X \oplus W_{g(e)}^X) = X \oplus W_{g(e)}^X \oplus W_e^{X \oplus W_{g(e)}^X}$$

so by Theorem 7.10(ii),

(15) $\forall e \in N\ \forall X \subseteq N(J_e J_{g(e)}(X) \equiv_T X')$.

By the Recursion Theorem, there is an $e \in N$ such that for all $X \subseteq N$, $W_e^X = W_{g(e)}^X$. Hence for all $X \subseteq N$, $J_e(X) = J_{g(e)}(X)$. For this e, we use $SJ(X)$ in place of $J_e(X)$. (SJ stands for semi-jump; two consecutive applications of the operator yield the jump. It is coincidence that this operator was discovered by Shore and Jockusch.) It follows from Theorem 7.10(i) and (15) that for all $X \subseteq N$:

(16) $X <_T SJ(X)$.

(17) $SJ^2(X) \equiv_T X'$.

The following computation uses (16) and (17):

$$X^{(n)} \equiv_T SJ^{2n}(X) <_T SJ^{2n+1}(X) = SJ^{(2n)}(SJ(X)) <_T SJ^{2n+2}(X) \equiv_T X^{(n+1)}.$$

Thus in particular, $SJ(\emptyset)$ has degree in **I**. ▯

7.12 Remarks. The finite injury priority method which is used in the proof of Theorem 7.4 was discovered by Friedberg [1957a] and Muchnik [1956]. Theorem 7.4, its relativizations, and the proofs of this section are due to Jockusch and Shore [1983]. Theorem 7.9 was initially proved as a corollary of the Sacks Jump Inversion Theorem [1963a]. Theorem 7.11 was proved independently by Lachlan [1965] and Martin [1966] using the Sacks Jump Inversion Theorem. Sacks [1967] subsequently found an easier proof using the Recursion Theorem and the Sacks Jump Inversion Theorem. The proof of the Sacks Jump Inversion Theorem is more difficult than those presented in this section.

7.13 Exercise. Let $e \in N$ be given, and let B be a recursively enumerable set of degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{0}'$. Show that there is a set $A >_T B$ such that $A \oplus W_e^A \equiv_T \emptyset'$. (*Hint*: Build A forcing its jump by a proof combining forcing and coding B into A. If an element enters B causing injury to a forcing requirement, then that requirement may again require attention and impose restraint on a location assigned to code $n \in \emptyset'$. In that case, assign a new coding location for $n \in \emptyset'$ beyond the new use, but keep the old coding location available. Always code at smallest unrestrained location, and code late if a location becomes unrestrained, even if coding has already been done at a different location.)

8. More Exact Pairs

In Sect. 3, we showed that if $\mathbf{a} \in \mathbf{D}$ and $A \subseteq N^2$ is a set of degree $\mathbf{a}$, then there is an exact pair $\langle \mathbf{b}, \mathbf{c} \rangle$ for the ideal generated by $\{A^{[i]}: i \in N\}$ such that $\mathbf{b}, \mathbf{c} \leqslant \mathbf{a}'$. Equivalently, if $X \subseteq N$ is a recursive set and $\mathbf{I}$ is the ideal generated by $\{\Phi_i^A: i \in X\}$, then there is an exact pair $\mathbf{b}, \mathbf{c} \leqslant \mathbf{a}'$ for $\mathbf{I}$. In this section, we improve this result by obtaining the same conclusion under the assumption that $X \in \Sigma_4^A$. Note that this result is best possible for X. For if $\mathbf{b}, \mathbf{c} \leqslant \mathbf{a}'$ is an exact pair for $\mathbf{I}$, then there is a sentence defining $\{i: \Phi_i^A \text{ is total } \& \ \Phi_i^A \leqslant_T B \ \& \ \Phi_i^A \leqslant_T C\}$ (where $B \in \mathbf{b}$, $C \in \mathbf{c}$) which is $\Sigma_3^{B \oplus C}$. Hence any ideal below $\mathbf{a}$ having an exact pair below $\mathbf{a}'$ must be generated as above by some $X \in \Sigma_3^{A'} = \Sigma_4^A$.

For the rest of this section, fix a recursive one-one correspondence of N with N^2, letting n correspond to $\langle n_0, n_1\rangle$.

We begin by analyzing sets $X \in \Sigma_3^A$.

8.1 Lemma. *Let $A, X \subseteq N$ be given such that $X \in \Sigma_3^A$. Then there is a set $B \subseteq N^2$ such that B is recursively enumerable in A and:*

$$\text{(i)} \qquad x \in X \Rightarrow \exists n(n_0 = x \,\&\, B^{[n]} = N).$$

$$\text{(ii)} \qquad x \notin X \Rightarrow \forall n(n_0 = x \to B^{[n]} \text{ is finite}).$$

Proof. Since $X \in \Sigma_3^A$, there is a relation $R \subseteq N^4$ which is recursive in A such that

$$x \in X \Leftrightarrow \exists n \,\forall y \,\exists z(R(x, n, y, z)).$$

Place

$$m \in B_s^{[n]} \Leftrightarrow \forall y \leqslant m \,\exists z \leqslant s(R(n_0, n_1, y, z)).$$

Let $B^{[n]} = \bigcup\{B_s^{[n]}: s \in N\}$ for all $n \in N$. Clearly, B is recursively enumerable in A. (i) and (ii) now follow from the Σ_3^A definition of X and the definition of B. ▯

Lemma 8.1, together with the priorities on $\mathscr{S}_2$ which we now define, are used to obtained the desired characterization of sets in Σ_3^A.

8.2 Definition. Given $\sigma, \tau \in \mathscr{S}_2$, we say that σ *has higher priority than* τ if σ precedes τ in the *lexicographical ordering* of binary strings, i.e., either $\sigma \subset \tau$ or $\sigma \mid \tau$ and $\sigma(x) < \tau(x)$ for the least x such that $\sigma(x) \neq \tau(x)$.

Given $X \in \Sigma_3^A$, we will want to define $\Gamma: N \to \{0, 1\}$ characterizing X. We thus view $\mathscr{S}_2$ as a tree ordered by inclusion, and Γ as a path through this tree. Γ is defined uniformly from a sequence of elements of $\mathscr{S}_2$ as follows:

8.3 Definition. Let $S = \{\gamma_s: s \in N\}$ be a sequence of elements of $\mathscr{S}_2$. For each $s \in N$, define β_s to be a string β of highest priority (if such a string exists) satisfying the following conditions:

(i) $\mathrm{lh}(\beta) = s$.

(ii) $\{s: \beta \subseteq \gamma_s\}$ is infinite.

(iii) $\{s: \gamma_s$ has higher priority than $\beta\}$ is finite.

Let $\Gamma_S = \bigcup\{\beta_s: s \in N\}$.

8.4 Lemma. *Γ_S is a function from an initial segment of N into $\{0, 1\}$. Furthermore, $\mathrm{lh}(\Gamma_S) < \infty$ if and only if there is some $\gamma \subset \Gamma_S$ such that $\{s: \gamma_s = \gamma\}$ is infinite.*

Proof. Note that $\beta_0 = \emptyset$. We proceed by induction on t. Let β_t be given such that $\beta = \beta_t$ satisfies 8.3(i)–(iii). Note that

$$\text{(1)} \qquad \{s: \gamma_s \supseteq \beta_t\} = \{s: \gamma_s = \beta_t\} \cup \{s: \gamma_s \supseteq \beta_t * 0\} \cup \{s: \gamma_s \supseteq \beta_t * 1\}.$$

Since, for $i \in \{0, 1\}$, β_t has higher priority than $\beta_t * i$, it follows from 8.3(iii) and (1) that β_{t+1} cannot be defined if $\{s: \gamma_s = \beta_t\}$ is infinite, in which case β_r is undefined for all $r \geqslant t + 1$. Hence in this case $\mathrm{dom}(\Gamma) = [0, t]$.

Suppose that $\{s: \gamma_s = \beta_t\}$ is finite. Then by 8.3(ii) and (1), there is a least i such that $\{s: \gamma_s \supseteq \beta_t * i\}$ is infinite. Let $\beta_{s+1} = \beta_s * i$. It easily follows that $\beta = \beta_{t+1}$ satisfies 8.3(i) and (iii). Furthermore, since, if γ has higher priority than β_t, then γ has higher priority than $\beta_t * i$ if and only if $\gamma = \beta_t$ or $\gamma \supseteq \beta_t * j$ for some $j < i$, 8.3(iii) follows from $\beta = \beta_{t+1}$. ▯

We are now ready to characterize $X \in \Sigma_3^A$.

8.5 Proposition. *Let $A, X \subseteq N$ be given such that $X \in \Sigma_3^A$. Then there is a set of binary strings $S = \{\gamma_s: s \in N\}$ which is recursively enumerable in A and has the following properties.*

(i) $\quad \mathrm{lh}(\Gamma_S) = \infty$.

(ii) $\quad \forall \gamma \subset \Gamma_S \exists s(\gamma_s = \gamma \,\&\, \forall t > s$ *(γ_t has lower priority than γ)*$)$.

(iii) $\quad x \in X \Leftrightarrow \exists n(x = n_0 \,\&\, \Gamma_S(n) = 0)$.

Proof. We will define S so that Γ_S is the characteristic function of $\{n: B^{[n]}$ is finite$\}$, where B is obtained from A and X as in Lemma 8.1. $S = \{\gamma_s: s \in N\}$ will be defined by induction on s. At each stage s of the induction, every $\sigma \in \mathscr{S}_2$ will either be *dormant*, *active*, or *discharged*. We begin by specifying that $\emptyset$ is active and that every other $\sigma \in \mathscr{S}_2$ is dormant.

Stage s. Each active $\sigma \in \mathscr{S}_2$ receives a *check* for $n \in N$ if $\mathrm{lh}(\sigma) \leqslant s$, $\sigma(n) = 0$ and $B_{s+1}^{[n]} - B_s^{[n]} \neq \emptyset$ (here $\{B_s: s \in N\}$ is an enumeration of B which is recursive in A and must exist since B is recursively enumerable in A). An active σ is *eligible* at stage s if for all $n < \mathrm{lh}(\sigma)$ such that $\sigma(n) = 0$, σ has received a check for n which has not been cancelled.

Let γ_s be the binary string of highest priority which is eligible at stage s. (There will always be an active string τ such that $\tau(n) = 1$ for all $n < \mathrm{lh}(\tau)$, so eligible strings will exist. Also, only finitely many strings will be active at stage s, so γ_s is well-defined.) γ_s becomes discharged. All checks assigned to strings of lower priority than γ_s are cancelled. The strings of higher priority than γ_s retain their designations. Each string of lower priority than γ_s becomes dormant unless it is of the form $\tau * i$ for some discharged τ and $i \in \{0, 1\}$, in which case it becomes active.

This completes the induction step. Let $S = \{\gamma_s: s \in N\}$. Let $\gamma \subset \Gamma_S$ be given, and fix a stage $s \in N$ so that for all $t \geqslant s, \gamma_t$ does not have higher priority than γ. If for some $t \geqslant s$ $\gamma_t = \gamma$, then γ_t is discharged at stage t, and by choice of s, remains discharged for all $r \geqslant t$. In any case, $\{r: \gamma_r = \gamma\}$ is finite so by Lemma 8.4, $\mathrm{lh}(\Gamma_S) = \infty$ and (i) holds.

For all $\delta \in \mathscr{S}_2$, if $\gamma_t \supset \delta$ then δ is discharged before stage t and this discharged status of δ is not changed before stage t. Hence for $\gamma \subset \Gamma_S$, by the above paragraph there must be a last stage s such that γ_s is discharged during stage s, i.e., $\gamma_s = \gamma$. If for some $t > s$ γ_t has higher priority than γ, then since $\{u: \gamma_u \supseteq \gamma\}$ is infinite, γ would again have to be discharged at some stage $r \geqslant t$ contrary to our assumption. Hence no such t can exist, and (ii) holds.

Let $x \in N$ be given such that $x \notin X$. Fix $n \in N$ such that $n_0 = x$. By Lemma 8.1(ii), $\{s: B_{s+1}^{[n]} - B_s^{[n]} \neq \emptyset\}$ is finite. Hence only finitely many σ such that $\sigma(n) = 0$ can receive checks for n. Thus for all but finitely many discharged σ, if $\mathrm{lh}(\sigma) > n$ then $\sigma(n) = 1$, so $\Gamma_S(n) = 1$.

Finally, let $x \in N$ be given such that $x \in X$. By Lemma 8.1(i), fix $n \in N$ such that $n_0 = x$ and $B^{[n]}$ is infinite. Let $\gamma \subset \Gamma_S$ be given such that $\mathrm{lh}(\gamma) = n$. To show that $\Gamma_S(n) = 0$, it suffices to show that $\{s: \gamma_s \supseteq \gamma * 0\}$ is infinite. Suppose this not to be the case in order to obtain a contradiction. Fix s such that for all $t \geqslant s$, γ_t has lower priority that $\gamma * 1$. Then for all $t \geqslant s$ and $\delta \supseteq \gamma * 0$, the status of δ remains unchanged during stage t. Fix δ of shortest length which is not discharged at stage s such that

$$\delta(x) = \begin{cases} \gamma(x) & \text{if} \quad x < n, \\ 0 & \text{if} \quad x = n, \\ 1 & \text{if} \quad n < x < \mathrm{lh}(\delta). \end{cases}$$

Then δ must be active at all sufficiently large stages since its predecessor in the priority ordering is discharged at all sufficiently large stages. Since $\gamma \subset \Gamma_S$ and since $B^{[n]}$ is infinite, δ must accumulate checks for all n such that $\delta(n) = 0$, and none of these checks can be cancelled. Thus δ is eligible at some stage $t \geqslant s$, so $\gamma_t = \delta$ or γ_t has higher priority than δ. But this contradicts the choice of s. ▯

We are now ready to prove a strong local version of the Exact Pair Theorem.

8.6 Theorem. *Let $A, X \subseteq N$ be given such that $X \in \Sigma_4^A$ and for all $x \in X$, Φ_x^A is total. Let A have degree $\mathbf{a}$ and for all $x \in X$ let $\mathbf{a_x}$ be the degree of Φ_x^A. Let $\mathbf{I} = \{\mathbf{d} \in \mathbf{D}$: $\exists F \subseteq N(F$ is finite & $\mathbf{d} \leqslant \cup\{\mathbf{a_i}: i \in F\})\}$. Then there is an exact pair $\langle \mathbf{c_0}, \mathbf{c_1} \rangle$ for $\mathbf{I}$ such that $\mathbf{c_0}, \mathbf{c_1} \leqslant \mathbf{a}'$.*

Proof. Fix notation as in the statement of the theorem. Let $S = \{\gamma_s$: $s \in N\}$ be obtained from Proposition 8.5 for $X \in \Sigma_3^{A'} = \Sigma_4^A$. For $j \in \{0, 1\}$, we construct a set $C_j = \bigcup\{\xi_j^s: s \in N\}$. The degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ of C_0 and C_1 form the exact pair for $\mathbf{I}$. We establish the usual requirements for an exact pair construction:

$$P_i\colon\ i \in X \Rightarrow \Phi_i^A \leqslant_T C_0 \;\&\; \Phi_i^A \leqslant_T C_1.$$

$$Q_i\colon\ \text{If } \Phi_{n_0}^{C_0} = \Phi_{n_1}^{C_1} \text{ and } \Phi_{n_0}^{C_0} \text{ is total}$$
$$\text{then } \Phi_{n_0}^{C_0} \leqslant_T \oplus\{\Phi_j^A: j \in F\} \text{ for some finite } F \subseteq X.$$

In order to prove the theorem, it suffices to show that P_i and Q_i are satisfied for all $i \in N$, and that $\mathbf{c_0}, \mathbf{c_1} < \mathbf{a}'$.

For each $j \in \{0, 1\}$, we will construct $C_j \subseteq N^2$. However, we identify N^2 with N recursively, so we can treat a string $\sigma \in \mathscr{S}_2$ as if it were a finite subset of N^2. We also have a one-one recursive correspondence $\{\sigma_i$: $i \in N\}$ of $\mathscr{S}_2$ with N; so for $\sigma \in \mathscr{S}_2$, we will speak about $C_j^{[\sigma]}$ in place of $C_j^{[i]}$, the correspondence being given by $C_j^{[\sigma_i]} = C_j^{[i]}$. Finally, we treat each $C_j^{[\sigma]}$ as if it consisted of elements of N^3.

We will code Φ_i^A into C_j for $i \in X$ as follows. Since $i \in X$, we will have some $n \in N$ such that $\Gamma(n) = 0$ and $n_0 = i$. Let $\gamma = \Gamma \upharpoonright n + 1$; Φ_i^A will be coded into $C_j^{[\gamma]}$. For each $x \in N$, there will be a triple of the form $\langle z, x, \Phi_i^A(x) \rangle \in C_j^{[\gamma]}$, and for all but finitely many triples $\langle z, x, y \rangle \in C_j^{[\gamma]}$, $y = \Phi_i^A(x)$. This strategy will allow us to satisfy P_i. The coding is done at those stages s such that $\gamma \subseteq \gamma_s$, and at the xth such stage, a triple of the form $\langle z, x, \Phi_i^A(x) \rangle$ is placed into $C_j^{[\gamma]}$, where z is chosen so that no decision has previously been made about placing this triple into $C_j^{[\gamma]}$.

In order to prevent unwanted triples from entering $C_j^{[\gamma]}$ when attempts to satisfy Q_m are made, we require that such attempts *respect* strings of sufficiently high priority. Thus at all sufficiently large stages, only strings containing information about $C_j^{[\gamma]}$ which is consistent with this coding procedure are allowed as extensions. This type of constraint allows us to satisfy P_i without imposing undue hardship on the satisfaction of Q_m.

We begin by setting $\xi_j^0 = \emptyset$ for $j \in \{0,1\}$, and assigning 0 to σ for each $\sigma \in \mathscr{S}_2$. (The number x assigned to σ_i in $\mathscr{S}_2$ designates the next pair $\langle x, \Phi_i^A(x)\rangle$ to be coded into $C_j^{[\sigma_i]}$.) Stage s of the construction has two steps: The first step attempts to make progress towards satisfying P_k for certain $k < \mathrm{lh}(\gamma_s)$; and the second step attempts to satisfy Q_k for $k = \mathrm{lh}(\gamma_s)$.

Stage s, Step 1. We proceed by induction on $\{k\colon k < \mathrm{lh}(\gamma_s)\}$. For each such k, let $\tau_k = \gamma_s \restriction k = \sigma_m$. Let $\eta_j^0 = \xi_j^s$ for $j \in \{0,1\}$. At the kth step of the induction, η_j^k will have previously been defined for $j \in \{0,1\}$. Fix $j \in \{0,1\}$. If $\gamma_s(k) = 0$ and $k_0 = i$, then we try to code $\Phi_i^A(x)$ into C_j by placing a number of the form $\langle z, x, \Phi_i^A(x)\rangle$ into $C_j^{[\tau_k]}$ where x is the number currently assigned to τ_k. (There will also be implicit restraints which try to keep numbers not of this form out of $C_j^{[\tau_k]}$.) We first use the A' oracle to determine whether $\Phi_i^A(x)\downarrow$. If not, or if $\gamma_s(k) \neq 0$, set $\eta_j^{k+1} = \eta_j^k$. If so, find an extension η_j^{k+1} of η_j^k for which there is a unique $y > \mathrm{lh}(\eta_j^k)$ such that $\eta_j^{k+1}(y)\downarrow = 1$, and this unique y is of the form $\langle m, \langle z, x, \Phi_i^A(x)\rangle\rangle$ for some $z \in N$. Assign $x+1$ to τ_k in place of x. When the induction is complete, go to Step 2. (Note that we must use the A' oracle, since it is possible that $\gamma_s(k) = 0$ but $k \notin X$, so Φ_k^A may not be total. Also, we do not want to code all of Φ_k^A into C_j during one stage, since if $k \notin X$, we may defeat the theorem by coding in too much.)

Step 2. We try to satisfy Q_k for $k = \mathrm{lh}(\gamma_s)$. Let $\langle k_0, k_1\rangle = \langle n, m\rangle$. We will try to find $x \in N$ and extend our definitions of C_0 and C_1 to force $\Phi_n^{C_0}(x) \neq \Phi_m^{C_1}(x)$. This must be done without violating the restraints alluded to in Step 1. Thus the extensions β_0 of η_0^k and β_1 of η_1^k which we define must *respect* γ_s, i.e., for all $i, y \in N$ and $j \in \{0,1\}$, if $\langle i, y\rangle \in \mathrm{dom}(\beta_j) - \mathrm{dom}(\eta_j^k)$ and $\beta_j(\langle i, y\rangle) = 1$ and σ_i has higher priority than γ_s, then $\sigma_i \subseteq \gamma_s$, $\gamma_s(i) = 0$, and if $y = \langle z, x, x^*\rangle$ then $\Phi_{i_0}^A(x) = x^*$. (In other words, β_j can only code elements into $C_j^{[\sigma_i]}$ for σ_i of relatively high priority if the information coded in by such an element is consistent with $\Phi_{i_0}^A$.)

If there are extensions β_0 of η_0^k and β_1 of η_1^k and $x \in N$ such that β_0 and β_1 respect γ_s and $\Phi_n^{\beta_0}(x) \neq \Phi_m^{\beta_1}(x)$, find the least such pair $\langle \beta_0, \beta_1\rangle$ (under some fixed recursive one-one correspondence of $\mathscr{S}_2^2$ with N) and let $\xi_j^{s+1} = \beta_j$ for $j \in \{0,1\}$. Otherwise, set $\xi_j^{s+1} = \eta_j^k$ for $j \in \{0,1\}$. Note that if β_0 and β_1 exist, then they can be found recursively in A'.

This completes the construction. We now verify that P_i and Q_i are satisfied for all $i \in N$.

Given $k \in N$, fix $\gamma \subset \Gamma_S$ such that $\mathrm{lh}(\gamma) = k$. Fix $s \in N$ as in Proposition 8.5(ii). Then for all $t \geqslant s$ and $j \in \{0,1\}$, ξ_j^{t+1} is an extension of ξ_j^t which respects γ. Let $\langle k_0, k_1\rangle = \langle n, m\rangle$. If $\Phi_n^{C_0} = \Phi_m^{C_1}$ and both $\Phi_n^{C_0}$ and $\Phi_m^{C_1}$ are total, then there can be no n-splitting $\langle \beta_0, \beta_1\rangle$ of ξ_0^s in which both strings respect γ. To determine whether an n-splitting of ξ_j^s respects γ requires knowledge only of $\oplus\{\Phi_i^A\colon \gamma_s(i) = 0\}$. Hence we see as in previous proofs that $\Phi_n^{C_0} \leqslant_T E$ for some E whose degree is in $\mathbf{I}$. Hence Q_i is satisfied.

Finally, suppose that $x \in X$. Then by Proposition 8.5(iii), there is an $n \in N$ such that $n_0 = x$ and $\Gamma_S(n) = 0$. Fix such an n, and fix $\gamma \subset \Gamma_S$ such that $\mathrm{lh}(\gamma) = n$ and $s \in N$ as in Proposition 8.5(ii) for γ. Then for all $t \geqslant s$ and $j \in \{0, 1\}$, ξ_j^{t+1} is an extension of ξ_j^t which respects γ. Hence $\{\langle z, y, y^* \rangle \in C_j^{[y]} : \Phi_x^A(y) \neq y^*\}$ is finite, and for all $y \in N$ there is a $z \in N$ such that $\langle z, y, \Phi_x^A(y) \rangle \in C_j^{[y]}$. Hence for all but finitely many $y \in N$ and all $j \in \{0, 1\}$,

$$\Phi_x^{C_j}(y) = y^* \Leftrightarrow \exists z(\langle z, y, y^* \rangle \in C_j^{[y]}) \Leftrightarrow \forall w \forall z(w \neq y^* \rightarrow \langle z, y, w \rangle \notin C_j^{[y]}).$$

Thus $\Phi_x^A \in \Delta_1^{C_j}$ so $\Phi_x^A \leqslant_T C_j$, and P_x holds. ▯

Theorem 8.6 can be extended in a different direction by finding an exact pair for **I** below certain $\mathbf{h} < \mathbf{a}'$. This extension is discussed in Exercise IV.4.17.

8.7 Remark. Theorem 8.6 was proved by Shore [1981] extending a result of Nerode and Shore [1980] in which it was assumed that $X \in \Sigma_3^A$.

8.8 Exercise. Show that the type of coding of Φ_i^A into the sets being constructed which was used in proofs of previous exact pair theorems cannot be used to prove Theorem 8.6. (For if $x \in X$ if and only if for all $j \in \{0, 1\}$ there is a $k \in N$ such that $\{y : C_j^{[k]}(y) \neq \Phi_x^A(y)\}$ is finite, then $X \in \Sigma_3^A$.)