Chapter XV
The Width of a Theory

Our discussion of independence has focused on individual types. We have defined the dimension of a type p in a model M and if the type is regular or even has weight one then this dimension is always well defined. We have seen, however, that we may need to consider the dimension of more than one type in order to specify a model. In Section 1 of this chapter we define the width of a theory so that when this width is a cardinal number, it is the number of types whose individual dimensions must be specified to determine a model. Unfortunately, the situation is not always that simple.

There are three important cases. If all types have the same dimension, we call the theory unidimensional. Unidimensional theories are a natural generalization of $\aleph_1$-categorical theories. If there is a cardinal $\delta(T)$ such that we must specify the dimension of $\delta(T)$ types to determine each model, we call the theory bounded. If no such $\delta(T)$ exists, we call the theory unbounded. Section 2 begins the study of unbounded theories. We prove there lower bounds on the number of models for an unbounded theory. The bounded case is the comparatively straightforward generalization of vector space theory to allow several dimensions. In Section 3 of this chapter we derive a number of properties of the spectrum of a bounded theory.

The unbounded case is much more complicated. It is no longer possible to define a single unordered set as a basis for a model. Rather, there is a skeleton which is, roughly speaking, partially ordered by dependence. Chapter XVI is devoted to a detailed analysis of such skeletons.

We consider in Section 4 an extension of the notion of homogeneity to obtain some more detailed information about countable models of bounded theories. In particular, we solve Vaught's conjecture for ω-stable bounded theories.

1. Classifying Theories By Width

We begin this section by formally defining the width of theory. In some respects the arguments here just reformulate those in Chapter XIII.2.

1.1 Notation. Let $\mathcal{R}(A)$ denote the set of non-algebraic regular types in $S(A)$.

Note that in Definition XIII.1, $R(M, A)$ referred to a set of points; here $\mathcal{R}(A)$ is a set of types.

1.2 Definition. Let X be a set of regular types. For $X_0 \subseteq X$ and $p \in X$ we say p *depends* on X_0 if for some $q \in X_0$, $q \mapsto_K p$. $\delta(X)$ is the cardinality of a maximal independent set of types in X under this dependence relation. We will write $\delta(A)$ for $\delta(\mathcal{R}(A))$.

The treatment of this notion as a dependence relation is somewhat artificial. For most purposes, we could just describe the $\delta(X)$ defined here as the maximal number of pairwise orthogonal types in X.

We could just as easily define p to depend on X_0 if p is realized when all the members of X_0 are realized because the restriction of $\mapsto_S$ to K-strongly regular types is totally trivial. That is, whenever a pair of strongly regular types compel a third, one of them does. This observation makes the next theorem easy.

1.3 Theorem. *For any A, the cardinal $\delta(\mathcal{R}(A))$ is well defined.*

1.4 Exercise. For any set $M \in S$, $\delta(M)$ is the supremum of the cardinals κ such that there exist regular types $\{p_i \in S(M) : i < \kappa\}$ such that if $i \neq j$ then $p_i \perp p_j$.

1.5 Definition. The *width* of a superstable theory T, which we denote by $\delta(T)$, is $\sup\{\delta(M) : M \in \mathbf{S}\}$ if such a supremum exists; otherwise $\delta(T) = \infty$.

i) If $\delta(T) = 1$, T is called *unidimensional.*
ii) If $1 \leq \delta(T) < \infty$, T is called *bounded.*
iii) If $\delta(T) = \infty$, T is called *unbounded.*

We differ here from Shelah's terminology. He uses *multidimensional* to refer to what we call unbounded and *non-multidimensional* for our bounded.

Recall that throughout Part D we assume the class K admits stationary regular types. We defined the width of T in terms of the acceptable class S. Ostensibly, we could have defined a K-width of T $\delta_K(T)$ for each acceptable class K. In fact, for superstable T all such $\delta_K(T)$ are equal. To see this, choose $M \in S$. Extend M to an $M' \in K$ and a basis for $\mathcal{R}(M)$ to a set of pairwise orthogonal regular types in $S(M')$. By Theorem XIII.3.4 $\delta_K(M') \geq \delta_S(M') \geq \delta_S(M)$. Thus, $\delta_K(T) \geq \delta_S(T)$. But the converse is even easier noting that every K-strongly regular type is regular.

In the remainder of the book we concentrate on superstable theories. Our major aim is the calculation of the spectrum of each theory and we know T has 2^λ nonisomorphic models of power λ if T is not superstable so we need not investigate unstable and strictly stable theories. There are, however, important questions about the width of strictly stable theories ([Shelah 1978] V.4, [Buechler unk], [Hrushovski 1986]).

1.6 Exercise. Show that if every type has U-rank at most 1 then T is bounded.

The following theorem describes the fundamental properties of bounded theories.

1.7 Theorem. *The following are equivalent for a superstable theory* T.

i) $\delta(T) \leq 2^{|T|}$.
ii) T *is bounded.*
iii) *For every type* p, $p \not\perp \emptyset$.
iv) *For every regular type* p, $p \not\perp \emptyset$.

If T *is* ω*-stable we can add a fifth equivalent.*

v) *For every* **AT***-strongly regular type* p, $p \not\perp \emptyset$.

Proof. Clearly i) implies ii). We show that ii) implies iii) and then that iii) implies i). Suppose iii) fails. Then, there is a type $p \perp \emptyset$ based on a finite set $\overline{a}$ which is not (Exercise VI.2.4) in the algebraic closure of $\emptyset$. For any cardinal κ, let $\langle \overline{a}_i : i < \kappa \rangle$ be a sequence of independent elements realizing $stp(\overline{a}; \emptyset)$. Then if p_i is the image of p under an automorphism taking $\overline{a}$ to $\overline{a}_i$, by Theorem VI.2.22 the p_i are pairwise orthogonal and we contradict ii). Thus, ii) implies iii). To show iii) implies i), note that if there are κ pairwise orthogonal types with $\kappa > 2^{|T|}$, we can assume without loss that each is based on a finite set $\overline{a}_i$ and that all the $\overline{a}_i$ have the same length. Then by Theorem V.1.28, we can assume that the $\overline{a}_i$ are independent. Finally, since there are only $2^{|T|}$ strong types over a finite set, two of the types must be conjugate by a strong automorphism (Definition IV.3.9) mapping their base sets from one to the other. By Theorem VI.2.22 this contradicts iii) and yields the theorem. The equivalence of iii) and iv) is easy using Theorem XIII.2.15.

Now v) follows easily by decomposing an arbitrary type in an ω-stable theory as a product of **AT**-strongly regular types. (Apply Theorem XIII.3.6 (with $K =$ **AT**), Exercise XIII.2.14, and Corollary XIII.2.24.)

The following theorem shows that if T is unbounded this fact is witnessed by a particular type. The next definition picks out such a witness and provides a useful notation for discussing its conjugates.

1.8 Definition. If the regular type p is orthogonal to the empty set, we say p is an *unbounded type.* We say the unbounded type p is *based on the type* r if p is strongly based on the set A and $t(A; \emptyset) = r$.

Although for most of our applications T will be superstable and A a finite set this is not essential to the definition.

Examination of the argument for Theorem 1.7 shows that we have in fact proved.

1.9 Theorem. *If T is a bounded theory and $stp(\overline{a};\emptyset) = stp(\overline{b};\emptyset)$ then for each $p = p_{\overline{a}} \in S(\overline{a})$, $p_{\overline{a}} \not\perp p_{\overline{b}}$.*

The last theorem restates Theorem VI.2.22 in this important context.

1.10 Corollary. *If T is superstable and bounded then $\delta(T)$ is at most the number of strong types over $\emptyset$ times the number of types over a finite set. In particular, if T is ω-stable and bounded then $\delta(T) \leq \aleph_0$.*

It follows from [Cherlin, Harrington, & Lachlan 1985] that if T is $\aleph_0$-categorical, bounded, and ω-stable then $\delta(T)$ is finite.

If T is bounded the computation of $\delta(T)$ does not require the taking of suprema.

1.11 Theorem. *If T is bounded and admits K-strongly regular types then for all $M \in K$, $\delta(M) = \delta(T)$.*

Proof. If possible, choose $M \in K$ with $\delta(M) < \delta(T)$. Imbed M in N with $\delta(N) > \delta(M)$. Let X be a basis for $\mathcal{R}(M)$. Then X', the collection of nonforking extensions to N of the types in X, is an independent set of regular types over N but it is not maximal such. Let $q \in S(N)$ be regular and orthogonal to each member of X' and hence to each member of X. By Theorem 1.7 ii), $q \not\dashv \emptyset$ so $q \not\dashv M$. Therefore by Corollary XIII.3.4, $q \not\perp r$ for some regular $r \in S(M)$. Since X is a basis for $\mathcal{R}(M)$ this implies q is not orthogonal to some member of X. This contradiction yields the result.

1.12 Exercise. Let T be the theory of seven disjoint, infinite, unary predicates. Find a basis for the strongly regular types over the prime model of T.

1.13 Exercise. Let T be the theory of infinitely many disjoint, infinite, unary predicates. Find a basis for the strongly regular types over the prime model of T.

1.14 Historical Notes. The division of theories into multidimensional and non-multidimensional ones was made by Shelah in [Shelah 1978]. His definition is more complicated and applies to stable theories. The definition here is due to Lascar and agrees with Shelah's on superstable theories. The preparation of this chapter also depended heavily on the surveys by Pillay [Pillay 1983f] and Saffe [Saffe 1983].

2. Unbounded Theories

In this section we deal with unbounded or multidimensional superstable theories. At this time we will only provide lower bounds on the number of models in various classes of unbounded theories. In the next section we will determine the spectra of bounded theories. In later chapters we will return to a more detailed discussion of unbounded theories.

The simplest example of an unbounded theory is the theory of a single equivalence relation with infinitely many infinite classes. This theory is manifestly $\aleph_0$-categorical. There are $\aleph_0$ models of power $\aleph_1$. Each one is determined by the number of blocks of size $\aleph_1$ and the number of blocks of size $\aleph_0$. The first number can be any cardinal between 1 and $\aleph_1$; the second any cardinal between 0 and $\aleph_1$. By iterating this kind of argument one can show the following exercise.

2.1 Exercise. Let T be the theory of a single equivalence relation with infinitely many infinite classes. Show $I(\aleph_\alpha, \mathbf{AT}) = |\alpha + \omega|^{|\alpha+1|}$.

2.2 Exercise. Let T be the theory of an equivalence relation with infinitely many infinite classes each of which is a model of $\mathrm{Th}(Z, S)$. Show $I(\aleph_\alpha, \mathbf{AT}) = |\alpha + \omega|^{|\alpha+\omega|}$.

2.3 Exercise. Show that in these examples $I^*(\aleph_\alpha, \mathbf{AT}) = I(\aleph_\alpha, \mathbf{AT})$.

The calculation of the spectrum function given in this section specializes to a number of different cases. On first reading K should be taken to be $\mathbf{S}$ and we are just computing the number of strongly $\aleph_0$-saturated models of a countable superstable theory. To strengthen the conclusion take K to be your favorite acceptable class, e.g., the $\mathbf{SET}_{\aleph_3}$-saturated models of an uncountable theory. The argument also applies to give the number of models of an ω-stable theory but since we are proving lower bounds this reading provides no new information.

We strike here a theme of continuing importance. The number of models in a class K which have cardinality $\lambda = \aleph_\alpha$ is often computed not as function of λ but, since it depends on the number of cardinals less than λ, as a function of α. Recall that $\lambda_0(\mathbf{I})$ tells us the cardinality of the smallest $\mathbf{I}$-saturated model. Thus the calculation of the number of models in a class K depends also on β when $\lambda_0(\mathbf{I}) = \aleph_\beta$. When $K = \mathbf{AT}$, $\beta = 0$.

In the first major result of this section we show that the number of K-models with cardinality $\aleph_\alpha$ is at least as large as the number of functions from the set of cardinals between $\lambda_0(\mathbf{I})$ and $\aleph_\alpha$ to the set of cardinals less than $\aleph_\alpha$. Roughly speaking, for each dimension between $\aleph_\beta$ and $\aleph_\alpha$ we can fix the number of copies of a given type which have that dimension as any cardinal $< \aleph_\alpha$.

2.4 Theorem. *Suppose T is a countable superstable unbounded theory and $\lambda_0(\mathbf{I}) = \aleph_\beta$. Then $I(\aleph_\alpha, K) \geq |\alpha + \omega|^{|\alpha-\beta+1|}$.*

The proof will require several steps. We begin by fixing some notation. By Theorem 1.7, a theory is unbounded just if it contains a regular type which is orthogonal to $\emptyset$. Since T is superstable we can assume that this type is based on a finite set.

2.5 Notation. Let $p = p_{\overline{a}}$ be a K-strongly regular type which is strongly based on the finite sequence $\overline{a}$. Let q denote $stp(\overline{a}; \emptyset)$. For any $M \in K$, let $D_{p,q}(M) = D(M) = \{p_{\overline{b}}: \models q(\overline{b})$ and $\overline{b} \in M\}$

By Theorem 1.3, we know that $\delta(D(M))$, the maximal number of pairwise orthogonal copies of $p_{\overline{a}}$ based on M, is an invariant of M. That is, $\delta(D(M))$ does not depend on the choice of basis for $D(M)$. For the proof of Theorem 2.4 we refine this invariant somewhat.

2.6 Definition. For any $M \in K$ with $|M| = \aleph_\alpha$, any $\gamma < |\alpha - \beta + 1|$, and any basis J for $D(M)$, let $\delta_M^J(\gamma) = \delta(\gamma) = |\{p \in J : \dim(p, M) = \aleph_{\beta+\gamma}\}|$.

Ostensibly, the function δ depends on p, q, M, and J. The dependence on p and q is real. We simply keep these parameters constant throughout the discussion and suppress them for ease of reading. The next lemma shows δ does not really depend on J.

Note that δ maps the set of cardinals between $\aleph_\beta$ and $\aleph_\alpha$ into the set of cardinals less than $\aleph_\alpha$. Thus it can be viewed as a member of $|\alpha+\omega|^{|\alpha-\beta+1|}$. Recall that we do not distinguish in notation between the set of functions from one cardinal to another and the cardinality of that set of functions.

2.7 Lemma. *Let $|M| = \aleph_\alpha$ and suppose $M \in K$. For any I and J which are bases for $D(M)$, and for each $\gamma < |\alpha + \omega|$, $\delta_M^I(\gamma) = \delta_M^J(\gamma)$.*

Proof. (Fig. 1). There is a $1-1$ correspondence between I and J which assigns to each $p \in I$ the unique $q \in J$ such that $p \not\perp q$. By Theorem XIV.2.8, $\dim(p, M) = \dim(q, M) \bmod(\lambda(\mathbf{I}))$ and the result follows.

Thus we can simplify our notation by writing δ_M for δ_M^J. Now we return to the proof of Theorem 2.4.

Proof of Theorem 2.4. Let C be the set of cardinals less than or equal to $\aleph_\alpha$. Observe that $|C| = |\omega + \alpha|$. For each $f \in C^{|\alpha-\beta+1|}$ such that $f(\alpha - \beta) = \aleph_\alpha$ we will define a model M_f such that $\delta_{M_f} = f$. First, choose for each i with $\aleph_\beta \leq \aleph_i \leq \aleph_\alpha$ a set E_i of sequences realizing q such that $|E_i| = f(i)$ and $E = \bigcup_{i<\aleph_\alpha} E_i$ is an independent set. Then for each $\overline{e} \in E_i$ choose an independent set $A_{\overline{e}}$ of $\aleph_i$ realizations of $p_{\overline{e}}$. Let $A = \bigcup_{\overline{e}\in E} A_{\overline{e}}$. Let $p'_{\overline{e}}$ be the nonforking extension of $p_{\overline{e}}$ to $\overline{e} \cup A_{\overline{e}}$. Let $S = \{p'_{\overline{e}} : \overline{e} \in E\}$. Since each $p \in S$ is orthogonal to $\emptyset$, p is $(\lambda(\mathbf{I})^+, K)$-tractable by Lemma XIV.3.2. As $\{A_{\overline{e}} : \overline{e} \in E\}$ is an independent set, by Theorem XIV.3.5 there is a model N such that for each $\overline{e} \in E_i$, $\dim(p'_{\overline{e}}, N) \leq \lambda(\mathbf{I}) + \aleph_i$ and $\dim(q, N) = \aleph_\alpha$ if q is irrelevant to S. Since $\dim(p_{\overline{e}}, N) \geq \aleph_\beta \geq \lambda(\mathbf{I})$, $\dim(p_{\overline{e}}, N) = |A_{\overline{e}}| + \dim(p'_{\overline{e}}, N) = \aleph_i$. Extend $\{p_{\overline{e}} : \overline{e} \in E\}$ to a basis I for $D(N)$. Note that $\delta_N^I = f$ and the theorem follows.

The following exercises are just computations demonstrating the meaning of the theorem.

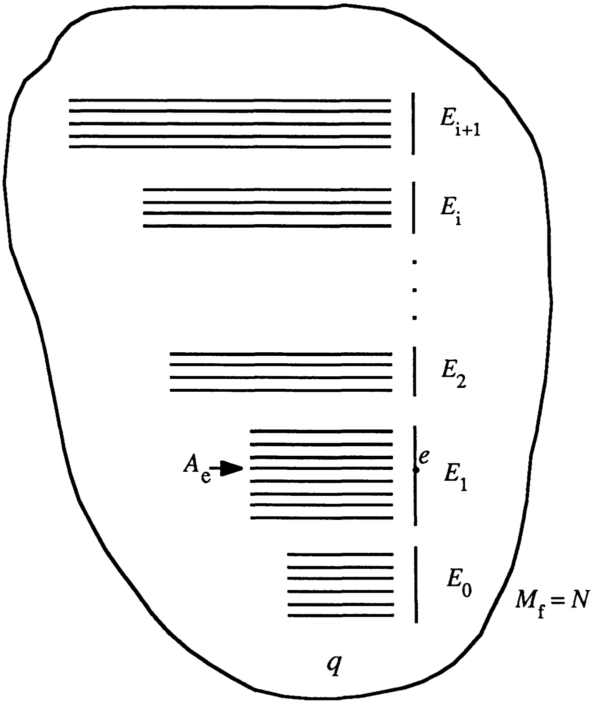

Fig. 1. Lemma XV.2.7: $|E_i| = f(i)$; $e \in E_i$ implies $A_e = \aleph_1$

2.8 Exercise. Show that if T is a countable ω-stable unbounded theory, then $I(\aleph_\alpha, \mathbf{AT}) \geq |\alpha + \omega|^{|\alpha+1|}$ if $\aleph_\alpha > \aleph_0$. In particular, T has infinitely many models in every uncountable power.

2.9 Exercise. Show that if T is a countable superstable unbounded theory, then $I(\aleph_\alpha, S) \geq |\alpha + \omega|^{|\alpha+1|}$ if $\aleph_\alpha \geq 2^{\aleph_0}$.

The next exercise outlines the means to extend the last sentence of Exercise 2.8 to the situation of Exercise 2.9.

2.10 Exercise (Saffe). Show that every countable superstable unbounded theory has infinitely many models in every uncountable power. Hint:

1) Note that T is small; otherwise it has $\geq 2^{\aleph_0}$ models in each power.
2) Thus T has prime models over countable sets of indiscernibles by Corollary V.1.26.
3) Let M be the prime model of T and choose $p \in S(M)$ and $\bar{e} \in M$ such that for some ϕ over $\bar{e}$, (p, ϕ) is **AT**-strongly regular, $p \dashv \emptyset$, and p does not fork over $\bar{e}$. Let $q = t(\bar{e}; \emptyset)$. (Note that one can not guarantee that $p|\bar{e}$ is stationary.)
4) Let $E = \{\bar{e}_i : i < \omega\}$ be a strongly independent set of realization of q. Let N be prime over E and let p_i be a nonforking extension of the conjugate of p obtained by mapping $\bar{e}$ to $\bar{e}_i$. Thus the p_i are pairwise orthogonal.

5) Construct for each $n < \omega$ a model of T with cardinality $\aleph_\gamma$ such that exactly n nonorthogonality classes of the p_i are realized only countably many times. This construction is done by iterating the following observation. If N_1 is **L**-constructible over $N \cup \overline{a}$ where $\overline{a}$ realizes p_i then for $i \neq j$, $p_j|\overline{e}_j$ is omitted in N_1. (This is a little subtle since $p_j|\overline{e}_j$ is not stationary.)

It is not entirely clear how to extend this argument to get the full force of the Exercise 2.8. Saffe suggests that one can construct elementary submodels of the distinct **S**-models constructed in the last exercise so that the **S**-models are **S**-prime over the submodels. Then the uniqueness of prime models will guarantee the submodels are distinct.

It is tempting to think that an argument similar to that for Theorem 2.4 would show that if a countable ω-stable theory has a nonisolated type which is unbounded then the lower bound in Exercise 2.8 could be improved to $I(\aleph_\alpha, \mathbf{AT}) \geq |\alpha + \omega|^{|\alpha+\omega|}$. The difficulty is that if we assign to each of a family $\langle p_i : i < \omega \rangle$ of orthogonal copies of a nonisolated type a finite dimension (by saying $p_i \in S(\overline{a}_i)$ is realized by a set X_i of independent elements and taking the prime model M over all the $\overline{a}_i$ and X_i) then we know $\dim(p_i, M)$ is finite but we can not control its exact value. This problem is the fundamental difficulty of the Vaught conjecture (for ω-stable T).

The preceding argument shows in particular that an unbounded theory T has at least $2^{|\alpha|}$ models in power $\aleph_\alpha$. We will see later that this is a very weak lower bound. We show now that with one further requirement we can force T to have 2^λ models of power λ.

We now consider one consequence of assuming that forking is trivial on the realizations of a type p. We show here, by methods similar to the proof of the last theorem, that the existence of nontrivial types of a certain kind implies there are many models. In Section XVI.2 we will develop many useful properties of trivial types and connect this notion with the dimensional order property.

2.11 Definition. The stationary type $p \in S(A)$ is *trivial* if the dependence relation of forking is trivial on $p'(\mathcal{M})$ for any nonforking extension p' of p. That is, if $\overline{a}$, $\overline{b}$, $\overline{c}$ realize p' and are pairwise independent then they are independent. If $\overline{a}, \overline{b}, \overline{c}$ are pairwise independent but not independent we say they form a *triangle.*

The remainder of this section is devoted to the proof of the following theorem.

2.12 Theorem. *Let K be an acceptable class and p an unbounded (λ, K)-tractable type. Suppose p is based on $\overline{d}$ and $\overline{b} \rhd_\phi \overline{d}$ where $\overline{b}$ realizes a nontrivial, weight one type r. Then $I(\lambda, K) = 2^\lambda$ for every $\lambda \geq \lambda_0(\mathbf{I})$.*

The following special case includes the essential ideas shorn of the extra parameters.

Let T be ω-stable and suppose $p \in S(\overline{a})$ is an unbounded, stationary, nonprincipal type. If $t(\overline{a};\emptyset)$ has weight one and is nontrivial then T has $2^{\aleph_0}$ countable models.

We may assume in proving Theorem 2.12 that $\overline{d}$ realizes a nontrivial weight one type r. For, $\overline{b} \rhd_{\emptyset} \overline{d}$ implies $\mathrm{wt}(t(\overline{d};\emptyset)) = 1$. But if $t(\overline{b};\emptyset)$ and $t(\overline{d};\emptyset)$ both have weight one, domination and Theorem XIII.2.9 transfer nontriviality of $t(\overline{b};\emptyset)$ to nontriviality of $t(\overline{d};\emptyset)$.

To make the following argument easier to read, we use single letters such as a, b, c to refer to realizations of r and p rather than the vector notation $\overline{a}, \overline{b}, \overline{c}$ we have held to so far. We write $A^{(2)}$ for the set of two element subsets of A.

Proof of Theorem 2.12. (Fig. 2). Let r_1 be the type of a triangle of realizations of r. Let G be a graph of power λ which is a disjoint union of graphs $\langle G_i : i < \lambda \rangle$ such that for each i, $|G_i| < \lambda$, every element of G_i is connected to at two least others and G_i contains no triangles (i.e. no complete subgraph on three vertices). Let the vertices of G be a set $A = \bigcup_{i<\lambda} A_i$ of independent realizations of r. Apply Lemma II.2.26 to construct a set $C = \bigcup_{i<\lambda} C_i$ where $C_i = \{c_{\{a,b\}} : \langle a, b \rangle \in G_\mu\}$ such that, writing c_{ab} for $c_{\{a,b\}} = c_{\{b,a\}}$, each triple $\langle a, b, c_{ab} \rangle$ realizes r_1 and such that $D = A \cup C$ is independent relative to the partial ordering which has only the relations $a < c_{ab}$ and $b < c_{ab}$ for $a, b \in A$. Thus, $\langle a, b, c_{ab} \rangle$ is a triangle.

For each $d \in D$, let p_d denote the copy of p over d. To simplify notation, we write p_{ab} for $p_{c_{ab}}$. For each triple $\langle a, b, c_{ab} \rangle$ we call a and b *roots* of c_{ab}.

Now let $\mathcal{S} = \{p_a : a \in A\} \cup \{p_{ab} : \langle a, b \rangle \in G\}$. Note that each p_{ab} is over some $A_i \cup C_i$ with $|A_i \cup C_i| < \lambda$. Take each of κ, λ, and μ as in Theorem XIV.3.5 and form a model M^G such that every type in $\mathcal{S}$ has dimension less than λ and all types irrelevant to $\mathcal{S}$ have dimension λ. We can apply Theorem XIV.3.5 since, if D^i denotes $\bigcup_{j<\lambda}(A_j \cup C_j) - (A_i \cup C_i)$ then by the independence relative to $<$ of D, $D^i \downarrow_{\emptyset} A_i \cup C_i$.

Since r has weight one, forking is an equivalence relation on $r(M^G)$. We will recover G from M^G by identifying the elements of G as equivalence classes mod $\not\downarrow$ of $r(M^G)$. For e realizing r, let $[e]$ denote the equivalence class of e under this relation. An equivalence class, $\mathcal{E}$, is *standard* if for some $e \in \mathcal{E}$, $\dim(p_e, M^G) < \lambda$. Note that $\mathcal{E}$ is standard just if for some $d \in D = A \cup C$, $[e] = [d]$. This follows because $p \dashv \emptyset$, so if $e \downarrow d$ for all $d \in D$, by Theorem VI.2.21 $p \dashv d$, and so p_e is irrelevant to $\mathcal{S}$.

A *triangle*, $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$, is a set of three equivalence classes such that some set of representatives (e_0, e_1, e_2) is pairwise independent but dependent. We showed in Theorem XIII.2.9 that a map $a \mapsto \hat{a}$ such that $a \not\downarrow_{\emptyset} \hat{a}$ preserves independence if its domain and range are contained in sets of realizations of weight one types. Thus, any set of representatives for a triangle of equivalence classes will be a triangle of elements. Note also that each equivalence class intersects D in at most one element. A *standard triangle* is one such that each $\mathcal{E}_i$ is standard.

The following proposition contains the main technical remarks necessary to finish the proof.

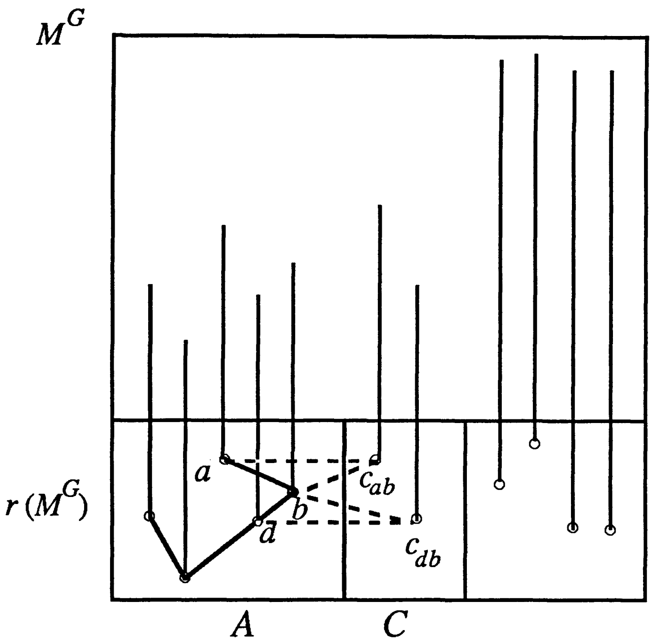

Fig. 2. Theorem XV.2.12.

2.13 Proposition. If $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ is a standard triangle then for some $a, b \in A$ and the associated $c_{ab} \in C$, $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) = ([a], [b], [c_{ab}])$.

The proof of the proposition relies on a number of observations about M^G which we record in the following exercises. The first depends only on the partial order defined on D.

2.14 Exercise. Show that if $\{a, b, d, e\} \subseteq A$ then i) if $\{a, b\} \cap \{d, e\} = \emptyset$ then $c_{ab} \downarrow_{\emptyset} c_{de}$ and ii) if $\{a, b\} \cap \{d, e\} = \{b\} = \{d\}$ then $c_{ab} \downarrow_{b} c_{de}$.

Now we can improve the second part of the last exercise using the fact that the triples $\langle a, b, c_{ab} \rangle$ were all required to be triangles.

2.15 Exercise. Show that if $\{a, b, d, e\} \subseteq A$ and $\{a, b\} \cap \{d, e\} = \{b\} = \{d\}$ then $c_{ab} \downarrow c_{de}$.

2.16 Exercise. Show that for each $c \in C$ there is a unique pair $\{a, b\}$ in $A^{(2)}$ such that $c \not\downarrow_{\emptyset} a^\frown b$.

Proof of 2.13. Since $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ is a standard triangle we know $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) = ([d_0], [d_1], [d_2])$ for some $d_0, d_1, d_2 \in D$. There are four cases.

Case 1: All the d_i are in A. Then the d_i do not form a triangle since A is independent.

Case 2: All the d_i are in C. Then the d_i do not form a triangle since by Exercises 2.14 and 2.15 C is an independent set.

Case 3: Two d_i, say d_0 and d_1, are in C, the third is in A. If d_2 is a common root of d_0 and d_1, we have (since C is independent relative to $<$) that $d_0 \downarrow_{d_2} d_1$ and since the elements of D are pairwise independent that $d_0 \downarrow_{\emptyset} d_1{}^\frown d_2$. Thus the d_i do not form a triangle.

If d_2 is not a common root of d_0, d_1 then d_2 is not in the downward closure of $\{d_0, d_1\}$ under the order $<$ so $\{d_0, d_1, d_2\}$ is an independent set and again the d_i do not form a triangle.

Case 4: Thus we have two of the d_i, say d_0, d_1 in A and the third in C. By Exercise 2.16 $(d_0, d_1, d_2) = (a, b, c_{ab})$ for some $a, b \in A$.

This finishes the proof of the proposition.

Now we can finish the proof of the theorem. Note that by Exercise 2.16 and Proposition 2.13 no class $[c]$ with $c \in C$ can be in two standard triangles. Thus, we recover $\{[a] : a \in A\}$ as the standard classes which are contained in at least two standard triangles and we recover the relation G on A by $\langle [a], [b] \rangle$ is in G just if there is a triangle containing both $[a]$ and $[b]$.

2.17 Exercise. Let T be a superstable theory. Show that if $p \in S(\overline{a})$, $p \dashv \emptyset$ and $t(\overline{a}; \emptyset)$ is a nontrivial weight one type then for every $\lambda \geq 2^{|T|}$, T has 2^λ **S**-models of power λ.

2.18 Historical Notes. Theorem 2.4 is due to Shelah (Section V.4 of [Shelah 1978]), but the proof here has been influenced by some notes of Lascar, [Bouscaren & Lascar 1983], and [Makkai 1984]. Theorem 2.12 is part of the Harrington-Makkai proof [Harrington & Makkai 1985] of Shelah's theorem that a theory with the dimensional order property has 2^λ models of power λ. We place it in this section to emphasize that the proof does not involve the dimensional order property. However, we will show in Chapter XVI that the hypothesis of Theorem 2.12 does imply the dimensional order property.

3. Bounded Theories

In this section we solve the spectrum problem for theories that are bounded but not unidimensional. We also handle some cases of unidimensional theories which fit easily into the general framework. We want to simultaneously compute the spectrum function $I(\lambda, K)$ for a number of different classes K. The uniform proof requires more restrictive assumptions about the class K than we have previously made.

The following classes will fall under this rubric: The class of all models of a countable ω-stable theory with $\lambda \geq \omega$, the class of **S**-models of a superstable theory with $\lambda \geq 2^{|T|}$, and the class of μ-saturated models of a superstable theory for $\lambda \geq \sup(\mu, 2^{|T|})$.

The next result gives a normal form for any model of a bounded theory. To simplify the discussion we fix some notation at the beginning. We will continue to accumulate special notations throughout this section as we refine this normal form to make the calculation of the spectrum function more precise.

3.1 Notation. Let $\mathcal{P}$ denote the K-prime model of the theory T and let $\mathcal{B} = \langle r_i : i < \delta(T)\rangle$ be a basis for $\mathcal{R}(\mathcal{P})$, the K-strongly regular types over $\mathcal{P}$. Recall that for $p \in S(M)$, if p is strongly based on $\bar{a} \subseteq M$, we write $p_{\bar{a}}$ for the restriction $p|\bar{a}$ of p to $\bar{a}$. In this situation we will say $\bar{a}$ *sustains* $p_{\bar{a}}$.

3.2 Theorem. *Let $N \in K$. Then N is K-prime and in fact K-minimal over $\mathcal{P} \cup X$ where X is a maximal independent subset of N such that each $x \in X$ realizes some r_i.*

Proof. Let N_1 be K-prime over $M \cup X$. If $N_1 \neq N$ then since K admits regular types there is a $q \in S(N_1)$ which is K-strongly regular and realized in $N - N_1$. By the maximality of $\mathcal{B}$, $q \not\perp r$ for some $r \in \mathcal{B}$. But then the nonforking extension r' of r to $S(N_1)$ must be realized in $N_1[q] \subseteq N$. This contradicts the maximality of X.

3.3 Exercise. Show, by considering the theory of an equivalence relation with infinitely many infinite classes, that Theorem 3.2 may fail when T is unbounded.

Let $T' = \mathrm{Th}(\mathcal{M}, \mathcal{P})$ (i.e. name the elements of $\mathcal{P}$). Clearly the spectrum function for T' dominates that for T and we can now easily calculate an upper bound on the spectrum function for T'. Let C be the set of cardinals less than or equal $\aleph_\alpha$. If $\{r_i : i < \delta(T)\}$ enumerates a basis for $\mathcal{R}(\mathcal{P})$ there is a function from $C^{\delta(T)}$ onto $K_{\leq\alpha}$ given by $s \mapsto M$ if M is K-prime over $\mathcal{P} \cup X$ where X is an independent set containing $s(i)$ independent realizations of r_i, for $i < \delta(T)$ and $s \in C^{\delta(T)}$. Two refinements are necessary to convert this upper bound into an exact computation.

First, we replace $\mathcal{P}$ with a set B on which all the types in a basis for $\mathcal{R}(\mathcal{P})$ are based. The second refinement is motivated by the following simple example. Let T be the theory of an equivalence relation with two infinite classes. The theory T', which is obtained from T by naming the equivalence classes (that is, naming elements which sustain a basis for $\mathcal{R}(\mathcal{P})$) has four models of power at most $\aleph_1$, given by assigning the two equivalence classes the cardinalities $(\aleph_0, \aleph_0)$, $(\aleph_0, \aleph_1)$, $(\aleph_1, \aleph_0)$, or $(\aleph_1, \aleph_1)$. Of course, as models of T, the second and third choices are isomorphic. These two models are 'accidentally' isomorphic; there are two orthogonal types from the basis for $\mathcal{R}(\mathcal{P})$ which happen to be assigned the same dimension. To give a general description of this phenomenon we have to be more careful in our choice of a basis for the K-strongly regular types.

3.4 Notation. We impose some further constraints on $\mathcal{B} = \langle r_i : i < \delta(T)\rangle$, a basis for $\mathcal{R}(\mathcal{P})$. If $i \neq j$ then either r_i is a conjugate of r_j or r_i is orthogonal to every conjugate of r_j.

A basis satisfying this condition can easily be chosen by induction. Suppose $\langle r_i : i < k\rangle$ are chosen. If there is a type $r \in S(M)$ which is orthogonal to the r_i for $i < k$ and which is conjugate to one of them let $r_k = r$. If not, let

r_k be any regular type over $\mathcal{P}$ orthogonal to each of those already chosen. Note that in this case r_k is in fact orthogonal to all conjugates of the r_i for $i < k$. For, if $r_k \not\perp r'_i$ for some conjugate r'_i of r_i then by Lemmas XIV.2.13 and XIV.2.10 r'_i is nonorthogonal to some conjugate r''_i of r_i which is based on $\mathcal{P}$ and orthogonal to the r_i for $i < k$.

With this choice, Theorem 1.9 implies that if r_i and r_j are conjugate the strong types of $\bar{b}_i$ and $\bar{b}_j$ are distinct. Thus, we have a set $B = \bigcup_{i<k} B_i$ where each B_i is a set of realizations of a type q_i over the empty set and each $p \in \mathcal{B}$ is based on some $\bar{b} \in B$.

Let G be the group of permutations $\hat{\alpha}$ of $\delta(T)$ such that for some α in $\mathrm{Aut}(\mathcal{M})$ there is a $B' = \alpha(B)$ such that for each $r_i \in \mathcal{B}$,

$$\alpha(r_i) \not\perp r_{\hat{\alpha}(i)}.$$

We have fixed a $1-1$ correspondence between $\mathcal{B}$ and $\delta(T)$ by our indexing so we can regard G as acting on either $\mathcal{B}$ or $\delta(T)$.

The following exercise justifies our replacing $\mathcal{P}$ by B.

3.5 Exercise. Modify the proof of Theorem 3.2 to show that if $M \in K$ and B and r_i are chosen according to 3.4 then M is a prime over B union a maximal independent set of realizations of the r_i.

With the discussion in Paragraph 3.4 in mind we make the following definition of a combinatorial structure to which we reduce the calculation of the spectrum problem.

3.6 Definition. Let μ and δ be cardinal numbers and let G be a group of permutations of δ. Then

$$(\mu^\delta/G)$$

denotes the result of partitioning the set of functions from δ into μ by the equivalence relation: $f \simeq g$ if there exists an $\alpha \in G$ with $g = f \circ \alpha$.

3.7 Exercise. Verify $\simeq$ is an equivalence relation.

Now we can calculate the spectra of $\mathbf{S}_\lambda$-saturated and $\mathbf{SET}_\lambda$-saturated models of a superstable theory. Calculation of the spectra of all models will require some further considerations. Note, however, that if T is countable and $\aleph_0$-categorical then every model of T is an $\mathbf{S}$-model and so this theorem applies.

3.8 Theorem. *Let T be a bounded superstable theory and suppose K is $\mathbf{SET}_\lambda$ or $\mathbf{S}_\lambda$ with $\lambda = \aleph_\beta \geq \lambda_0(\mathbf{I})$ (for the relevant $\mathbf{I}$). Then there is a group G of permutations of $\delta(T)$ such that for any α with $\aleph_\alpha \geq \lambda_0(\mathbf{I})$*

$$I^*(\aleph_\alpha, K) = |(|\alpha - \beta + 1|^{\delta(T)})/G|.$$

Proof. Fix B and G as in Notation 3.4. Let C be the set of cardinals κ with $\aleph_\beta \leq \kappa \leq \aleph_\alpha$. Clearly, $|C| = |\alpha - \beta + 1|$. For any $f \in C^{\delta(T)}$ define M_f to be K-prime over $B \cup X$ where $X = \bigcup_{i<\delta(T)} X_i$ and each X_i is an independent set of $f(i)$ realizations of r_i. By Exercise 3.5 each $M \in K$

must be isomorphic to some M_f. We must show $M_f \approx M_g$ if and only if $f \simeq g$. Suppose α is an isomorphism between M_f and M_g; write B' for $\alpha(B)$. Note that if $r_i \in \mathcal{B}$ is based on the empty set then α fixes r_i and $f(i) = g(i)$. Now consider an r_i based on some $\overline{a} \in B$. Then $\alpha(r_i)$ is based on $\overline{a}' = \alpha(\overline{a}) \in M_g$. Since $\mathcal{B}$ is a basis, $\alpha(r_i) \not\perp r_j$ for some $r_j \in \mathcal{B}$. For each i, let $\hat{\alpha}(i)$ be the j given by this procedure; then $\hat{\alpha} \in G$. We must show $f = g \circ \hat{\alpha}$. Since $\alpha(r_i) \not\perp r_j$, the careful selection of $\mathcal{B}$ guarantees that r_j is a conjugate of r_i and thus of $\alpha(r_i)$. Since α is an isomorphism and invoking Theorem XIV.2.15 we have

$$\dim(r_i, M_f) = \dim(\alpha(r_i), M_g) = \dim(r_j, M_g).$$

Thus, $f = g \circ \hat{\alpha}$ and $f \simeq g$. If $f \simeq g$ via a permutation γ then the automorphism of $\mathcal{M}$ which induces γ induces, via the uniqueness of K-prime models, an isomorphism of M_f and M_g.

We have reduced the calculation of the spectra of **S**-models to the calculation of $|\mu^\delta/G|$. Note that each equivalence class under $\simeq$ has at most $|G|$ elements. Thus for infinite μ if $\mu^\delta > |G|$ and especially if $\mu > |G|$, $|\mu^\delta/G| = |\mu|^\delta$. Thus G is relevant to the calculation of $I^*(\aleph_\beta, K)$ only for relatively small values of μ.

3.9 Exercise. Suppose that T is a countable superstable theory and that $\aleph_\alpha > 2^{\aleph_0} = \aleph_\beta$, then $I^*(\aleph_\alpha, S) = |\alpha - \beta + 1|^{\delta(T)}$.

The following exercise can be solved by analyzing the various possibilities for $\delta(T)$.

3.10 Exercise. Find all the possibilities for the function $I^*(\aleph_\alpha, S)$ when T is a countable superstable bounded theory.

One important fact simplifies the situation in Theorem 3.8. We have a lower bound of $\aleph_\beta$ on the dimension in a member of K of a strongly regular type over a finite set. When we deal with arbitrary models of a theory, some types may have only finite dimension in some models and we must distinguish this situation.

The following discussion is only relevant when T admits **AT**-prime models over finite sets and it will usually be applied when T is ω-stable. In fact, the analysis we now describe can be completely carried out only for a countable ω-stable theory. However, in view of the importance of these concepts for studying Vaught's conjecture for superstable theories we will make the definitions as general as possible and carry out as much of the analysis as we can for small superstable theories. The essential difficulty is that we can not guarantee that a small superstable theory admits *stationary* regular types.

We want to distinguish between those types which are '$\aleph_0$-categorical', that is, have infinite dimension in every model, and those to which we can assign finite dimensions. Roughly speaking, this is the difference between an isolated and a nonisolated type. However, noting that the only 1-type over the empty set in $\mathrm{Th}(Z, S)$ is isolated but every nonforking extension of it is not isolated, shows the situation is a little more complex. The following definition handles this difficulty.

3.11 Definition. Let $p_{\overline{a}} \in S(\overline{a})$ be **AT**-strongly regular.

i) $p_{\overline{a}}$ is *eventually nonisolated* if for some finite $\overline{b} \supset \overline{a}$, some nonforking extension $q_{\overline{b}}$ of $p_{\overline{a}}$ to $S(\overline{b})$ is not **AT**-isolated.
ii) $p_{\overline{a}}$ is *persistently isolated* if for every finite $\overline{b} \supset \overline{a}$, all nonforking extensions $q_{\overline{b}}$ of $p_{\overline{a}}$ to $S(\overline{b})$ are **AT**-isolated.

We extend this definition from types over finite sets to arbitrary types in the following way.

iii) We say an arbitrary type p is eventually nonisolated (persistently isolated) if it is a nonforking extension of an eventually nonisolated (persistently isolated) type.

These two concepts are the negations of each other and we have two separate names only to avoid dissonance. These notions are referred to in [Shelah, Harrington, & Makkai 1984] as eni and neni types. We may occasionally lapse into this usage. Contrary to [Shelah, Harrington, & Makkai 1984], we do not require that eventually nonisolated or persistently isolated types be stationary.

3.12 Exercise. If p is a nonisolated type over a finite set then every nonforking extension of p is also nonisolated.

3.13 Exercise. If $p \in S(\overline{a})$ is eventually nonisolated then every nonforking extension of p is eventually nonisolated.

3.14 Exercise. Determine the eventually nonisolated and the persistently isolated types in i) the theory of seven disjoint, infinite, unary predicates and ii) the theory of infinitely many disjoint unary predicates.

The next lemma shows the notion of eventual nonisolation is invariant under parallelism. The lemma after it shows that it is also invariant under nonorthogonality.

3.15 Lemma. *If $p \parallel q$ and p is eventually nonisolated then so is q.*

Proof. Suppose not and assume that p is eventually nonisolated but q is not. Without loss of generality we may assume that p is not isolated. Let $A = \operatorname{dom} p \cup \operatorname{dom} q$. Since p and q are parallel, there is a global type r which is a nonforking extension of both p and q. Since p is not isolated, the open mapping theorem guarantees that $r|A$ is not isolated. But since q is persistently isolated, $r|A$ is isolated. This contradiction yields the theorem.

In this case we have established that a concept defined for types which do not need to be stationary is invariant under parallelism.

3.16 Lemma. i) *If $p \in S(P)$ is persistently isolated and based on $\overline{a} \in P$ then for every $M \models T$, $\dim(p_{\overline{a}}, M) \geq \omega$.*

ii) *Suppose T is countable and ω-stable. If $p \in S(M)$ is persistently isolated and $q \not\perp p$ then q is persistently isolated.*

Proof. i) It suffices to show that $\dim(p_{\overline{a}}, \mathcal{P})$ is infinite. Clearly, $p|A$ is realized in $\mathcal{P}$ for any finite $A \subseteq \mathcal{P}$. But letting $A_0 = \overline{a}$ and choosing a_i to realize $p|A_i$ for $i < \omega$ we have the result.

ii) Suppose not. Without loss of generality we can replace p and q by parallel types over a single finite set A which are respectively isolated and not isolated. To simplify notation we call the resulting types p and q as well. Since T is ω-stable, we can assume p and q are stationary. Let M be prime over A. By i) $\dim(p, M) = \omega$ while by the omitting types theorem $\dim(q, M) = 0$. Let J be a proper subset of an infinite independent set I of realizations of p in M and choose N prime over $A \cup J$. By Corollary X.4.4, if $\overline{a} \in I - J$, $\overline{a}$ realizes $\mathrm{Av}(J, N)$, so $\overline{a} \downarrow_A N$. Thus, N is a proper elementary submodel of M. Since $p \not\perp q$, if q' is a nonforking extension of q to $S(N)$, $q' \not\perp t(\overline{a}; N)$. Thus q', and a fortiori q, is realized in $N[\overline{a}] \prec M$. This contradiction yields the theorem.

One of the major applications of these concepts will be to the proof of Vaught's conjecture for an ω-stable theory. It is natural to ask if we can extend that proof to superstable theories. For such an extension, we would like to prove the results here for small superstable theories. The assumption that T is ω-stable was used in two ways in the preceding proof: first, to guarantee that p and q were stationary and second, to find the model prime over $A \cup J$. It does not seem necessary to require p and q to be stationary. We could find a model prime over $A \cup J$ by invoking Corollary V.1.26 if T were a small superstable theory. In the next lemma we need again that a persistently isolated type is stationary. This implicit appeal to ω-stability seems harder to avoid.

The choice of $\overline{b}$ to witness that a type $p_{\overline{a}}$ is eventually nonisolated seems fairly arbitrary. In fact, it can be made in any model prime over $\overline{a}$.

3.17 Lemma. *Let T be a countable ω-stable theory and let $p_{\overline{a}}$ be a stationary eventually nonisolated type.*

i) *If $\overline{a} \subseteq M$, where M is prime over a finite set then* $\dim(p_{\overline{a}}, M) < \omega$.
ii) *There exists $\overline{b} \supseteq \overline{a}$ with $t(\overline{b}; \overline{a})$* **AT**-*isolated so that every nonforking extension of $p_{\overline{a}}$ to $S(\overline{b})$ is nonisolated.*

Proof. i) Suppose M is prime over the finite set $A' \supseteq \overline{a}$ and $\dim(p_{\overline{a}}, M) = \omega$. We will prove that for any finite $B \supseteq A'$ and any nonforking extension q of p to $S(B)$, q is isolated. Since any extension of $p_{\overline{a}}$ has a nonforking extension whose domain contains A', the open mapping theorem then yields that $p_{\overline{a}}$ is not eventually nonisolated. Assume for contradiction that q is nonisolated and let N be prime over B. Without loss of generality, $M \subseteq N$. Let I be an infinite independent set witnessing $\dim(p_{\overline{a}}, M) = \omega$. Since T is superstable there is a finite $I_0 \subset I$ such that $B \downarrow_{A \cup I_0} I - I_0$. Now, since q is stationary, any $c \in I - I_0$ realizes q. But if q is not isolated, $\dim(q, N) = 0$.

ii) Let M be **AT**-prime over $\bar{a}$. By i) $\dim(p, M) < \omega$. Let I_0 be a maximal independent subset of M realizing p. Then no nonforking extension of p to $S(\bar{a} \cup I_0)$ is realized in M. Thus, any such type is nonisolated and $I_0 \cup \bar{a}$ is the required $\bar{b}$.

3.18 Exercise. Show that Lemma 3.17 i) holds for a small superstable theory under the additional hypothesis that p has finite multiplicity.

3.19 Notation. Let T be a theory with dimension $\delta = \delta(T)$. Let δ_1 be the number of persistently isolated types in a basis $\mathcal{B}$ for $\mathcal{R}(\mathcal{P})$ and δ_2 the number of eventually nonisolated types. For simplicity, assume that $\mathcal{B} = \langle r_i : i < \delta \rangle$ is arranged with the persistently isolated types first. Let G_1 be the restriction of the group G of permutations of $\mathcal{B}$ defined in 3.4 to persistently isolated types and G_2 the restriction to eventually nonisolated types.

The next three theorems describe the spectra of a countable bounded ω-stable theory.

3.20 Theorem. *If T is a countable bounded ω-stable theory then for any uncountable $\aleph_\alpha$*

$$I^*(\aleph_\alpha, \mathbf{AT}) = ||\alpha + 1|^{\delta_1}/G_1| \times ||\alpha + \omega|^{\delta_2}/G_2|.$$

Proof. Let $\langle c_i : i < \alpha + \omega \rangle$ be an enumeration of the cardinals less than $\aleph_\alpha$. Map $|\alpha + 1|^{\delta_1} \times |\alpha + \omega|^{\delta_2}$ onto $I^*(\aleph_\alpha, \mathbf{AT})$ as follows. For each pair of functions $f \in (\alpha + 1)^{\delta_1}$ and $g \in (\alpha + \omega)^{\delta_2}$, choose $M_{f,g}$ such that $\dim(p_i, M_{f,g}) = c_{f(i)}$ for $i < \delta_1$ and $\dim(p_i, M_{f,g}) = c_{g(i)}$ for $\delta_1 \leq i < \delta$. Just as in Theorem 3.8, we can see that $M_{f,g} \approx M_{f',g'}$ if and only if there exist $\alpha_1 \in G_1$ and $\alpha_2 \in G_2$ with $f' = f \circ \alpha_1$ and $g' = g \circ \alpha_2$.

The formula in Theorem 3.20 is misleading when $\delta(T)$ is infinite. In this case we can compute $I^*(\aleph_\alpha, K)$ without recourse to G. The computations in Theorems 3.20 and 3.21 are not contradictory. The group action is still present in the following situation; it just isn't reflected in the spectrum calculation.

3.21 Theorem. *Let T be a countable bounded ω-stable theory.*

i) *If $\delta_1(T) = \aleph_0$ then $I^*(\aleph_\alpha, \mathbf{AT}) \geq |\alpha + 1|^{\aleph_0}$.*
ii) *If $\delta_2(T) = \aleph_0$ then $I^*(\aleph_\alpha, \mathbf{AT}) \geq |\alpha + \omega|^{\aleph_0}$.*

Proof. The proof for the two parts is virtually identical so we do only i). Suppose first that there exists a type q and infinitely many r_i which witness that δ_1 is infinite and are all based on realizations $\bar{b}_i$ of q. Then, the r_i can be taken over a single realization $\bar{b}$ of q. For, since T is ω-stable, q has finite multiplicity so without loss of generality all the $\bar{b}_i$ realize the same strong type. But then for each i, j Theorem XIV.2.10 guarantees that $r_i \not\perp r_{ij}$ where r_{ij} denotes the conjugate of r_i over $\bar{b}_j$. So if $r_j \not\perp r_{ij}$ we obtain the contradiction $r_i \not\perp r_j$ by the transitivity of orthogonality on regular types. In this situation we obtain the required estimate by passing to the finite inessential extension of T obtained by naming $\bar{b}$. Thus, we can assume only finitely many of the r_i are based on realizations of any fixed type over $\emptyset$. Since each type has finite multiplicity, each set B_i defined in 3.4 is finite. Thus, there are infinitely many distinct B_i. For each $i < \omega$ choose an $r_i \in \mathcal{R}(\mathcal{P})$ which is strongly based on some $\bar{b} \in B_i$. Now, not only the r_i but any of their conjugates are pairwise orthogonal. Thus we can choose a model M fixing the dimensions of r_i in M independently and obtain the theorem.

The following exercise illustrates the loss of information inherent in describing structure results by spectrum functions.

3.22 Exercise. Show that the two spectrum functions in Theorem 3.21 have the same values.

3.23 Exercise. Show that by using the techniques of 2.6 and 2.7 we could obtain this result without recourse to the careful arrangement of $\mathcal{R}(\mathcal{P})$.

In Section 2 we showed that an unbounded ω-stable theory had infinitely many models in every uncountable power. The next exercise and following theorem give two successive weakenings of the hypothesis necessary to obtain this result.

3.24 Exercise. If T is a countable ω-stable theory and $\delta(T)$ is infinite then T has infinitely many models in every uncountable cardinal.

3.25 Theorem. *Let T be a countable ω-stable theory. If T is not unidimensional and $\delta_2(T) > 0$ then $I(\mu, \mathbf{AT}) \geq \omega$ for every uncountable μ.*

Proof. By Theorem 3.21 we can assume that $\delta(T)$ is finite. Let r_0 be an eventually nonisolated type and r_1 an orthogonal strongly regular type. Let M_n be prime over μ independent realizations of r_1 and n independent realizations of r_0. Now for each $m < \omega$ there are only finitely many models M_n such that $M_n \approx M_m$. For, such an M_n must have $\dim(r_i, M_n) = m$ for some i but there are only finitely many choices for i. Thus $I(\mu, \mathbf{AT})$ is infinite.

The following tables summarize some of the consequences for the spectrum problem of the work in this section. For $\mathbf{S}$, the spectrum depends on $\delta(T)$ and the group G. For $\mathbf{AT}$, we must also consider the role of δ_1 and δ_2.

The table compiles the results of Theorems 3.8, 3.20, and 3.21 with one minor adjustment. We have switched from describing the cumulative spectrum I^* to calculating the exactly the number of models in power $\aleph_\alpha$. This requires a modification of the definition of $(|\alpha + \omega|^\delta)/G$. Instead of considering all functions from δ into the set of cardinals less than or equal α, consider those functions which have $\aleph_\alpha$ in their range. Of course, (see the remark before Exercise 3.9) the group G is really interesting only when δ and α are both finite. In this case, it is smoother to use the original definition of (μ^δ/G) and calculate $I(\aleph_{m+1}, K)$ as $I^*(\aleph_{m+1}, K) - I^*(\aleph_m, K)$.

3.26 Theorem. *Let T be a countable bounded ω-stable theory. The following tables give the number of* S*-models (models) of T with cardinality $\aleph_\alpha$ for $\alpha > 0$.*

Number of S-models

	$\delta(T) = 1$	$1 < \delta(T) < \omega$	$\delta(T) = \omega$
$I(\aleph_\alpha, \mathbf{S}_{\aleph_0})$	1	$\|\alpha+1\|^\delta/G$	$\|\alpha+1\|^\omega$

Number of models

$I(\aleph_\alpha, \mathbf{AT}_{\aleph_0})$	$\delta_1 = 0$	$\delta_1 = 1$	$1 < \delta_1 < \omega$	$\delta_1 = \omega$
$\delta_2 = 0$	impossible	1	$\|\alpha+1\|^{\delta_1}/G$	$\|\alpha+1\|^\omega$
$\delta_2 = 1$	1	$\|\alpha+\omega\|$	$\|\alpha+\omega\|$	$\|\alpha+1\|^\omega$
$1 < \delta_2 < \omega$	$\|\alpha+\omega\|$	$\|\alpha+\omega\|$	$\|\alpha+\omega\|$	$\|\alpha+1\|^\omega$
$\delta_2 = \omega$	$\|\alpha+\omega\|^\omega$	$\|\alpha+\omega\|^\omega$	$\|\alpha+\omega\|^\omega$	$\|\alpha+\omega\|^\omega$

In fact, [Cherlin, Harrington, & Lachlan 1985] shows that the case $\delta_2 = 0$ but δ_1 is infinite does not occur. We can read off from the second table and Lemma 3.28 (below) a number of the major early results on the spectrum problem.

3.27 Theorem. i) (Morley's Theorem) *If T is categorical in one uncountable power then T is categorical in every uncountable power.*

ii) (Baldwin-Lachlan Theorem) *If T is categorical in some uncountable power then T has either* 1 *or* $\aleph_0$ *countable models.*

iii) (Lachlan, Shelah) *If T is ω-stable and has finitely many models in some uncountable power then T is $\aleph_0$-categorical and there exists an integer m and a finite group G such that for $n < \omega$,*

$$I^*(\aleph_n, \mathbf{AT}) = |n^m/G|.$$

iv) (Shelah) *If T is a bounded ω-stable theory then T has 1, $\aleph_0$ or $2^{\aleph_0}$ countable models.*

Proof. For i), ii), and iii) we see by Theorem 2.4 that T must be bounded. In the first two cases, by Theorems 3.21 and 3.25, T must be unidimensional. In case iii), Theorem 3.21 implies T is finite dimensional. The remainder of the theorem can be read off from the chart and the following refinement of iv).

3.28 LEMMA. *The number of countable models of an ω-stable bounded theory is determined by the value of $\delta_2(T)$. More specifically,*

i) *If $\delta_2 = 0$ then T is $\aleph_0$-categorical.*
ii) *If $0 < \delta_2 < \omega$ then T has $\aleph_0$ countable models.*
iii) *If δ_2 is infinite then T has $2^{\aleph_0}$ countable models.*

Proof. For i) note that if there is a nonisolated type over some finite set, the preservation of eni by nonorthogonality guarantees that some member of the basis must be eventually nonisolated; that is $\delta_2 > 0$. Part ii) is immediate from Theorem 3.20 and part iii) from Theorem 3.21 ii).

We quote without proof Theorem IX.1.20 of [Shelah 1978].

3.29 Theorem. *Let T be superstable and λ the least cardinal in which T is stable. Then, for any μ with $\mu \geq \aleph_1 + |T|$, $I(T, \mathbf{AT}) \geq \min(2^\mu, 2^\lambda)$.*

Combining this result with Theorem 3.26 we have the following theorem of Lachlan.

3.30 Theorem. *Suppose T is a countable superstable theory which has finitely many models of power $\aleph_k$ for some $k < \omega$. Then, T is a bounded ω-stable theory. If T is not $\aleph_1$-categorical then T is $\aleph_0$-categorical and for $k < \omega$, $I^*(\aleph_k, \mathbf{AT}) = |(k+1)^m/G|$ for some m and G.*

Since we gave Theorem 3.29 without proof a crucial step in our argument for this theorem is missing: the move from T is superstable to T is ω-stable.

3.31 Historical Notes. Most of the results in this section are contained in the last theorem of the first edition of [Shelah 1978]. The discussion of eventually nonisolated and persistently isolated types is taken from [Shelah, Harrington, & Makkai 1984]. Our exposition depends very heavily on the treatments in [Bouscaren & Lascar 1983], [Saffe 1983], and especially [Pillay 1983f]. We thank David Kueker and the University of Maryland logic seminar for detecting some serious flaws in an earlier version of this section.

Morley proved not only Theorem 3.27 i) [Morley 1965], but also that an $\aleph_1$-categorical theory has at most $\aleph_0$ countable models which is subsumed in Theorem 3.27 ii). The other direction of Theorem 3.27 ii) is from [Baldwin & Lachlan 1971]. The results in Theorem 3.27 iii) appear in X.2.7 of [Shelah 1978] and, more identifiably, in [Lachlan 1975] and [Lachlan 1978].

There has been considerable further work on unidimensional theories. Kueker [Kueker 198?] conjectured that a countable theory whose every uncountable model is ω-saturated is either $\aleph_0$ or $\aleph_1$-categorical. Lachlan (unpublished) proved the conjecture for ω-stable theories. Buechler [Buechler 1984a] extended the result to superstable theories. Shelah [Shelah 1978] asked if every stable unidimensional theory is superstable. Hrushovski ([Hrushovski 1986], [Hrushovski 198?], [Hrushovski 198?a]) answered this question positively and also obtained Kueker's conjecture for stable theories. Shelah [Shelah 1986c] provided a rather straightforward argument that a countable unidimensional theory has either 1, 2^λ or $\min(2^\lambda, 2^{2^{\aleph_0}})$ models in power λ for each uncountable λ.

4. Almost Homogeneous Models

In this section we discuss work of Pillay, Bouscaren, and Lascar on a generalization of the notion of an ω-homogeneous model, an almost homogeneous model. As pointed out by Pillay, the most natural context for this concept is the class of those stable theories in which all types have finite multiplicity.

4.1 Definition. i) The theory T is *extra-stable* if T is stable and every type has finite multiplicity.

ii) The model M is *almost homogeneous* if for each pair of finite sequences $\overline{a}, \overline{b} \in M$ with $stp(\overline{a}; \emptyset) = stp(\overline{b}; \emptyset)$ and each $c \in M$ there exists a $d \in M$ with $stp(\overline{a}^\frown c; \emptyset) = stp(\overline{b}^\frown d; \emptyset)$.

4.2 Exercise. Show every ω-stable theory is extra-stable.

There is an inherently slippery feature of almost homogeneity. Namely, $stp(\overline{a}; \emptyset)$ is well defined only when we regard $\overline{a}$ as sitting in a model of T. It is this problem which makes condition i) of Theorem 4.10 necessary.

The first major result is that if T is extra-stable a countable model of T is determined by the types over the empty set which it realizes. The following more abstract situation allows us to combine two arguments from the original proof. We slightly generalize the notion of almost homogeneity. The purpose of this generalization is not so much to cover more cases as to emphasize the properties of almost homogeneity which dominate the following proofs.

4.3 Definition. Let $\mathcal{G}$ be a normal subgroup of $\mathrm{Aut}(\mathcal{M})$ and for each submodel M of $\mathcal{M}$ denote by G_M the subgroup of $\mathrm{Aut}(M)$ consisting of the restrictions of elements of $\mathcal{G}$ to automorphisms of M. We say $\overline{a}, \overline{b} \in M$ are *$\mathcal{G}$-conjugate* and write $\overline{a} \sim_{\mathcal{G}} \overline{b}$ if there is an $\alpha \in \mathcal{G}$ with $\alpha(\overline{a}) = \overline{b}$. We say $\overline{a}, \overline{b} \in M$ are *M-conjugate* and write $\overline{a} \sim_M \overline{b}$ if there is an $\alpha \in G_M$ with $\alpha(\overline{a}) = \overline{b}$. The structure M is *$\mathcal{G}$-homogeneous* if for each $\overline{a}, \overline{b} \in M$, $\overline{a} \sim_M \overline{b}$ if and only if $\overline{a} \sim_{\mathcal{G}} \overline{b}$.

There are two natural exemplars of this notion, ordinary ω-homogeneity where $\mathcal{G} = \mathrm{Aut}(\mathcal{M})$ and almost homogeneity where $\mathcal{G} = \mathrm{Saut}(\mathcal{M})$.

We say $\mathcal{G}$ is a *closed subgroup* if every model M and every automorphism α of M satisfy the following condition. If every restriction of α to a finite set can be extended to a member of $\mathcal{G}_M$ then $\alpha \in \mathcal{G}_M$. Both $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Saut}(\mathcal{M})$ are closed.

4.4 Exercise. Show that if M is countable and $\mathcal{G}$ is closed then M is $\mathcal{G}$-homogeneous if and only if for each pair of finite sequences $\overline{a}, \overline{b} \in M$ with $\overline{a} \sim_{\mathcal{G}} \overline{b}$ and each $c \in M$ there exists a $d \in M$ with $\overline{a}^\frown c \sim_{\mathcal{G}} \overline{b}^\frown d$.

The next lemma collects the conditions on $\mathrm{Saut}(\mathcal{M})$ which are sufficient to prove most of the results presented here on almost homogeneous models.

4.5 Lemma. *Suppose $\mathcal{G}$ is* $\mathrm{Saut}(\mathcal{M})$ *and T is extra-stable.*

i) *For each* $p \in S(\emptyset)$, *the number of orbits of realizations of* p *under the action of* G_M *is finite.*

ii) *If* $\{\overline{a}_0, \ldots, \overline{a}_n\}$ *is a set of representatives for distinct orbits of* $p(M)$ *under* $\mathcal{G}$ *and* $t(\overline{a}_0, \ldots, \overline{a}_n; \emptyset) = t(\overline{b}_0, \ldots, \overline{b}_n; \emptyset)$ *then* $\{\overline{b}_0, \ldots, \overline{b}_n\}$ *is a set of representatives for distinct orbits of* $p(M)$ *under* $\mathcal{G}$.

Proof. Condition i) of Lemma 4.5 follows immediately from the definition of extra stable; condition ii) can be easily verified.

4.6 Exercise. Show that if T is extrastable and $\mathcal{G} = \mathrm{Saut}(\mathcal{M})$ then condition ii) holds.

4.7 Exercise. Show, in fact, that condition ii) depends only on the hypothesis that $\mathcal{G}$ is normal in $\mathrm{Aut}(\mathcal{M})$.

4.8 Exercise. Give an example of a type p and two models M and N of an extrastable theory such that the realizations of p split into a different number of strong types in M than in N.

4.9 Exercise. Show the phenomenon of the last exercise is impossible if p is an isolated type.

The following theorem yields that two countable almost homogeneous models which realize the same types are isomorphic.

4.10 Theorem. *Let* T *and* $\mathcal{G}$ *satisfy the conditions listed in Lemma 4.5.*

i) *If* M *and* N *realize the same types over the empty set,* M *is countable and* N *is* $\mathcal{G}$*-homogeneous then there is an elementary embedding of* M *into* N.

ii) *If* M *and* N *are countable,* $\mathcal{G}$*-homogeneous, and realize the same types over the empty set then they are isomorphic.*

Proof. i) Let $M = \{a_i : i < \omega\}$ and let $q_n = t(a_0, \ldots, a_{n-1}; \emptyset)$. We will construct by induction a set $S \subseteq \omega^{<\omega}$ and for each $s \in S$ an element $b_s \in N$. We write $\overline{b}_s$ for $\langle b_{s|1}, \ldots, b_s \rangle$ and let S_n denote the elements of S with length n. The construction will satisfy the following conditions.

a) S is closed under the formation of subsequences and each S_n is finite.
b) For each $s \in S_n$, $\overline{b}_s$ realizes q_n.
c) If $\overline{c} \in M$ realizes q_n, there is an $s \in S_n$ with $\overline{b}_s \sim_{\mathcal{G}} \overline{c}$.

Suppose for $m < n$ we have defined S_m and b_s for each $s \in S_m$; we must define S_n and for each $s \in S_n$, b_s. Let $C = \{\overline{c}_0, \ldots, \overline{c}_r\}$ be a complete list of representatives of the orbits of $q_n(N)$ under $\mathcal{G}$. Partition C by $\overline{c} \sim \overline{c}'$ if $\overline{c}|(n-1) \sim_{\mathcal{G}} \overline{c}'|(n-1)$. For each class D of this partition, choose by c) of the induction hypotheses an $s_D \in S_{n-1}$ such that for all $\overline{d} \in D$, $\overline{d}|(n-1) \sim_N \overline{b}_{s_D}$. By the $\mathcal{G}$-homogeneity of N, for each $\overline{d} \in D$ there exists a $b^{\overline{d}} \in N$ such that $\overline{d} \sim_N \overline{b}_{s_D}{}^\frown b^{\overline{d}}$. Fix an ordering of D and if $\overline{d}$ is the ith member of this ordering place $s_D{}^\frown i$ into S_n and set $b_{s_D{}^\frown i} = b^{\overline{d}}$.

Now S is an infinite finitely branching tree so by König's lemma it has an infinite branch, say η. Now, mapping a_i to $b_{\eta|(i+1)}$ yields the required imbedding of M into N.

ii) By i) we may assume $M \prec N$. We show that for $\overline{a} \in M$ and $\overline{b} \in N$, the relation $\overline{a} \simeq \overline{b}$ if and only if $\overline{a} \sim_{\mathcal{G}} \overline{b}$ is a 'back and forth' which yields the required isomorphism. The 'forth' is immediate so we prove only the 'back'. Suppose $\overline{a} \sim_{\mathcal{G}} \overline{b}$, $\lg(\overline{b}) = n$, and $d \in N$. We must find $c \in M$ with $\overline{a}^\frown c \sim_{\mathcal{G}} \overline{b}^\frown d$. Let $\overline{c} = \langle \overline{c}_0, \ldots, \overline{c}_r \rangle$ be a complete set of representatives for the orbits of $t(\overline{b}^\frown d; \emptyset)$ under $\mathcal{G}$ which intersect N and let $q = t(\overline{c}; \emptyset)$. Choose $\overline{c}' \in M$ to realize q. By condition 4.5 ii) $\overline{c}'$ is a set of representatives of distinct orbits in M of $t(\overline{b}^\frown d; \emptyset)$ under the action of $\mathcal{G}$. Since no more orbits can be represented in M than in N, for some $i < r$, $\overline{c}_i \sim_{\mathcal{G}} \overline{b}^\frown d$. In particular, $\overline{c}_i|(n-1) \sim_{\mathcal{G}} \overline{b}$. Since, by hypothesis, $\overline{a} \sim_{\mathcal{G}} \overline{b}$, we have $\overline{c}_i|(n-1) \sim_{\mathcal{G}} \overline{a}$. By the $\mathcal{G}$-homogeneity of M, there is a $c \in M$ with $\overline{a}^\frown c \sim_{\mathcal{G}} \overline{c}_i$. By transitivity $\overline{b}^\frown d \sim_{\mathcal{G}} \overline{a}^\frown c$ as required.

The remaining results in this section use the assumption that T is a countable ω-stable theory. This assumption is essential for the first result but it is open whether the hypothesis in the second case can be weakened to extra-stable.

4.11 Theorem. *Let T be countable, ω-stable, and bounded. Then every countable model of T is almost homogeneous.*

Proof. We first note that if T is ω-stable and bounded and $stp(\overline{a}; \emptyset) = stp(\overline{b}; \emptyset)$ then T satisfies the following condition.

$$\begin{array}{l}\text{For every } c \in M \text{ with } q = t(c; \overline{a}) \text{ strongly regular} \\ \text{there is a } d \in M \text{ with } stp(\overline{a}^\frown c; \emptyset) = stp(\overline{b}^\frown d; \emptyset).\end{array} \qquad (*)$$

For this, let $\alpha \in \mathrm{Saut}(\mathcal{M})$ map $\overline{a}$ to $\overline{b}$. Then since T is bounded, Lemma XIV.2.9 shows $\dim(q, M) = \dim(\alpha(q), M)$. In particular, $\alpha(q)$ is realized in M and we have the claim.

To see that the theorem follows from $(*)$, fix $\overline{a}, \overline{b}$, and $c \in M$. Since $\mathrm{Saut}(\mathcal{M})$ is closed, it suffices to find $d \in M$ with $stp(\overline{a}^\frown c; \emptyset) = stp(\overline{b}^\frown d; \emptyset)$. Choose $M_1 \subseteq M$ to be prime over $\overline{a}$ and choose $\overline{a}_1 \subseteq M_1$ with $\overline{a} \subseteq \overline{a}_1$ so that $t(c; M_1)$ is strongly based on $\overline{a}_1$. Moreover, let $t(c; M) \sqsubseteq^e \otimes r_i$ where each r_i is strongly regular and based on $\overline{a}_1$. By Theorem 3.2, we may suppose further that for some $\overline{e} \in M_1[c] \subseteq M$ realizing $\otimes r_i$, $t(c; M \cup \overline{e})$ is isolated over $\overline{a}_1 \cup \overline{e}$. This implies, in particular, that $t(c; \overline{a}_1 \cup \overline{e})$ is isolated by some $\phi(\overline{x}, \overline{a}_1, \overline{e})$. Since $\overline{a}_1 \subseteq M_1$, $t(\overline{a}_1; \overline{a})$ is principal so applying Lemma XIV.2.12, there is a $\overline{b}_1 \in M$ with $stp(\overline{a}^\frown \overline{a}_1; \emptyset) = stp(\overline{b}^\frown \overline{b}_1; \emptyset)$. Now if $\alpha \in \mathrm{Saut}(\mathcal{M})$ takes $\overline{a}_1$ to $\overline{b}_1$ there is, by $(*)$, for each e_i an $e'_i \in M$ realizing $\alpha(r_i)$. Since the e_i are independent, $\overline{e}'$ realizes $t(\alpha(\overline{e}); \overline{b})$. Now since $\phi(\overline{x}, \overline{b}_1, \overline{e}') \vdash stp(\alpha(c); \overline{b}_1)$ and $\overline{b}_1 \supseteq \overline{b}$, so M is almost homogeneous.

From Theorems 4.10 and 4.11 we have another proof of Theorem 3.27 iv).

4.12 Corollary. *Let T be countable ω-stable and bounded. Then T has $1, \aleph_0$ or $2^{\aleph_0}$ countable models.*

Proof. Each model of T is determined by the set of types over the empty set which are realized in it.

Finally, we show that for ω-stable theories, countable ω-homogeneous models are almost homogeneous. The following exercise shows where the difficulties of the theorem lie.

4.13 Exercise. Show that if M is a countable ω-saturated model of an extra-stable theory then M is almost homogeneous.

4.14 Theorem. *Suppose T is a countable ω-stable theory. Every countable ω-homogeneous model of T is almost homogeneous.*

Proof. Let $\overline{a}, \overline{b} \in M$ with $stp(\overline{a}; \emptyset) = stp(\overline{b}; \emptyset)$ and let $c \in M$. Choose $M_1 \subseteq M$ **AT**-prime over $\overline{a}$ and choose $\overline{a}_1 \in M_1$ with $t(c; M_1)$ strongly based on $\overline{a} \cup \overline{a}_1$. By Lemma XIV.2.12, there is a $\overline{b}_1 \in M$ so that $\overline{b}^\frown \overline{b}_1$ realizes $stp(\overline{a}^\frown \overline{a}_1; \emptyset)$. Since M is ω-homogeneous, there is a $d \in M$ with $t(\overline{a}^\frown \overline{a}_1{}^\frown c; \emptyset) = t(\overline{b}^\frown \overline{b}_1{}^\frown d; \emptyset)$. Since $stp(\overline{a}^\frown \overline{a}_1; \emptyset) = stp(\overline{b}^\frown \overline{b}_1; \emptyset)$ there is an $\alpha \in \mathrm{Saut}(\mathcal{M})$ which maps $\overline{a}^\frown \overline{a}_1$ to $\overline{b}^\frown \overline{b}_1$. If $q = stp(c; \overline{a}^\frown \overline{a}_1)$, d realizes $\alpha(q)$. Thus, by Lemma IV.3.20, $stp(\overline{a}^\frown \overline{a}_1{}^\frown c; \emptyset) = stp(\overline{b}^\frown \overline{b}_1{}^\frown d; \emptyset)$ and we finish.

There are a number of interesting questions raised by this section. We adopted an abstract exposition of Theorem 4.10 to emphasize that it is an assertion about certain families of group representations with little model theoretic content. In contrast, Theorem 4.11 makes heavy use of the stability machinery. This use seems intrinsic to the proof but we have no example of any model which is homogeneous without being almost homogeneous. There are several further speculations and insightful examples in [Pillay 1982a].

4.15 Historical Notes. This section is a mild rewrite of [Pillay 1982a] and one theorem from [Bouscaren & Lascar 1983]. Bouscaren and Lascar proved a more general version of Theorem 4.11 by induction on U-rank. Their argument applies when α_T is finite without assuming that T is bounded.