Chapter XII
Regular Types

In this chapter we introduce the most important concepts involved in assigning dimension to models. We saw in Section 1.3 that the simplest sort of $\aleph_1$-categorical theory is one in which the universe of each model is strongly minimal. A strongly minimal set D has two crucial features: i) its dimension is well-defined, ii) if $M \subseteq N$, every element of $D(N) - D(M)$ is independent from M over the empty set. The possession of both of these features is no accident; we will show that properly formulated versions of the two properties are equivalent. For any acceptable class K, we define the concept of a K-strongly regular type. The definition given in Section 1 is less intuitive but technically more useful than either of the properties just described. We then recast this definition in terms of the second of these properties. This recasting simplifies the construction of regular types in Section 2. In Section 3 we analyze invariance of dimension in terms of the transitivity properties of the forking relation. This approach makes it clear that for any K-strongly regular (stationary) type based on $A \subseteq M$, $\dim(p, M)$ is well defined. Finally, in Section 4 we show that the two approaches to defining regular types yield the same class of types. With this we can describe the relation between $\not\perp$ and $\triangleright^e$. Two types are orthogonal if and only if they are disjoint in the partial ordering imposed by domination. We further conclude that $\not\perp$ is an equivalence relation on the K-strongly regular types. Moreover, if $M \prec N$ and q is the nonforking extension of p to $S(N)$ then $\dim(p, N) = \dim(p, M) + \dim(q, N)$.

Throughout this chapter we assume that K is an acceptable class of models.

1. Weak Isolation and Regular Types

The usual notion of regular type is primarily associated with **S**-models and superstable theories; strongly regular types are similarly associated with arbitrary models of an ω-stable theory. We develop in this chapter a common framework for the two notions.

The first three paragraphs of this section define the notions of regularity, strong regularity and weak isolation. The next three pages expound technical properties of these notions and establish their parallelism invariance. These technicalities arise again, for example in Theorem XIII.3.3. The theories described in Example 1.11 illustrate the fine distinctions made earlier in the discussion. The last three results concern the structure of $M[p]$ where p is regular.

We will describe a collection of types with the following property. On realizations of a type in the class the dependence relation arising from forking has a good notion of dimension. That is, we will be able to find a 'basis' for the set of realizations of such a type. Intuitively, a basis is both a maximal independent set and a minimal generating set. Suppose that a, b, c are the realizations of such a type in M and that a is independent from b. If both c depends on a and b depends on c over a, we have that $\{a, b\}$ is a maximal independent set but $\{a\}$ is a smaller generating set. Thus, we need to guarantee that if a and b are independent and c depends on a then b does not depend on c over a or, by symmetry, that c does not depend on b over a. The following definition is, a priori, somewhat stronger than is required. We will see in Section 4 that the weaker intuition suffices but the chosen definition is more useful in practice.

1.1 Definition. The non-algebraic type $p \in S(A)$ is *regular* if and only if every extension of p is orthogonal to p or parallel to p.

If $q = stp(\overline{b}; B)$ then q is regular if the unique nonforking extension of q to a complete type over $B \cup \overline{b}$ is regular.

There are two difficulties with this definition. First, it makes the verification of the existence of regular types inconvenient. More important, a surprising but crucial consequence of this definition—that regular types are **S**-minimal—does not extend to the ω-stable case. For this, we must sharpen the definition of regularity. The sharpened version deals with a pair (p, q) where p is a complete type and q is an **I**-formula.

1.2 Definition. i) For any acceptable class K, any non-algebraic type $p \in S(A)$, and any **I**-formula q over A with $q \subseteq p$, (p, q) is *K-strongly regular* if for every $B \supseteq A$ and for every $r \in S(B)$ which extends q, either r is a nonforking extension of p or $r \perp p$.

ii) The pair (p, q) is called *stationary and strongly regular* if, in addition, p is stationary.

iii) We may write $p \in S(A)$ is *K-strongly regular* to mean that for some **I**-formula over A, (p, q) is K-strongly regular.

Unless explicitly declared otherwise, when we refer to a K-strongly regular pair (p, q), we denote $\operatorname{dom} p$ by A.

Following [Shelah 1978] and in contrast to [Makkai 1984], we do not require that a K-strongly regular type be stationary. Note that in Definition 1.2 i) the requirement is that r does not fork over A, not that r does not fork over $\operatorname{dom} q$. Neither p nor r is required to be a nonforking extension of q.

This distinction is illustrated in Example 1.11 iv). In fact, this requirement could be added without invalidating the existence proof. We omitted it to maintain uniformity with [Shelah 1978].

The role of q in the definition of strong regularity is suggested by the relation between a strongly minimal formula, $\phi(x)$, and the unique non-algebraic type p containing ϕ. That is, ϕ 'weakly isolates' p in the sense that if the parameters of ϕ are from a model M and N is a proper elementary extension of M, then every realization of ϕ in $N - M$ actually realizes p. The following definition of one type weakly isolating another extends this concept to an arbitrary acceptable class K. After showing this relation is a property of the parallelism class of p, we will prove that (p, q) is K-strongly regular if and only if p is weakly isolated by q.

1.3 Definition. (Fig. 1). Let $M \subseteq N \models T$. Suppose M is **I**-saturated, q is an **I**-formula over $A \subseteq M$, and p is a nonforking extension of q to $S(M)$.

i) q *weakly isolates* p in N if $q(N) - M \neq \emptyset$ and every $\bar{b} \in q(N) - M$ both satisfies $p|A$ and is independent from M over A.
ii) q weakly isolates the type $p \in S(M)$ if for some N containing M with $q(N) - M \neq \emptyset$, q weakly isolates p in N.

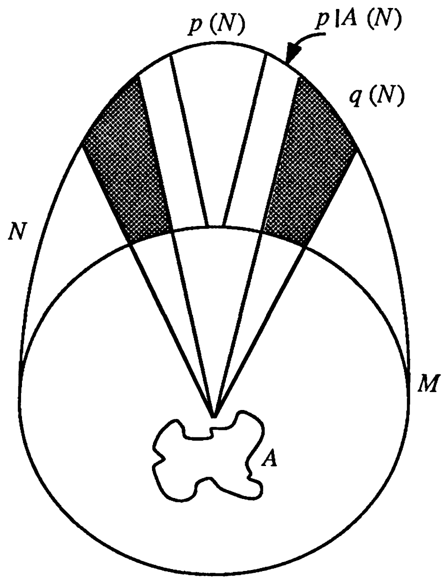

Fig. 1. q weakly isolates p in N. If $p|A$ is stationary then $p|A\,(N) - p(N)$ is empty.

We will need in Section XIII.3 the following more complicated version of weak isolation.

iii) q *weakly isolates* p in (N_1, N), where $M \subseteq N_1 \subseteq N$, if $q(N) - N_1 \neq \emptyset$ and for every $\bar{b} \in q(N_1) - N$, $\bar{b} \downarrow_A M$ and $\bar{b}$ realizes $p|A$.

Note that i) is the special case of iii) obtained by identifying N_1 and N.

The definition of weak isolation is actually relative to the class K. Since this class is invariably fixed by the context we have not made it explicit in the definition.

In considering this definition recall that the notation $\bar{b} \in q(N) - M$ means $\bar{b} \in N$ realizes q and some member of the sequence $\bar{b}$ is not in M. The conclusion $\bar{b} \downarrow_A M$ implies $\bar{b} \cap M \subseteq cl(A)$. Thus, a minor consequence of q weakly isolating p in N is that no realization $\bar{b} \in N$ of q can contain points from M which are not algebraic over A.

Suppose that a nonalgebraic type $p \in S^2(A)$ is realized by $\langle a, b\rangle$ with $a \downarrow_A b$. Then we can choose $\langle c, d\rangle$ such that $a^\frown b \downarrow_A c^\frown d$ and each of $\langle a, b\rangle$, $\langle c, b\rangle$, and $\langle c, d\rangle$ realize p. This choice quickly yields that p is not regular. Thus, although we consider regular n-types for $n > 1$, any sequence realizing such a regular type must be dependent.

When K is the class of **AT**-saturated (i.e. all models) we have the notion called strongly regular in Shelah [Shelah 1978]. In discussing ω-stable theories, we will frequently write strongly regular for **AT**-strongly regular.

1.4 Exercise. Show that if q weakly isolates $p \in S(M)$, where q is an **I**-formula and $M \in K$, then q weakly isolates p in $M[\bar{a}]$ for any $\bar{a}$ realizing p.

The next lemma is a technical point to show that weak isolation is a property of the parallelism class of p where $p \in S(M)$. Note that if q weakly isolates p in N then q weakly isolates p in the image of N under any isomorphism which fixes M. We will show that if q weakly isolates p then q weakly isolates every p' which extends q and is parallel to p. We first show this if p' is a nonforking extension of p (or vice versa). The result then extends to arbitrary parallel p and p' by passing through their common nonforking extension.

1.5 Theorem. *Let q be an* **I**-*formula and M be an* **I**-*saturated model. Suppose $p \in S(M)$ and $p' \in S(M')$ is a nonforking extension of p. Then q weakly isolates p if and only if q weakly isolates p'.*

Proof. Suppose first that q weakly isolates p' and let $\bar{a}$ realize p'. Suppose that $N' \supseteq M' \cup \bar{a}$ witnesses the weak isolation of p' by q. Let $N = M[\bar{a}]$. It suffices to show that q weakly isolates p in N. Let $\bar{b} \in q(N) - M$. Let $\bar{b}_1$ denote $\bar{b} \cap (N - M)$ and $\bar{b}_2$ denote $\bar{b} \cap M$. Then $\bar{a} \rhd_M \bar{b}_1$. Since $\bar{a} \downarrow_M M'$ it follows that $\bar{b}_1 \downarrow_M M'$. By monotonicity and since $\bar{b}_2 \in M$, $\bar{b}_1 \downarrow_M M' \cup \bar{b}_2$ and thus $\bar{b} \downarrow_M M'$. So, $\bar{b} \in q(M'[\bar{a}]) - M'$. Now, as q weakly isolates p' we have $\bar{b}$ realizes $p'|A = p|A$ and $\bar{b} \downarrow_A M'$. By monotonicity $\bar{b} \downarrow_A M$, so q weakly isolates p in N

Now suppose that q weakly isolates p and let $\bar{a}$ realize p'. Let $N' = M'[\bar{a}]$. Assume for contradiction that $\bar{b} \in N'$ realizes q but either $\bar{b}$ does not realize $p|A$ or $\bar{b} \not\downarrow_A M'$. We prove the following strong result which not only yields the theorem at hand but will be applied again to prove Theorem 1.7 and Corollary 1.8.

1.6 Lemma. *Suppose the type q over $A \subseteq M$ weakly isolates the type $p \in S(M)$ where q is an $\mathbf{I}$-formula and M is $\mathbf{I}$-saturated. Let $\overline{a}$ realize p and suppose $\overline{a} \downarrow_A M$ and $\overline{a} \downarrow_M D$. For any $\overline{b}$ realizing q, if $\overline{b} \not\downarrow_{M \cup D} \overline{a}$ then $t(\overline{b}; M \cup D)$ is a nonforking extension of $p|A$.*

Proof. By monotonicity, $\overline{a} \not\downarrow_A M \cup D \cup \overline{b}$ so applying the local character of forking, there is a formula $\chi_0(\overline{x}, \overline{y}, \overline{z}) \in F(A)$ such that $\chi_0(\overline{a}, \overline{b}, \overline{m})$ for some $\overline{m} \in M \cup D$ and for any $\overline{b}'$ and $\overline{m}'$, if $\models \chi_0(\overline{a}, \overline{b}', \overline{m}')$ then $\overline{a} \not\downarrow_A \overline{b}' \frown \overline{m}'$. We argue by contradiction. If $t(\overline{b}; A) \neq p|A$ let $\chi_1(\overline{x})$ be a formula over A which is satisfied by $\overline{b}$ but is not in $p|A$. Otherwise, choose $\chi_1(\overline{x}, \overline{y}) \in F(A)$ and $\overline{d} \in M \cup D$ so that $\models \chi_1(\overline{b}, \overline{d})$ and for every $\overline{b}'$ realizing $t(\overline{b}; A)$, if $\models \chi_1(\overline{b}', \overline{d}')$ then $\overline{b}' \not\downarrow_A \overline{d}'$. Now apply Corollary X.1.13 to the $\mathbf{I}$-formula $\{\chi_0(\overline{a}, \overline{y}, \overline{z}) \wedge \chi_1(\overline{y}, \overline{w})\} \cup q(\overline{y})$ to obtain $\overline{m}', \overline{d}' \in M$ and $\overline{b}'$ such that $\chi_0(\overline{a}, \overline{b}', \overline{m}') \wedge \chi_1(\overline{b}', \overline{d}') \wedge q(\overline{b}')$. Let N be any model in K containing $M \cup \overline{a} \cup \overline{b}'$. As $\overline{a} \not\downarrow_A \overline{b}' \frown \overline{m}'$, $\overline{b}' \notin M$. Since q weakly isolates p, $t(\overline{b}'; A) = p|A$. This contradicts the choice of χ_1 in the case $t(\overline{b}; A) \neq p|A$. In the second case, we have $\overline{b}' \not\downarrow_A \overline{d}'$ which also contradicts the weak isolation of p by q. This concludes the proof of the lemma.

To finish the proof of Theorem 1.5, let $M' = D$. Then by FI_2, $\overline{b} \not\downarrow_{M'} M' \cup \overline{a}$ so $t(\overline{b}; M) \not\perp p$. Hence, by Lemma 1.6, $t(\overline{b}; M')$ is a nonforking extension of p.

With these technical preliminaries disposed of we can recast the K-strong regularity in terms of weak isolation. We will profit from this reformulation in showing that strong regularity is a parallelism invariant and when constructing K-strongly regular types in Section 2.

1.7 Theorem. *Let $p \in S(A)$ and $q \subseteq p$ be an $\mathbf{I}$-formula over A. The pair (p, q) is K-strongly regular if and only if there is an $M \in K$ which contains A and an $\overline{a}$ realizing p with $\overline{a} \downarrow_A M$ such that q weakly isolates $t(\overline{a}; M)$.*

Proof. Suppose that p is K-strongly regular. Let $A \subseteq M \in K$ and let p' be a nonforking extension of p to $S(M)$. Let $\overline{a}'$ realize p'.

We claim q weakly isolates p' in $M[\overline{a}']$. To see this, let $\overline{b} \in M[\overline{a}']$ realize q and let r denote $t(\overline{b}; M)$. By the definition of K-strong regularity, $r \perp p$ or r is a nonforking extension of p. The first is impossible. For, $r \not\perp p'$ because r is realized in $M[\overline{a}']$; thus, r is a nonforking extension of p as required.

Note that we are not required to conclude that $\overline{b}$ realizes p'. Without assuming that p is stationary we could not reach that conclusion.

Suppose the second condition holds. We must show that if $B \supseteq A$, $r \in S(B)$, and $r \supseteq q$ then $r \perp p$ or r is a nonforking extension of p. By taking a nonforking extension of r we can assume without loss of generality that B is the universe of a model $M \in \mathbf{S}$. Then, if $r \not\perp p$ there exist a $\overline{b}$ realizing r, an $\overline{a}$ realizing p such that $\overline{a} \downarrow_A M$ and a set D such that $\overline{a} \downarrow_M D$, $\overline{b} \downarrow_M D$ but $(\overline{a} \not\downarrow \overline{b}; M \cup D)$. By Lemma 1.6, r is a nonforking extension of p.

An easy application of Theorem 1.7 yields that if the $\mathbf{I}$-formula q weakly isolates $p \in S(M)$ then (p, q) is K-strongly regular. We now give sufficient conditions for the preservation of regularity and strong regularity 'up' and 'down' under nonforking extensions. We have so far avoided requiring regular types to be stationary. That requirement is essential for most of the preservation results. The proof of Theorem 1.8 ii) is a natural extension of the second paragraph of the proof of Theorem 1.7.

1.8 Corollary. *Let $p' \in S(A')$ be a nonforking extension of $p \in S(A)$.*

i) *If p is regular then p' is regular. If p is stationary and p' is regular then p is regular.*
ii) *Suppose the stationary type p contains the* **I***-formulas q and for some $M \in K$, q weakly isolates p^M. Then (p', q) is K-strongly regular.*
iii) *Suppose p is stationary. Then for any* **I***-formula $q \subseteq p$, (p', q) is K-strongly regular if and only if (p, q) is K-strongly regular.*

Proof. i) Preservation 'up' is immediate from the definitions. Suppose for the converse that $p' \in S(A')$ is a nonforking extension of the stationary type p and p' is regular. To show p is regular, let $A \subseteq C$ and suppose $\bar{b}$ realizes a forking extension r of p to $S(C)$. Without loss of generality, we may choose $C \frown \bar{b} \downarrow_A A'$. Thus, $t(\bar{b}; C \cup A')$ is a forking extension of p' (since p is stationary and $\bar{b} \downarrow_A A'$). So $t(\bar{b}; C \cup A')$ is orthogonal to p' and thus to p. But, by monotonicity, $\bar{b} \downarrow_C A' \cup C$ so $r \perp p$ as required.

ii) Let $r \in S(B)$ be an extension of p'. Suppose N contains $M \cup B$ and r' is a nonforking extension of r to $S(N)$. Now if $r \not\perp p$, the preservation of nonorthogonality by parallelism guarantees $r' \not\perp p^N$. Choose $\bar{b}'$ realizing r', $\bar{a}$ realizing p^N and D so that $\bar{b}' \downarrow_N D$ and $\bar{a} \downarrow_N D$ but $\bar{b}' \not\downarrow_{D \cup N} \bar{a}$. By Lemma 1.5, q weakly isolates p^N. So, by Lemma 1.6 r' is a nonforking extension of p. Since p is stationary, r' is a nonforking extension of p as required.

iii) Suppose (p, q) is K-strongly regular. By Theorem 1.7, for some M in K, q weakly isolates p^M. By ii) we conclude (p', q) is K-strongly regular. Conversely, if (p', q) is K-strongly regular then for some $M \in K$, q weakly isolates $(p')^M$. But, $(p')^M = p^M$ so the result follows by ii).

1.9 Exercise. Prove the 'up' direction of Corollary 1.8 i). Deduce from Corollary 1.8 that if $p \in S(A)$ is regular and $\bar{a}$ realizes p then $stp(\bar{a}; A)$ is regular.

1.10 Exercise. Show that the hypothesis that p is stationary is necessary to prove that regularity is preserved downwards by nonforking extensions. (Hint: Consider the following theory with three equivalence relations E_1, E_2, E_3. Restricted to E_1 and E_2, T is the theory CEF_2. E_3 has two classes, one coincides with a single E_1 class. In this theory there are two types over the empty set: p_1 contains $(\exists y)[E_3(x, y) \wedge \neg E_1(x, y)]$ while p_2 does not. However, it is easy to see that if there were a unique type over the empty set it would have both a nonforking extension which is regular and one which is not. David Marker has suggested the following device to modify the theory and obtain a single type over the empty set. Add to the language three unary predicates A, B, and C. Let the extension of C in a model of T

be the structure described above. Now consider a ternary relation R with the following properties.

i) $R(x,y,z) \rightarrow A(x) \wedge B(y) \wedge C(z)$.
ii) For each $a \in A$, there are infinitely many $b \in B$ (this subset of B is called U_a^1) such that $R(a,b,c)$ if and only if b realizes p_1.
iii) For each $a \in A$, there are infinitely many $b \in B$ (this subset of B is called U_a^2) such that $R(a,b,c)$ if and only if b realizes p_2.
iv) For each $a \in A$, there are infinitely many $b \in B$ (this subset of B is called U_a^3) such that there is no $c \in C$ satisfying $R(a,b,c)$.
v) If $a_0 \neq a_1$, $U_{a_0}^1 \cup U_{a_1}^2 \subseteq U_{a_1}^3$.

The first four conditions tranform the pair of types into a single type. The fifth condition makes the theory complete and ω-stable.)

The following examples illustrate a number of the subtleties of the notion of K-strong regularity.

1.11 Examples. i) If T is ω-stable and K is the class of **AT**-saturated models then for any strongly minimal set D, if p is the type of an unrealized element of D, the pair (p,D) is K-strongly regular.

ii) Let T be the theory of an equivalence relation with infinitely many infinite classes each of which is a model of $\mathrm{Th}(Z,S)$. Consider the 2-type, p, of a pair of elements, $\langle a,b\rangle$, which are equivalent but not on the same Z-segment. Whether p is K-strongly regular depends on our choice of K. If we fix K_1 to be **AT** and thus **I**-formulas are single formulas then p is not K_1-strongly regular. For, if $\phi(x,y)$ is any formula in p, there is a nonforking extension of ϕ realized by two elements which are only finitely distant as well as one which is realized by two points which are infinitely far apart. Thus, there is no choice of an **I**-formula ϕ to make (p,ϕ) K_1-strongly regular.
On the other hand, letting K_2 be the class of (strongly)-$\aleph_0$-saturated models of T (thus the **I**-formulas are types which are (almost) over finite sets) it is easy to see that $(p, p|\emptyset)$ is K_2-strongly regular.

Both example i) and ii) (when $K = \mathbf{S}$) have the property that if we pass to $\mathcal{M}^{eq}$ there is an obvious 'quotient' type of the K-strongly regular type which contains a strongly minimal formula. The next example does not have that property.

iii) Let T be the theory REI_ω of countably many refining equivalence relations with infinite splitting. If M is a model of T, $p \in S(M)$, and p specifies all the equivalence classes, then (p',p) is $\mathbf{AT}_{\aleph_1}$-strongly regular for any nonforking extension p' of p. But no strongly minimal set is associated with p (even in $\mathcal{M}^{eq}$).

The next two examples illustrate some difficulties in trying to strengthen the properties of the weakly isolating formula in a strongly regular pair.

iv) Let T be the theory of two crosscutting equivalence relations, E_1, E_2, and let p be the complete type over a containing the formula

$\{E_1(x;a) \wedge E_2(x;a)\}$. Then $(p, x = x)$ is **AT**-strongly regular but p forks over the empty set. (This example was suggested by Steve Buechler.)

It is essential for Theorem 1.7 that p be a complete type over A. Moreover, it is impossible to revise the definition of strongly regular type to make it a property of one type rather than a pair of types. The next example illustrates the dependence on p.

v) Let T be the theory of infinitely many disjoint infinite unary predicates. If M is the prime model of T and p is any non-algebraic type over M, $(p, x = x)$ is **AT**-strongly regular. But there is no stationary formula q so that (p, q) is **AT**-strongly regular. Here, we say a formula, ϕ, is stationary if the type $\{\phi\}$ is stationary.

The following result emphasizes the importance of the **S**-saturated models.

1.12 Theorem. *The following are equivalent.*

i) *There is an* **S**-*formula* q *such that* (p, q) *is* **S**-*strongly regular.*
ii) *For some acceptable class* K, *there is an* **I**-*formula* q *such that* (p, q) *is* K-*strongly regular.*
iii) *The type* p *is regular.*

Proof. Clearly, i) implies ii). Definition 1.2 yields immediately that ii) implies iii). To see that iii) implies i), let $p \in S(B)$ and choose $A \subseteq B$ with $|A| < \kappa(T)$ such that p does not fork over A. For some $\overline{a}$ realizing p, let $q = stp(\overline{a}; A)$. By Theorem 1.8 i) q is regular. Now, let $M \supseteq B$ be **S**-saturated and $p' \in S(M)$ be a nonforking extension of p. If $\overline{a}'$ realizes p' then for any $\overline{b}'$ in $q(M[\overline{a}']) - M$, $t(\overline{b}'; M) \not\perp p'$ by FI_2. Thus by the definition of regular, $t(\overline{b}'; M)$ is a nonforking extension of q and thus of p as required.

K-strongly regular types should be viewed as a particuarly good sort of K-minimal type. The next theorem verifies that they are, in fact, K-minimal. Sections 3 and 4 will reveal their sterling qualities.

1.13 Theorem. *If* $M \in K$, $p \in S(M)$ *and* (p, q) *is* K-*strongly regular then* p *is* K-*minimal.*

Proof. Suppose $r \in S(M)$, r is not realized in M, and $p \mapsto_K r$. We must show $r \mapsto_K p$. To this end, let $\overline{b}$ realize p and $\overline{a} \in M[\overline{b}]$ realize r. By Corollary X.1.22 (letting $D = \emptyset$) there exists a $\overline{b}' \in M[\overline{a}] - M$ which realizes q. By the definition of a strongly regular pair, $\overline{b}'$ realizes p and we finish.

The following theorem of Pillay shows a good deal about the structure of the K-models prime over K-strongly regular types.

1.14 Theorem. *If* $M \in K$, $p \in S(M)$ *and* (p, p_0) *is* K-*strongly regular then for any* $\overline{a}$ *realizing* p *and any* $\overline{b} \in M[\overline{a}] - M$, $t(\overline{a}; M \cup \overline{b})$ *is* **I**-*isolated.*

Proof. Since $\overline{a} \not\downarrow_M \overline{b}$, there is a formula $\chi(\overline{x};\overline{b})$ such that $\models \chi(\overline{a};\overline{b})$ and $\chi(\mathcal{M};\overline{b}) \cap M = \emptyset$. Let the **I**-formula $q_0(\overline{a};\overline{y})$ isolate $t(\overline{b}; M \cup \overline{a})$. Then the type $p(\overline{x}) \cup q_0(\overline{x};\overline{y})$ implies a complete type in $\overline{x}^\frown\overline{y}$ over M. Let r_0 be the **I**-formula, $p_0(\overline{x}) \cup q_0(\overline{x};\overline{b}) \cup \chi(\overline{x};\overline{b})$. Now, any point in $M[\overline{a}]$ which realizes r_0 (in particular, which realizes $p_0 \cup \chi(\overline{x};\overline{b})$) is in $M[\overline{a}] - M$ and thus realizes p. Since $M[\overline{a}]$ is **I**-saturated, this implies that $r_0 \vdash p$. We deduce $r_0(\overline{x}) \vdash p(\overline{x}) \cup q_0(\overline{x};\overline{b}) \vdash t(\overline{a}; M \cup \overline{b})$ and finish.

We can immediately conclude

1.15 Corollary. *If $M \in K$, $p, q \in S(M)$ are K-strongly regular, and $p \underset{\sim}{\vdash}_K q$ then $M[p] \approx M[q]$.*

Proof. Let $\overline{a}$ realize p, $\overline{b}$ realize q. Applying Theorem 1.14 it is easy to see that $M[\overline{a}]$ is **I**-constructible over $M \cup \overline{b}$. By Theorem IX.4.12, $M[\overline{a}]$ is isomorphic to $M[\overline{b}]$.

1.16 Historical Notes. Shelah introduces the notion of regular type in Section V.1 and that of strongly regular type in Section V.3 of [Shelah 1978]. Our treatment has been greatly influenced by [Makkai 1984] which in turn was influenced by [Lascar 1984]. The direct argument for Theorem 1.13 is due to Lascar [Lascar 1984]. Theorem 1.14 is due to Pillay, [Pillay 1982] (also quoted in [Lascar 1984]). We also learned a good deal in conversations with Buechler and Pillay. Another approach is taken in [Pillay 1984]. Exercise 1.10 provides a negative answer to question 1.1 on page 224 of [Shelah 1978].

2. *Existence of Strongly Regular Types*

This section is devoted to finding strongly regular types for various acceptable classes K. We will see that for superstable theories regular types are plentiful. One of the difficulties in investigating theories which are stable but not superstable is that analogs of the next several results do not hold.

For the existence theorems beginning this section we do not assume that T admits K-prime models. In particular, one of the exercises shows that a small superstable theory admits **AT**-strongly regular types (which may not be stationary). We will reap some benefits from this added generality in Chapter XIII. However, the main cases are **S**-models of superstable theories and arbitrary models of ω-stable theories.

The exercise after the following definition describes the 'useful' version of that definition. That is, it provides for the existence of all regular types needed for most of the arguments in this chapter. For Theorem XIII.3.3 a more refined existence theorem is necessary. The definition itself, which relies on part iii) of Definition 1.3, formulates this condition. Since the definition only yields a $b \in N - N_1$ which is weakly isolated over M, not N_1, the definition is not a very significant strengthening of the exercise.

The extremely difficult Theorem XIII.3.3 is necessary to get any benefit from the extension.

2.1 Definition. The acceptable class K *admits* (K*-strongly*) *regular types* if for every triple of K-models $M \subseteq N_1 \subseteq N$ with M properly contained in N there is an **I**-formula, p_0, over $A \subseteq M$ with $|A| < \lambda(I)$ and a $b \in N - N_1$ such that p_0 weakly isolates $t(b; M)$ in (N_1, N).

We say K *admits stationary regular types* if we can demand in addition that $t(b; A)$ is stationary. The theory T admits K-strongly regular types if the class of K-models of T does so.

Although we spoke in Section 1 of regular type of arbitrary finite length, this definition specifically provides for the existence of regular 1-types.

2.2 Exercise. Show that by taking $N_1 = N$ we can deduce from the assertion that K admits regular types the following simpler assertion. For every M, $N \in K$ with N a proper extension of M, there is a $b \in N$ such that for some $A \subseteq M$ and some **I**-formula, q, over A, $(t(b; M), q)$ is K-strongly regular.

2.3 Theorem. *If T is a superstable theory then* **S** *admits stationary regular types.*

Proof. (Fig. 2). Let $M \subseteq N_1 \subseteq N$ with N_1 properly contained in N. For as

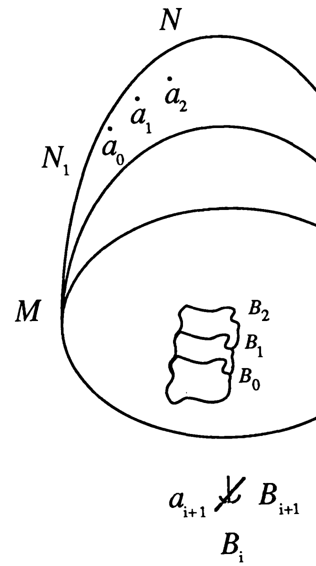

Fig. 2. Theorem XII.2.3.

long as possible, choose a coherent sequence $\langle \overline{a}_i : i < \alpha \rangle$ and $\langle B_i : i < \alpha \rangle$ with $B_i \subseteq B_{i+1} \subseteq M$ such that $\overline{a}_i \in N - N_1$, $t(a_i; B_i)$ is stationary, $|B_i| < \kappa(T)$, $|B_{i+1} - B_i| < \omega$, and $\overline{a}_{i+1} \not\downarrow_{B_i} B_{i+1}$. If continued infinitely this sequence contradicts the assumption that $\kappa(T) = \omega$; when it stops $stp(\overline{a}_i; B_i)$ weakly isolates $t(a_i; M)$ in (N_1, N).

This argument does not continue past a limit ordinal. Thus, we are unable to prove that a countable stable theory admits regular types for $K = \mathbf{AT}_{\aleph_1}$, which is the natural candidate when T is merely stable. It is not easy to find an example to illustrate this problem. There may be one in the second edition of [Shelah 1978].

2.4 Exercise. Modify the argument for Theorem 2.3 to show that if M is $\mathbf{S}$-saturated, T is superstable and M is a proper submodel of the $\mathbf{S}$-model N then there is a $b \in N - M$ with $t(b; M)$ regular.

An alternative proof of Theorem 2.3 invokes rank to find the element realizing the regular type.

2.5 Exercise. Show that if T is superstable and $R_C(t(a; M))$ is minimal among all types over M realized in N then $t(a; M)$ is regular.

In the remaining arguments in this chapter we shall use several of the notions of rank discussed in Chapter VII. We begin by extending the last result to the class of all models if T is a countable ω-stable theory.

2.6 Theorem. *If T is a countable ω-stable theory then* $\mathbf{AT}$ *admits stationary* $\mathbf{AT}$*-strongly regular types.*

Proof. Let $M \subseteq N_1 \subseteq N$ with M properly contained in N. Choose a in $N - N_1$ to minimize the Morley rank, $R_M(t(a; M))$, and choose a formula $\phi(x; \overline{b}) \in t(a; M) = p$ with the same rank, say α, and degree as p. Clearly, the degree is one. Now if $a' \in N - M$ and $\models \phi(a'; \overline{b})$ then $R_M(t(a'; M)) \geq \alpha$ so $t(a'; \overline{b}) = p|\overline{b}$ and $\overline{a}' \downarrow_{\overline{b}} M$. Thus, $\phi(x, \overline{b})$ weakly isolates $t(a; M)$ in (N_1, N) and $(p, \phi(x, \overline{b}))$ is $\mathbf{AT}$-strongly regular as required.

The following exercise shows the necessity of assuming ω-stability for this theorem.

2.7 Exercise. Show that the theory of infinitely many independent unary predicates does not admit $\mathbf{AT}$-strongly regular types.

2.8 Exercise. Show, by forming a disjoint union of the last example with a strongly minimal set, that one can have (p, q) K-strongly regular and r a nonforking extension of q without (r, q) being K-strongly regular. (This differs from Theorem 1.8 as we require only $r \supseteq q$ rather than $r \supseteq p$.)

Two contrasting properties of Morley rank were used in the proof of Theorem 2.6. On the one hand, every type contains a formula with the same rank; on the other, a type has only finitely many distinct extensions of the same rank. Possessing both of these properties characterizes Morley rank. We can extend the scope of this result slightly by using different ranks to serve the two functions. However, we lose the ability to demand that p is stationary.

2.9 Exercise. Let T be a small superstable theory. Show T admits **AT**-strongly regular types. (Hint: First choose $a \in N - M$ to minimize the Shelah degree of $t(a; M) = p$. Then choose $\phi(x; \bar{b}) \in p$ with the same degree. Then choose a' satisfying $\phi(x; \bar{b})$ to minimize the Cantor-Bendixson rank in $S(\bar{b})$ of $t(a'; \bar{b})$. Show this is an appropriate choice.)

This exercise may seem illegitimate since our development of K-strongly regular types assumed that T admits K-prime models. However, the proof that weak isolation implies strong regularity in Theorem 1.7 does not require this hypothesis. The current proof of the converse does need the hypothesis.

The previous set of exercises shows that if T is a small superstable theory then T admits **AT**-strongly regular types. However, while one may find $p \in S(M)$ and ϕ over M with (p, ϕ) **AT**-strongly regular, there is no guarantee that there is a finite subset $A \subseteq M$ with $p|A$ stationary and $(p|A, \phi)$ **AT**-strongly regular. Indeed, Example VI.1.33 ii) shows this property may fail.

2.10 Exercise. Find an example to show the result of Exercise 2.9 can not be improved to require p to be stationary. (Hint: Consider the theory REF_ω.)

2.11 Exercise. Contrast the theories CEF_ω and REF_ω with respect to the existence of **AT**-strongly regular types.

The following characterization of K-minimal types requires the existence of regular types and so could not have been proved at the end of Section 1.

2.12 Theorem. *Let $M \in K$ and $p \in S(M)$. If K admits regular types then p is K-minimal if and only if for some K-strongly regular $q \in S(M)$, $p \underset{\sim}{\mapsto} q$.*

Proof. Suppose p is K-minimal. Since K admits regular types we can find a K-strongly regular q realized in $M[p]$. Thus, $p \mapsto q$. But since p is K-minimal, $q \mapsto p$. The converse follows quickly from Theorem 1.13.

We can combine this result with Theorem 1.14 to extend Corollary 1.15 and conclude that all K-minimal types yield isomorphic prime models.

2.13 Exercise. If $M \in K$, $p, q \in S(M)$ are K-minimal and $p \underset{\sim}{\mapsto} q$ then $M[p] \approx M[q]$.

The following construction provides another source of regular types as well as illustrating their connection with strongly minimal sets.

2.14 Definition. The type $p \in S(A)$ is *minimal* if for every formula $\phi(x)$, either $p \cup \phi$ or $p \cup \neg\phi$ has only finitely many solutions.

The following property of minimal types is straightforward.

2.15 Exercise. Every minimal type is regular.

This notion of minimality is quite distinct from K-minimality. Although by Theorem 2.12 and Exercise 2.15 every minimal type is K-minimal the converse is far from true.

2.16 Exercise. Find a regular type which is not minimal (say for **AT**).

2.17 Historical Notes. The existence arguments are derived from Theorem V.3.5 in [Shelah 1978], [Shelah 1982] and [Makkai 1984]. There is a considerably deeper treatment of minimal types in Section V.1 of [Shelah 1978]. They also play an important role in [Buechler 1984a].

3. Some Variants on Transitivity

At the outset of this chapter, we noted that the notion of regularity generalized two properties of strongly minimal sets. So far we have focused on the weak isolation aspect. Now, we consider the property, 'having a well defined dimension'. We showed in Section V.2 that the dimension of a set of indiscernibles is well defined if the set of indiscernibles has at least $\kappa(T)$ elements. To extend this result to sets with finite dimension depends, as in the theory of vector spaces, on a more detailed analysis of the independence notion. But in the theory of vector spaces, the extra care arises from the exchange principle which, in the form of the symmetry axiom, we always have. The extra difficulty arises here in guaranteeing that dependence is transitive. Accordingly we develop in this section, perhaps over pedantically, some variations on the notion of transitivity. Several of these will be seen in later sections to play an important role in our dimension theory. The notion of *transitivity of dependence* was formulated by Van der Waerden as follows.

If a depends on X and each $x \in X$ depends on Y then a depends on Y.

In Chapter II we extended the notion of dependence to deal directly with n-tuples (or, as we saw later, n-element sets). The proper generalization of this axiom is somewhat problematical. The most obvious generalization is

$$\text{If } (\overline{a} \not\downarrow B \cup C; A) \text{ and each } c \in C \text{ satisfies } c \not\downarrow_A B \text{ then } \overline{a} \not\downarrow_A B. \qquad (*)$$

Unfortunately, this version doesn't quite work. The basic problem is that if C is the union of the ranges of a collection of sequences which realize an n-type, we may not have enough information about individual members of C to carry through the proofs. This problem shows up in the proof of Theorem 4.1. The proper notion requires our conventions (see Convention X.1.29) about the treatment of n-tuples and the relation between $p^*(M)$ and $p(M)$.

3.1 Definition. Let p be a type over A. The relation of forking is *fully transitive* on $p(\mathcal{M})$ if for any sets B and C with $B \cup C \subseteq p(\mathcal{M})$ and any $\overline{a}$ realizing p, if $\overline{a} \not\downarrow_A (B \cup C)$ and each $\overline{c} \in C$ satisfies $\overline{c} \not\downarrow_A B$ then $\overline{a} \not\downarrow_A B$.

This would seem a perfectly natural extension of the first version of transitivity to n-tuples, but for one anamoly. Formally, the assertion $\overline{a} \not\downarrow_A (B \cup C)$ makes sense only when $B \cup C$ is thought of as a subset of $p^*(\mathcal{M})$ rather than $p(\mathcal{M})$. However, the symmetry axioms and Theorem II.10 guarantee the equivalence of $\overline{a} \downarrow_A p(M)$ with $\overline{a} \downarrow_A p^* M)$.

3.2 Exercise. Prove the notion of transitivity in Definition 3.1 implies that mentioned in the paragraph above the definition.

3.3 Exercise. Prove that if p is an n-type over A, $\overline{c} \in p^*(\mathcal{M})$, and $\lg(\overline{c}) < n$ then letting $q = t(\overline{c}; A)$, $q^*(\mathcal{M}) \subseteq p^*(\mathcal{M})$.

3.4 Exercise. Prove for p a type over A and $p \subseteq q \in S(A)$ that if forking is fully transitive on $p(\mathcal{M})$ then forking is fully transitive on $q(\mathcal{M})$.

Clearly, by the finite character of dependence we can reduce any question about transitivity to one about finite B and C. In the next lemma we observe that it is enough to check the case where C has one element to prove full transitivity.

3.5 Lemma. *Let $p \in S(A)$. Suppose for any $B \cup \overline{c} \subseteq p(\mathcal{M})$ and for any $\overline{a} \in p(\mathcal{M})$, if $\overline{a} \not\downarrow_A (B \cup \overline{c})$ and $\overline{c} \not\downarrow_A B$ then $\overline{a} \not\downarrow_A B$. Then forking is fully transitive on $p(\mathcal{M})$.*

Proof. Without loss of generality we assume $B \cup C$ is finite and induct on the cardinality of C. The hypothesis of the lemma is the case $|C| = 1$. Suppose $|C| = n + 1$. Let $\overline{d}$ denote the sequence $\langle \overline{c}_0, \ldots, \overline{c}_{n-1} \rangle$. We have $\overline{a} \not\downarrow_A B \cup \overline{d} \cup \overline{c}_n$ and $\overline{c}_n \not\downarrow_A B$. By monotonicity, $\overline{c}_n \not\downarrow_A B \cup \overline{d}$. Now $\overline{a} \not\downarrow_A (B \cup \overline{d})$ follows by using the hypothesis of the lemma with $B \cup \overline{d}$ playing the role of B. By induction, we have $\overline{a} \not\downarrow_A B$ as required.

There is another variant of transitivity which plays an important role.

3.6 Definition. Let $p \in S(A)$. The forking relation is *mildly transitive* on $p(\mathcal{M})$ if for each $\overline{a}$, $\overline{b}$, $\overline{c} \in p(\mathcal{M})$ whenever $\overline{a} \not\downarrow_A \overline{b}$ and $\overline{b} \not\downarrow_A \overline{c}$ then $\overline{a} \not\downarrow_A \overline{c}$.

This definition just asserts that for non-algebraic p forking is an equivalence relation on $p(\mathcal{M})$. Note that if $p(\mathcal{M})$ is mildly transitive then for any M we can assign an invariant to $p(M)$: the number of pairwise independent realizations of p in M. The exact properties of this invariant are unclear. It is unclear, for example, that if we defined a notion of dimension in terms of the maximal number of pairwise independent realizations of a type on which forking is mildly transitive whether the vitally important Theorem 4.4 (below) would still hold.

We have introduced these notions as variations of transitivity in this section to indicate one way in which they are related. Later we will revert to the more standard usage, p is regular if forking is fully transitive on $p(\mathcal{M})$; p has weight one if forking satisfies a condition slightly stronger than mild transitivity on $p(\mathcal{M})$. We will show that the term 'weight one', at least, is extremely descriptive. The next lemma formulates the minimal amount of transitivity necessary for a well defined dimension.

3.7 Definition. Let X be a set of n-tuples. The forking relation on X is *weakly transitive over* A if for any $\overline{a}, B, C$ contained in X the following condition holds. If B and C are each independent sets over A and if each $\bar{c} \in C$ satisfies $\bar{c} \not\downarrow_A B$ and $\bar{a} \not\downarrow_A (B \cup C)$ then $\bar{a} \not\downarrow_A B$.

If $A = \emptyset$ we just write 'weakly transitive'.

3.8 Exercise. Show that full transitivity implies weak transitivity implies mild transitivity.

We show now that weak transitivity implies that dimension is well defined. This proof is exactly as in linear algebra. We just note that the application of transitivity uses only weak transitivity. To ease readability we prove this theorem for X such that forking is weakly transitive over the empty set. The extension to an arbitrary A is routine.

3.9 Theorem. *Suppose the forking relation on X is weakly transitive over the empty set. For any Y and Z which are maximal independent subsets of X, $|Y| = |Z|$.*

Proof. By the finite character of forking the result is obvious unless one of Y and Z is finite. Suppose $|Z| < \omega$. We show by induction on $|A|$ for $A \subseteq Y$ that there exists a subset B of Z such that $|B| = |A|$ and, letting Z_0 denote $Z - B$, that $A \cup Z_0$ is a maximal independent set. This implies $|Y| \leq |Z|$. If $A = \emptyset$ there is nothing to prove. Suppose we have the result for $|A| \leq n$ and suppose $A = A' \cup \{a\}$ with $|A'| \leq n$. By induction, there exists B' with $|B'| = |A'|$ such that, letting Z_0' denote $Z - B'$, $A' \cup Z_0'$ is a maximal independent set. Thus $a \not\downarrow A' \cup Z_0'$. By the generalized symmetry principle (Corollary II.1.12), letting Z_b denote $Z_0' - \{b\}$, for any $b \in Z_0'$, $b \not\downarrow (A' \cup Z_b \cup \{a\})$. If $a \not\downarrow A' \cup Z_b$ then by weak transitivity, $b \not\downarrow A' \cup Z_b$ contradicting the induction hypothesis. (We don't need full transitivity since a and $A' \cup Z_b$ are separately independent.) Thus $A' \cup Z_b \cup \{a\}$ is independent as required.

The following example illustrates the difference between these variants on transitivity.

3.10 Example. Let T be the theory of the group $Z_4^{\aleph_0}$. Let r be the type over the empty set asserting x has order 4. Let M be a countable model of T and let N be $M \oplus Z_4 \oplus Z_4$. Consider the following three elements of N

	M	Z_4	Z_4
a	0	1	0
b	0	1	2
c	0	0	1

Now $c \not\downarrow_M a^\frown b$ and $b \not\downarrow_M a$ but $c \downarrow_M a$ so forking is not fully transitive nor even weakly transitive on $r(\mathcal{M})$. It is mildly transitive.

Our goal, of course, is to be able to attach a dimension to large subsets of models of T (large enough eventually so that they determine the model). The next lemma separates the part of our construction which is on the level of linear algebra from that which is model theory. We will apply this lemma in Section XIII.1.

3.11 Lemma. *Let X be a subset of $\mathcal{M}^n$. Suppose that there is an equivalence relation E on X such that*

i) *Forking over A is fully transitive on each equivalence class of E.*
ii) *If $I \subseteq X$ and $\bar{a} \not\downarrow_A I$ then $\bar{a} \not\downarrow_A I_{\bar{a}}$ where $I_{\bar{a}} = \{\bar{b} \subseteq I : \bar{b} E \bar{a}\}$.*

Then forking over A is weakly transitive on X.

Proof. Let $I, J \subseteq X$ be separately independent over A. Suppose $\bar{a} \not\downarrow_A (I \cup J)$ and for each $\bar{b} \subseteq I$, $\bar{b} \not\downarrow_A J$. By ii) $\bar{a} \not\downarrow_A (I_{\bar{a}} \cup J_{\bar{a}})$. Clearly, if $\bar{b} \in I_{\bar{a}}$ then $\bar{b} E \bar{a}$ implies $\bar{b} \not\downarrow_A J_{\bar{a}}$. Since forking is fully transitive on the equivalence class of $\bar{a}$, $\bar{a} \not\downarrow_A J_{\bar{a}}$ and the result is immediate by monotonicity.

With these notions of transitivity in hand we can draw some conclusions about the dimensions of types.

3.12 Definition. Let $p \in S(A)$ and $A \subseteq M$. Then $\dim(p, M)$ is the cardinality of any maximal independent set of realizations of p in M.

The following result is obvious from the considerations in Theorem 3.9.

3.13 Theorem. *If forking is weakly transitive on $p(\mathcal{M})$ then for any M which contains* $\operatorname{dom} p$, $\dim(p, M)$ *is well defined.*

3.14 Historical Notes. This section was motivated by Shelah's loose assertion in V.1 of [Shelah 1978] that on the realizations of regular types forking satisfies the axioms for vector spaces. The actual result proved is that the other axioms and weak transitivity (rather than full transitivity) are satisfied. The intent was that invariance of dimension holds. These distinctions were explored in [Baldwin 1984].

4. Strongly Regular Types and Compulsion

In this section we combine the two views of regularity discussed in Sections 1 and 3 to establish the role of orthogonality between K-strongly regular types in the ordering $\mapsto_K$. The first theorem establishes that we have been looking at the same notion from two viewpoints. Although the assumption that forking is fully transitive on $p(\mathcal{M})$ seems to be a purely local assumption (dealing only with realizations of p), it has consequences, e.g. those in Definition 1.1 of regularity, involving points which do not realize p.

4.1 Theorem. *Let $p \in S(A)$. The following are equivalent.*

i) *p is a regular type.*
ii) *Forking is fully transitive on $p(\mathcal{M})$.*

Proof. To show ii) implies i), it suffices, as in the proof of Theorem 1.7, to show that for any $\mathbf{S}$-saturated M with $A \subseteq M$, if $\bar{a}$ realizes p then $\bar{a} \downarrow_A M$ or $t(\bar{a}; M) \perp p$. By Theorem VI.1.37, if $t(\bar{a}; M) \not\perp p$ there is a $\bar{b}$ realizing p such that $\bar{b} \downarrow_A M$ but $\bar{a} \not\downarrow_M \bar{b}$. By Axiom FI_3 and Proposition

X.1.37, $(\overline{a}^\frown\overline{b}\downarrow M; A \cup p^*(M))$. Now if $\overline{a} \downarrow_A A \cup p^*(M)$ then $\overline{a} \downarrow_A M$ by transitivity of independence and we finish. But if $\overline{a} \not\downarrow_A A \cup p^*(M)$ note that since $(\overline{a}^\frown\overline{b}\downarrow M; A \cup p^*(M))$ we have $(\overline{b}\downarrow M \cup \overline{a}; A \cup p^*(M) \cup \overline{a})$. Hence if $\overline{b} \downarrow_A A \cup p^*(M) \cup \overline{a}$, we conclude by transitivity of independence that $\overline{b} \downarrow_A M \cup \overline{a}$, contrary to the hypothesis that $\overline{a} \not\downarrow_M \overline{b}$. We are thus left with the case $\overline{b} \not\downarrow_A A \cup p^*(M) \cup \overline{a}$ and $\overline{a} \not\downarrow_A A \cup p^*(M)$ but then invoking the transitivity of dependence we have $\overline{b} \not\downarrow_A A \cup p^*(M)$. But $\overline{b} \downarrow_A M$ and this contradiction yields the first assertion.

To show i) implies ii) we show for any $B \cup \{\overline{a}, \overline{c}\} \subseteq p(M)$, if $\overline{c} \not\downarrow_A B$ and $\overline{a} \not\downarrow_A (B \cup \overline{c})$ then $\overline{a} \not\downarrow_A B$. To see this note that $\overline{c} \not\downarrow_A B$ implies by Definition 1.2 that $t(\overline{c}; B) \perp p$. Thus, if $\overline{a} \downarrow_A B$ (i.e. $t(\overline{a}; B) \parallel p$) we have $t(\overline{a}; B) \perp t(\overline{c}; B)$. In particular, $\overline{a} \downarrow_A (\overline{c} \cup B)$. We now deduce ii) from Lemma 3.5.

In the preceding proof we were careful to follow the notation of Chapter X which distinguishes between $p^*(M)$ and $p(M)$. Hereafter, we often take advantage of the remark after Definition and 3.1 and identify these two sets when speaking of independence. Note that this circumlocution is not formally correct if we speak of implication rather than independence. (Recall that if $a \downarrow_M b$ and $t(a; A)$ is stationary $t(a; A) \vdash t(a; A \cup b)$.)

The following version of regularity extends transitivity to domains which are not contained in $p(\mathcal{M})$.

4.2 Exercise. If $p \in S(A)$ is regular, $A \subseteq B$, $\overline{a} \cup X$ is a set of realizations of p such that $\overline{a} \not\downarrow_B X$ and for each $\overline{x} \in X$, $\overline{x}$ depends on B over A then $\overline{a} \not\downarrow_A B$. (Note this also holds if we replace $\overline{a} \not\downarrow_B X$ in the hypothesis by $\overline{a} \not\downarrow_A X \cup B$.)

We would like to have an analogue of Theorem 4.1 for strong regularity. Our success depends on the direction of the implication. Part i) of the following exercise verifies that one half of Theorem 4.1 (regularity implies transitivity) does go through for any acceptable K. This can be easily deduced from Theorem 1.12. In fact, this part of the proof of Theorem 4.1 applies to any class K such that if $p, q \in S(M)$ and $M \in K$ then $p \not\perp q$ implies $p \not\perp^a a$. The converse is harder; it is difficult to formulate a smooth characterization of strong regularity in terms of transitivity. The suggestion of Shelah embodied in part ii) of Exercise 4.3 seems to be the best.

4.3 Exercise. i) Show that if $p \in S(A)$ and (p, q) is K-strongly regular then forking is fully transitive on $p(\mathcal{M})$.

ii) Show (p, q) is K-strongly regular iff the dependence relation on $q^*(\mathcal{M})$ defined by '$\overline{a}$ depends on B over A if and only if $t(\overline{a}; A) \neq p$ or $(\overline{a} \not\downarrow B; A)$' is fully transitive.

Now we use both weak isolation and full transitivity to prove an important theorem about strongly regular types.

4.4 Theorem. *Let $M, N \in K$ and $p \in S(A)$ with $A \subseteq M \subseteq N$. Suppose that (p, p_0) is K-strongly regular. If C is a set of realizations of p_0 which* *is independent over M but $C \not\downarrow_A M$ then $C \not\downarrow_A p(M)$. Moreover,*

$$\dim(p, N) = \dim(p, M) + \dim(p^M, N).$$

Proof. (Fig. 3). If the first assertion fails, Proposition X.1.37 yields that $C \not\downarrow_A p_0(M)$ but $C \downarrow_A p(M)$. Without loss of generality, C is finite. Thus, for some finite $D \subseteq p_0(M) - p(M)$, $C \not\downarrow_{A \cup p(M)} D$. We show this is impossible by induction on the cardinality of the set of $\lg(p)$-tuples D. By the definition

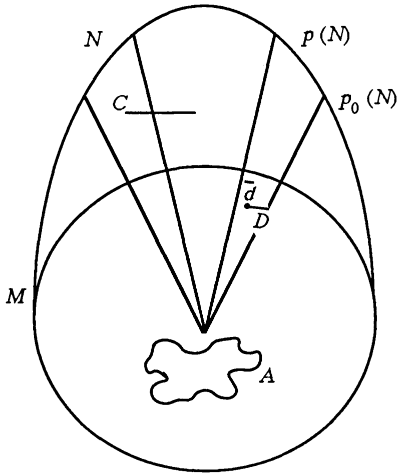

Fig. 3. Lemma XII.4.4.

of strong regularity, for each $\overline{d} \in D$, $t(\overline{d}; A \cup p(M)) \perp p^{A \cup p(M)}$. Invoking the triviality of orthogonality and the fact that C is independent over M, this establishes the result if $|D| = 1$. Suppose the conclusion holds when $|D| \leq k$ and consider D with $|D| = k + 1$, say, $D = D' \cup \overline{d}$ and D' satisfies the induction hypothesis. Since $\overline{d}$ does not realize p, the definition of strong regularity implies that $\overline{d} \not\downarrow_A A \cup p(M) \cup D'$ and that $t(\overline{d}; A \cup p(M) \cup D') \perp p$. By the triviality of orthogonality we have $t(\overline{d}; A \cup p(M) \cup D') \perp t(C; A \cup p(M) \cup D')$ and we finish.

To prove the final statement, just extend a basis, X, for $p(M)$ to a basis, $X \cup Y$, for $p(N)$. We now show the new elements are a basis for $p^M(N)$. Since forking is fully transitive on $p(N)$, $Y \downarrow_A X$ implies $Y \downarrow_A p(M)$. By the first conclusion of the theorem, this implies $Y \downarrow_A M$ and we finish.

Theorem 1.13 showed that K-strongly regular types are K-minimal. Thus, two K-strongly regular types are either incomparable or equivalent modulo $\mapsto_K$. The next theorem and its corollaries provide a more syntactic description of this relation.

4.5 Theorem. *Let M be* **S***-saturated and suppose $p \in S(M)$ is a regular type. If $q \in S(M)$ and $q \not\perp p$ then $q \mapsto_S p$. Thus $q \rhd p$.*

Proof. Let $\bar{b}$ realize q and let N be **S**-prime over $M \cup \bar{b}$. Choose $A \subseteq M$ with $|A| < \lambda(I)$, such that p is strongly based on A and let $p_1 = p|A$. Since $p \not\perp q$, and over **S**-models $\perp^w$ implies $\perp$ (Theorem VI.1.40), there is $\bar{a}$ realizing p such that $\bar{a} \not\downarrow_M (M \cup \bar{b})$ and by monotonicity $\bar{a} \not\downarrow_M N$. By Axiom FI_3, $p|(A \cup p_1^*(N)) \vdash p$. So $(\bar{a} \not\downarrow A \cup p_1^*(N); A)$. Let I be a maximal independent subset of $p_1^*(N)$ such that $I \cap M$ is a maximal independent subset of $p_1^*(M)$. Since p (and therefore p_1) is regular, $\bar{a} \not\downarrow_A (A \cup I)$. If $I \cap (N - M) = \emptyset$ then $\bar{a} \not\downarrow_A M$, contrary to hypothesis. Thus there exists an $\bar{a}' \in I \cap (N - M)$. Now $\bar{a}' \downarrow_A (I \cap M)$ implies by the choice of $I \cap M$ and regularity again that $\bar{a}' \downarrow_A (A \cup p_1^*(M))$ and thus using FI_3 once more that $\bar{a}' \downarrow_A M$. Since p_1 is stationary, this yields $\bar{a}'$ realizes p as required.

Note that this argument gives a different proof (for the **S**-case) of Theorem 1.13: if $p \in S(M)$, M is **S**-saturated and p is regular then p is S-minimal.

The following exercise points out a slightly stronger version of Theorem 4.5 which we actually established in the preceding proof.

4.6 Exercise. Let M be **S**-saturated and suppose $p \in S(M)$ is a regular type, $q \in S(M)$, and $q \not\perp p$. If $\bar{a}$ realizes p, $\bar{b}$ realizes q, and $\bar{a} \not\downarrow_M \bar{b}$ then there is an $\bar{a}' \in M[\bar{b}] - M$ which realizes p and satisfies $\bar{a} \not\downarrow_M \bar{a}'$.

The next variant of Theorem 4.5 will be very useful in applications.

4.7 Exercise. Let $p \in S(A)$ and $q \in S(B)$. If $q \not\perp p$ and p is regular then $q \rhd^e p$.

Now comparing the models $M[p]$, $M[q_1]$, and $M[q_2]$ quickly yields

4.8 Corollary. i) *Suppose q_1 and q_2 are regular and $q_1 \not\perp q_2$. Then for any p, $p \perp q_1$ if and only if $p \perp q_2$.*

ii) *Nonorthogonality is the equivalence relation induced on the class of regular types by the relation $\underset{\sim}{\mapsto}_S$.*

Proof. We prove only i) as ii) follows immediately from i). Without loss of generality we may assume p, q_1, and q_2 are in $S(M)$ for some **S**-saturated M. If $p \not\perp q_1$, then by Theorem 4.5, q_1 is realized in $M[p]$. Since $q_1 \not\perp q_2$, q_2 is realized in $M[q_1]$ which without loss of generality is a submodel of $M[p]$. But then by Exercise X.1.21 there are realizations $\bar{a}$, $\bar{b}$ of p and q_2 with $(\bar{a} \not\downarrow \bar{b}; M)$. Thus $p \not\perp q_2$.

4.9 Corollary. *If q is regular then for any p and r, $p \not\perp q$ and $r \not\perp q$ implies $p \not\perp r$.*

Proof. Suppose not. Without loss of generality, p, q and $r \in S(M)$ for some **S**-model M. Let $\bar{a}$, respectively $\bar{b}$, realize p and r. By Theorem 4.5, q is realized by some $\bar{c} \in M[\bar{a}] - M$ and some $\bar{c}' \in M[\bar{b}] - M$. Let α map $\bar{c}'$ to $\bar{c}$ fixing M. Then $\alpha(\bar{b})$ realizes r and $\bar{c} \in M[\bar{a}] \cap M[\alpha(\bar{b})]$. But, $p \perp r$ implies

$\alpha(\bar{b}) \downarrow_M \bar{a}$ which implies $M[\alpha(\bar{b})] \downarrow_M M[\bar{a}]$ which implies $M[\alpha(\bar{b})] \cap M[\bar{a}] = M$. This contradiction yields the result.

To extend these results to K-strongly regular types for arbitary acceptable classes K, we must extend the result that weak orthogonality implies orthogonality from types over **S**-models to types over K-saturated models. We do that here for stationary K-strongly regular types. This allows us to conclude the analog of Theorem 4.5 and Corollary 4.8.ii).

4.10 Theorem. *Suppose $M \in K$, $p \in S(M)$, and (p, p_0) is K-strongly regular. If $q \in S(M)$ is K-strongly regular, $q|\operatorname{dom}^*(q_0)$ is stationary, and $q \not\perp p$ then $q \mapsto_K p$ and $p \not\perp^w q$.*

Proof. Let $A = \operatorname{dom}^*(q_0)$. Since $p \not\perp q$, there is a $D \supseteq M$ and $\bar{a}$, $\bar{b}$ realizing nonforking extensions of p and q to $S(D)$ such that $(\bar{a} \not\downarrow \bar{b}; D)$. By Corollary X.1.22 there is a $\bar{b}' \in q_0(M[\bar{a}])$ such that $(\bar{b}' \not\downarrow \bar{a}; M)$. Let $r = t(\bar{b}'; M)$. Then r is a regular type, so either $r \perp q|A$ or r is a nonforking extension of the stationary type $q|A$. In the second case, $\bar{b}'$ realizes q and witnesses both $p \not\perp^w q$ and $p \underset{\sim}{\mapsto}_K q$. But, $r \perp q|A$ is impossible. For, we have $r \not\perp p$ and $p \not\perp q|A$, since nonorthogonality is a parallelism invariant. Thus, by Corollary 4.8 i), $r \not\perp q|A$.

The contrapositive of this result yields: Suppose $p, q \in S(M)$ and p_0, q_0 are over A with (q, q_0) and (p, p_0) K-strongly regular and $q|A$ is stationary. If $p \perp^w q$ then $p \perp q$.

Note that we invoked Theorem 4.8 i) and thus, indirectly, Theorem 4.5 in the proof of Theorem 4.10. We are unable to give a uniform proof of this result for all acceptable classes K but must first prove the result for **S** and then transfer it to the other classes. This theme resounds through the rest of this book.

The proof of Theorem 4.10 did not require that K admit regular types. Thus, it applies for example to an arbitrary model and two stationary **AT**-strongly regular types in a superstable theory. Note, however, that in this situation the hypothesis that $q|A$ be stationary is not automatically satisfied. The extension of Theorem 4.10 to obtain the full analog of Theorem 4.5 depends on K admitting stationary regular types. With this hypothesis we will deduce the result for arbitrary types (with domain in K) in the next chapter and this will allow us to obtain the analog of Theorem 4.8 ii). Even without this hypothesis one can extend Theorem 4.8.ii) to any acceptable K.

4.11 Corollary. *Nonorthogonality is the equivalence relation induced on the class of K-strongly regular types by the relation $\underset{\sim}{\mapsto}_K$.*

4.12 Exercise. Prove Corollary 4.11.

While our definition of a pair (p, ϕ) being **AT**-strongly regular allows p to fork over $\operatorname{dom}\phi$, the following exercise shows that has a powerful consequence. The exercise follows easily from Corollary 4.11 and the fact (established in Exercise 2.14 and Theorem 2.15 that in an ω-stable theory questions of orthogonality can be reduced to regular types.

4.13 Exercise (Buechler). Let T be ω-stable, $M \models T$, and $p \in S(M)$. Suppose that for some $\overline{c} \in M$ and some formula $\phi(x;\overline{c})$ the pair (p,ϕ) is **AT**-strongly regular. Show that if p forks over $\overline{c}$ then $p \dashv \overline{c}$.

4.14 Historical Notes. This entire chapter is an elaboration of Theorem V.1.19 of [Shelah 1978] which gives five equivalent forms of the definition of a regular type. We have separated these various notions to indicate the roles that the various formulations play. In addition, we have integrated the treatment of strongly regular types. The proof that weak orthogonality implies orthogonality for K-strongly regular types is derived from [Lascar 1982a]. We have not discussed the notion of a *semi-regular* type which is defined in Section V.4 of [Shelah 1978] and plays an important role in extending the study to strictly stable theories. Hrushovski [Hrushovski 1986] has extensively exploited this notion.