Chapter IV

Finite Equivalence Relations, Definability, and Strong Types

We study in this chapter the nonforking extensions of a type $p \in S(A)$. In the first section we see that although two such types cannot be distinguished by formulas over A, they can be distinguished by formulas that are 'almost over' A. This leads to a straightforward proof of Lachlan's theorem that an $\aleph_0$-categorical superstable theory is ω-stable. In the second section we study in more detail the multiplicity of p - the number of nonforking extensions of p. We apply this analysis to reformulate the stability hierarchy in terms of definability. We introduce the important notions of the 'base' and the 'strong base' of a type. We show each type over a strongly $\kappa(T)$-saturated model is strongly based on a subset of power less than $\kappa(T)$. In Section 3 we summarise the fundamental properties of strong types. The strong type of a over A allows one to analyze the relation of a and A in terms of a stationary (i.e. multiplicity one) type. This is an essential tool for the further development of the theory. Finally we study the relation between the strong type (more generally, the multiplicity) of a pair and that of its components.

1. Finite Equivalence Relations

Unfortunately, the study of types will not provide as smooth a theory of independence as we would like. A simple example of the difficulty arises if we consider the theory of an equivalence relation with two infinite classes. Then the unique type, p, over the empty set has two extensions to a type over a model which do not fork over the empty set; namely, the type of a generic point in either equivalence class. Morley addressed this problem by saying 'the type has degree two' and the work of Lachlan [Lachlan 1975], [Lachlan 1978], successfully develops this approach. However, we find it more convenient to follow Shelah in introducing the notion of strong type. This approach describes the situation in the above example, by saying 'there are two strong types extending p', one for each equivalence class.

This approach also deals with another issue. Let A be a set and suppose $\overline{b}$ is not in A, but is algebraic over A. Consider a formula $\phi(\overline{x};\overline{b})$. Although $\phi(\overline{x};\overline{b})$ is not in $F(A)$, clearly it is 'close' to being so. We will formalize a generalization of this property by defining the notion that $\phi(\overline{x};\overline{b})$ is 'almost over' A.

Recall that for $X \subseteq \mathcal{M}$ we say X is definable over A (i.e. definable with parameters from A), or X is just over A if there is a formula $\phi(\overline{x};\overline{y})$ and a sequence $\overline{a}$ from A such that $X = \phi(\mathcal{M};\overline{a})$. Although all three expressions for this concept are commonly used we will eschew here this usage of 'definable over A' so as to distinguish this notion from the notion of a type being 'definable over A' considered in III.1 and below. We will say $X \subseteq \mathcal{M}$ is definable if it is definable with some parameters from $\mathcal{M}$ (i.e. if X is over $\mathcal{M}$). We recall from Chapter I a useful criterion for determining whether X is over A.

I.2.30 Lemma. *If X is a definable subset of $\mathcal{M}$ and X is fixed by every A-automorphism of $\mathcal{M}$ then X is over A.*

Recall that p is definable over A if for each $\phi(\overline{x};\overline{y})$ the set $\{\overline{c}{:}\phi(\overline{x};\overline{c}) \in p\}$ is over A. We distinguish the notions, 'p is over A' and 'p is definable over A' even though for stable T and $p \in S(A)$ they coincide. Generalizing the notion, 'X is over A', we replace the requirement that X be fixed by A-automorphisms with the requirement that it have 'few' conjugates. Since we are working in the monster model, a cardinal number is small or large depending on whether it is less than the cardinality of the monster model. In analogy with set theory, we say a collection (e.g. of formulas with parameters or of types) 'is a set' or 'has bounded cardinality' if its cardinality is less than that of the monster model and a class if its cardinality is the same as the cardinality of the monster model. Formally:

1.1 Definition. i) Let X be a definable subset of $\mathcal{M}$. Then X is *almost over A* if $|\{\alpha(X) : \alpha \in \mathrm{Aut}_A(\mathcal{M})\}| < \omega$.

ii) The formula $\phi(\overline{x};\overline{b})$ is *almost over A* if the set $\phi(\mathcal{M},\overline{b})$ is almost over A.

iii) The type p is *almost over A* if each formula in p is almost over A.

Note that if $\overline{b}$ is in the algebraic closure of A then for each formula $\phi(\overline{x};\overline{y})$, $\phi(\overline{x};\overline{b})$ is almost over A. Further, if T is the theory of an equivalence relation E with two classes, both infinite, then for any b, $E(x;b)$ is almost over the empty set. Virtually no complete type (not over A) is almost over A since if $b \notin \mathrm{cl}(A)$ the formula $x \neq b$ is not almost over A. However, the members of a very important class of incomplete types, the strong types over A (cf. Definition 3.1), are almost over A.

The following characterization (Theorem 1.3) underlines the significance of this notion.

1.2 Definition. i) A *finite equivalence relation* is a definable equivalence relation on $\mathcal{M}^m$ for some m which has only finitely many classes. We say E is over A if E is definable with parameters from A.

ii) The formula $\phi(\overline{x};\overline{b})$ *depends on the finite equivalence relation* E if the solution set of ϕ is a union of equivalence classes of E.

We denote by $FE^m(A)$ the collection of finite equivalence relations (on m-tuples) which are over A. We write $FE(A)$ to refer ambiguously to some $FE^m(A)$.

1.3 Theorem. *The following are equivalent.*

i) $\phi(\overline{x};\overline{c})$ *is almost over* A.
ii) $\phi(\mathcal{M};\overline{c})$ *has a bounded number of conjugates over* A.
iii) *There is an* $E \in FE(A)$ *and* $\overline{b}_0,\ldots,\overline{b}_{k-1}$ *such that:*

$$\phi(\overline{x};\overline{c}) \leftrightarrow \bigvee_{i<k} E(\overline{x};\overline{b}_i).$$

iv) $\phi(\mathcal{M};\overline{c})$ *is over* M *for every model* M *containing* A.

Proof. It is easiest to first argue that ii) and iii) are separately equivalent to i). Clearly i) $\rightarrow$ ii) and the converse is an easy compactness argument. If iii) holds, each conjugate of $\phi(\overline{x};\overline{c})$ is equivalent to a union of equivalence classes of E so it is easy to bound the number of conjugates in terms of the number of classes of E. Now we show i) $\rightarrow$ iii). Define $E(\overline{x};\overline{y})$ to hold just if $\phi(\overline{x};c_i) \leftrightarrow \phi(\overline{y};c_i)$ for each of the formulas $\phi(\overline{x};c_i)$ representing the finitely many conjugates of $\phi(\overline{x};\overline{c})$. Then E is a finite equivalence relation, definable with parameters from $\mathcal{M}$, and fixed under A-automorphisms. By Lemma I.2.30, $E \in FE(A)$. Now we show iii) implies iv) and iv) implies ii). By iii), the solution set in $\mathcal{M}$ of $\phi(\bar{x};\bar{c})$ is the union of equivalence classes of a finite equivalence relation over A. Thus for any M containing A there are k sequences $\overline{m}_i$ from M such that: $\models \phi(\overline{x};\overline{c}) \leftrightarrow \bigvee_{i<k} E(\overline{x};\overline{m}_i)$ and iv) holds. On the other hand, if $\phi(\overline{x};\overline{c})$ is over every M containing A, so are all the conjugates of $\phi(\overline{x};\overline{c})$. That is, all conjugates of $\phi(\overline{x};\overline{c})$ are over a single model M so $\phi(\overline{x};\overline{c})$ has a bounded, hence finite, number of conjugates and is almost over A.

Thus ϕ is almost over A iff ϕ depends on some $E \in FE(A)$.

The fourth characterization in the previous theorem reinforces a second meaning for the word 'almost' which we have seen before. Recall from Definition III.3.13 that $\phi(\overline{x};\overline{c})$ is said to be almost satisfiable (or almost satisfied) in A if for every M extending A, $\phi(\mathcal{M};\overline{c}) \cap M \neq \emptyset$. A property P is almost true in A if it is true in every model containing A. We will consider several other uses of the word almost in this sense.

1.4 Exercise. Show that the collection of formulas almost over A is closed under the logical operations (Boolean and quantifiers), permutation of variables and substitution of dummy variables. (This is a routine application of Theorem 1.3. Details are in III.2.2 of [Shelah 1978].)

1.5 Corollary. *If the consistent type* p *is almost over* A *then each formula in* p *is almost satisfiable in* A *and so* p *does not fork over* A.

Proof. By Theorem 1.3 each $\phi(\overline{x};\overline{c})$ in p defines the union of some equivalence classes of a finite equivalence relation over A. But each class must intersect every model containing A so p is almost satisfiable in A.

1.6 Exercise. If $A \subseteq B$, $p \in S(B)$ does not fork over A, and r is almost over A, show that if $r \cup p$ is consistent then $r \cup p$ does not fork over A.

1.7 Exercise. Show that $p \in S(A)$ is stationary iff every $\phi(\overline{x};\overline{y})$ represented in $\hat{p}$, the bound of p, is represented in p in the following weak sense. If $q \in [\hat{p}]$ and $\phi(\overline{x};\overline{b}) \in q$ there is a formula $\psi(\overline{x};\overline{a}) \in p$ such that $\models \phi(\overline{x},\overline{b}) \leftrightarrow \psi(\overline{x};\overline{a})$ (Baldwin-Prest).

1.8 Exercise. Suppose $\phi(\overline{x};\overline{b})$ is almost over A and $\phi(\overline{x};\overline{b}_1),\ldots,\phi(\overline{x};\overline{b}_n)$ is a list of representatives of all the A-conjugates of $\phi(\overline{x};\overline{b})$. Show that $\bigwedge_{i\leq n}\phi(\overline{x};\overline{b}_i)$ and $\bigvee_{i\leq n}\phi(\overline{x};\overline{b}_i)$ are each equivalent to a formula over A.

1.9 Exercise. Show that if the finite equivalence relation E is almost over A, then there is another finite equivalence relation E' which is over A such that E depends on E'. (Hint: Consider the conjunction of the conjugates of E.)

It is immediate from Theorem 1.3 that there are at most $2^{(|T|+|A|)}$ distinct types almost over A. In Lemma 1.12 we will reduce this bound to $\sup(2^{|T|},|S(A)|)$. Now we combine the concepts 'defining a type' and 'almost over' by saying that p is definable almost over A if p is definable in the sense of III.1.23 but the formulas $d\phi$ are almost over A.

1.10 Definition. The type p is *definable almost over A* if for each $\phi(\overline{x};\overline{y})$, $\{\overline{c} : \phi(\overline{x};\overline{c}) \in p\}$ is almost over A.

It is immediate from the definitions and the characterization of almost over in Theorem 1.3 iv) that for any global type $\hat{p}$, $\hat{p}$ does not fork over A if and only if $\hat{p}$ is definable almost over A. We gave in Exercise III.3.28 an example of an incomplete type which was definable (and so certainly almost definable) over the empty set but still forked over the empty set. Thus this remark does not extend from global types to arbitrary types. Exercise 1.15 notes that in one direction this extension is possible; Exercise 1.16 points out a strengthening of the hypothesis which regains the converse.

1.11 Exercise. Show that $\hat{p}$ is definable almost over A if and only if for each formula $\phi(\overline{x};\overline{y})$, $d\phi$ is almost over A.

The following lemma yields another characterization of the nonforking extensions of a type p.

1.12 Lemma. *If $\hat{p}$ does not fork over A then $\hat{p}$ has at most $2^{|T|}$ conjugates over A.*

Proof. Since $\hat{p}$ does not fork over A, there is a model M and a map $d : F(L) \mapsto F(\mathcal{M})$ so that $\hat{p}$ is defined by d and each $d\phi$ is almost over A. Then each conjugate of $\hat{p}$ is defined by conjugates of the $d\phi$. But each of the $|T|$ formulas $d\phi$ has only finitely many conjugates over A, so $\hat{p}$ has at most $2^{|T|}$ conjugates over A.

1.13 Exercise. Give a proof of Lemma 1.12 without appealing to definability. (Hint: First assume without loss of generality that $|A| < \kappa(T)$, then choose a model of cardinality at most $|T|$ containing A and note that the nonforking types over A are determined by their restrictions to that model.)

We can show a strong form of the converse.

1.14 Lemma. *If $\hat{p}$ has a bounded number of conjugates over A then $\hat{p}$ does not fork over A.*

Proof. If $\hat{p}$ has a bounded number of conjugates over A then for each ϕ, $d\phi$ has a bounded number of conjugates over A. So $\hat{p}$ is definable almost over A and so does not fork over A.

For any p we have shown the extensions of p to types in $S(\mathcal{M})$ are partitioned into conjugacy classes by the fundamental order (Corollary III.2.38). There are at most $2^{(|T|+|A|)}$ classes. One of them, the class of nonforking extensions has at most $2^{|T|}$ members. The others have a class (i.e. $|\mathcal{M}|$) of members.

1.15 Exercise. Let $A \subseteq B$ and suppose p is a type over B such that $p|A$ is complete and p does not fork over A. Show that if d is any defining scheme for p then each formula $d\phi$ is almost over A.

1.16 Exercise. Show that if $p \in S(M)$ is definable almost over A then p does not fork over A. Give an example to show the necessity of assuming M is a model.

We have shown that if p doesn't fork over A then p is definable almost over A. We can strengthen this result if $p|A$ is stationary or even stationary relative to $\operatorname{dom} p$ in the following sense.

1.17 Definition. i) Let $A \subseteq B$ and $p \in S(A)$. Then we say p is *stationary inside* B if p has a unique extension in $S(B)$ which does not fork over A.

The remainder of this definition makes precise two ways in which a type can be founded on a 'small' set. These notions will play an increasingly important role later on. We discuss them more fully after Definition 2.4.

ii) Let $A \subseteq B$ and $p \in S(B)$. Then

a) p is *based* on A if p does not fork over A.
b) p is *strongly based* on A if p does not fork over A and $p|A$ is stationary.

WARNING: Our terminology here differs with that of Shelah. He only defines 'based' for a type which is the average type of a sequence of indiscernibles and does so in such a way that his 'based' is equivalent to our 'strongly based'.

Note that if the set B in Definition 1.17 i) is a model then the type p is, in fact, stationary. (See Theorem III.2.23.)

1.18 Exercise. Give an example of $A \subsetneq B$ with $p \in S(A)$ which is stationary inside B but not stationary.

1.19 Exercise. Show that if $p \in S(A)$ is stationary then for every $B \supseteq A$, p is stationary inside B.

1.20 Lemma. i) *If $A \subseteq B$, $p \in S(B)$ does not fork over A, and $p|A$ is stationary inside B then p is definable over A.*

We will be most interested in the following special case.

ii) (Stationary types are definable). *Suppose $p \in S(A)$ is stationary. There is a definition d of p over A such that for every B containing A, if $p \subseteq q \in S(B)$ and q does not fork over A then $q = d(p, B)$.*

Proof. Let $\hat{p}$ be an extension of p to a global type which does not fork over B (and hence does not fork over A). Then $\hat{p}$ is definable almost over A by some map d taking ϕ to $d\phi(\overline{y}; \overline{c})$. Let $d^*\phi(\overline{y})$ be $\bigvee_{\alpha \in \mathrm{Aut}_A(\mathcal{M})} d\phi(\overline{y}; \alpha\overline{c})$. Then $d^*\phi(\overline{y})$ is invariant under automorphisms which fix A and so is equivalent by Theorem I.2.30 to a formula over A (thus justifying our omission of parameters). But since $d^*\phi(\mathcal{M}) \cap B$ is $\{\overline{b} : \phi(\overline{x}; \overline{b}) \in p\}$ we have defined p over A as required.

Note that the $d^*\phi$ found in the proof of Lemma 1.20 is a positive Boolean combination of instances of $d\phi$. We replace $d\phi$ by ϕ in Lemma V.2.6.

1.21 Exercise (Berline). Show $d^*\phi(\overline{c})$ holds if and only if $\phi(\overline{x}; \overline{c})$ is in every nonforking extension of p to a global type.

1.22 Exercise. Prove 1.20 ii) directly rather than deducing it from 1.20 i).

The following table summarises the relations among the notions considered in this section. In each column there are a number of statements equivalent to the assertion that a global type has the number of conjugates over A given at the head of the column. Each row represents a particular kind of description.

1	$\leq 2^{\lvert T \rvert}$	∞
definable over A	definable almost over A	
over A	does not fork over A	forks over A
strongly based on A	based on A	

A type may be almost over T — in that each formula in the type has only finitely many conjugates over A — but the type still has $2^{|T|}$ conjugates.

Suppose $A \subseteq B$, $p \in S(B)$, and p has no proper conjugates over A. Then by Theorem I.2.30 (The basic definability lemma) p is definable over A. We have just seen that if p has few conjugates then p does not fork over A and is definable almost over A. We now show that two distinct nonforking extensions of a type over A can be separated by a formula which is almost over A.

1.23 Theorem. *Let $A \subseteq M$, $p, q \in S(M)$. Suppose p and q do not fork over A and $p|A = q|A$. If $p \neq q$ then there is formula $\psi(\overline{x})$ almost over A with $\psi(\overline{x}) \in p$ and $\neg\psi(\overline{x}) \in q$.*

Proof. By Exercise II.2.6 we can choose $\overline{c}$ and $\overline{d}$ realizing p and q respectively with $\overline{c}\frown\overline{d} \downarrow_A M$. Choose $\phi(\overline{m};\overline{x}) \in F(M)$ so that $\phi(\overline{m};\overline{c})$ and $\neg\phi(\overline{m};\overline{d})$. Let $\hat{r} \in S(\mathcal{M})$ be an nonforking extension of $r = t(\overline{m}; A \cup \overline{c} \cup \overline{d})$ to a global type. By symmetry and monotonicity, $\hat{r}$ does not fork over A. Thus there is a formula $d\phi(\overline{x})$ which is over M, almost over A, and defines $\hat{r}_\phi$. But then we have $\models d\phi(\overline{c}) \wedge \neg d\phi(\overline{d})$. So this formula is in p but not in q as required.

We now establish one of the principal tools of stability theory.

1.24 Theorem (The Finite Equivalence Relation Theorem). Let $A \subseteq B$ and $p, q \in S(B)$. Suppose $p|A = q|A$ and neither p nor q forks over A. If $p \neq q$ then there is an $E \in FE(A)$ such that: $p(\overline{x}) \cup q(\overline{y}) \vdash \neg E(\overline{x};\overline{y})$.

Proof. We know by Theorem 1.23 that there is a formula $\phi(\overline{x};\overline{b})$ in p but not q which is almost over A. Choose an $E \in FE(A)$ so that $\phi(\overline{x};\overline{b})$ depends on E. Now clearly $p(\overline{x}) \cup q(\overline{y}) \vdash \neg E(\overline{x},\overline{y})$.

For the following exercise consult the definition of Morley rank in Chapter VII.1.

1.25 Exercise (Relations With Morley Rank). Show that if $p \in S(M)$ then p has degree 1. Show that if the rank of M is α there is a finite equivalence relation definable over the empty set with $\deg(M)$ classes each with rank α.

One of the most important corollaries of the Finite Equivalence Relation Theorem is the following proof of Lachlan's theorem that a countable $\aleph_0$-categorical superstable theory is ω-stable. Recall from III.3.3 that $N(M, A)$ denotes the complete types over M which do not fork over A.

1.26 Theorem. *If T is a countable $\aleph_0$-categorical superstable theory then T is ω-stable.*

Proof. If not, there is a model M of T with $|S(M)| > |M|$. Without loss of generality we may fix an integer m such that the number of m-types over M is greater than $|M|$. Since T is superstable, for each p in $S(M)$ there is a finite subset of M over which p does not fork. Since there are only $|M|$ finite subsets of M, the theorem follows if we can show that for any finite A contained in M, $N(M, A)$ is also finite. By Ryll-Nardzewski's Theorem, $S(A)$ is finite so it suffices to show that any fixed member r of $S(A)$ has only finitely many extensions in $N(M, A)$. Thus, suppose $p(\overline{x})$ and $q(\overline{x})$ are distinct members of $N(M, A)$ extending $r \in S(A)$. Then by Theorem 1.24 there is an $E(\overline{x},\overline{y}) \in FE^m(A)$ such that $p(\overline{x}) \cup q(\overline{y})$ implies $\neg E(\overline{x};\overline{y})$. (Fig. 1). Thus $|N(M, A)|$ is bounded by the product of $\{n(E) : E \in FE^m(A)\}$ where $n(E)$ denotes the number of equivalence classes of E. But the $\aleph_0$-categoricity of T implies by Ryll-Nardzewski's theorem that the number of formulas with $|A| + 2m$ free variables is finite and this number certainly bounds $|FE^m(A)|$. Thus $N(M, A)$ is finite and the theorem follows.

Lachlan [Lachlan 1974] conjectured that the hypotheses of this theorem could be weakened by assuming only that T is stable. This conjecture

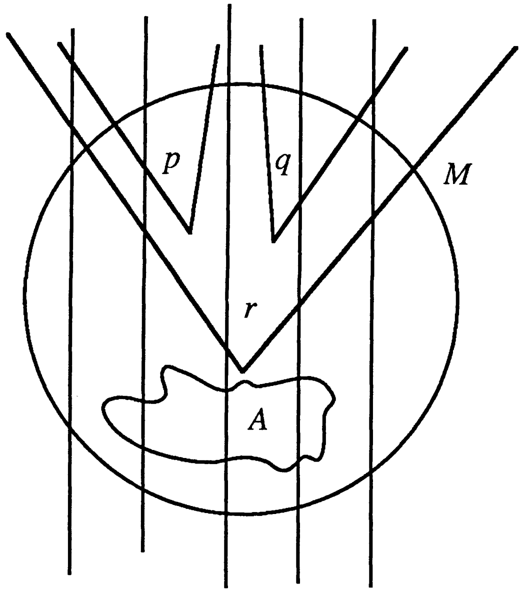

Fig. 1. Theorem 1.26. The vertical columns are the E-equivalence classes.

remains one of the critical unsolved problems in the area. This problem is the first of several which show that the investigation of stable but not superstable theories will require significant new tools. In his work on this problem Lachlan proved the normalization lemma which we discuss in detail in Chapter VIII. Moreover, he introduced the concept of pseudoplane which is also discussed in Chapter VIII. This notion has proved to be at the combinatorial core of the subject.

The following more algebraic version of the finite equivalence relation theorem is to due to John Vaughn [Vaughn 1985]. The terminology from Boolean algebras used in the next few paragraphs can be found in [Sikorski 1964]. To simplify the following proof we make the following definitions.

1.27 Definition. Let $B \subseteq \mathcal{M}$. Recall that $F(\mathcal{M})$ is the Lindenbaum algebra of formulas over $\mathcal{M}$. We fix the following notation. (Fig. 2).

i) F_B is the set of formulas in $F(\mathcal{M})$ which fork over B.
ii) A_B is the set of formulas in $F(\mathcal{M})$ which are almost over B.
iii) N_B is the set of formulas in $F(\mathcal{M})$ which do not fork over B.
iv) $\neg F_B$ is the set of formulas in $F(\mathcal{M})$ whose negations fork over B.

Recall that an ideal, I, in a Boolean algebra is a subset which is closed under joins and such that if $b \in I$ then for any a, $a \wedge b \in I$.

1.28 Exercise. Show that F_B is an ideal in $F(\mathcal{M})$.

We denote by $\phi \,\triangle\, \psi$ the symmetric difference of the formulas ϕ and ψ. Recall that an ideal, I, in a Boolean algebra determines a congruence on

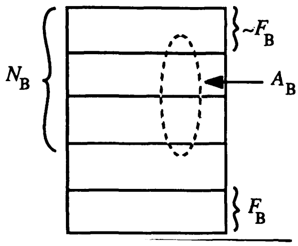

Fig. 2. Definition 1.27

the algebra where $\phi \simeq \psi$ if and only if $\phi \triangle \psi \in I$. Now we can reformulate the finite equivalence relation theorem as providing a normal form for $F(\mathcal{M})/F_B$.

1.29 Theorem. $F(\mathcal{M})/F_B \approx A_B$.

Proof. We show that for each $\phi(\overline{x};\overline{m}) \in F(\mathcal{M})$ there is a $\psi(\overline{x};\overline{b})$ which is almost over B such that $\phi(\overline{x};\overline{m}) \simeq \psi(\overline{x};\overline{b})$. As any formula in F_B is equivalent mod F_B to $x \neq x$ and and any formula in $\neg F_B$ is equivalent mod F_B to $x = x$, we can assume $\phi(\overline{x};\overline{m}) \in N_B - \neg F_B$. Thus, both $Y = U_{\phi(\mathcal{M},\overline{m})} \cap N(\mathcal{M}, B)$ and $X = U_{\neg\phi(\mathcal{M},\overline{m})} \cap N(\mathcal{M}, B)$ are nonempty closed, and thus compact sets. Now for each $q \in Y$ and $p \in X$, the finite equivalence relation theorem implies there is a clopen $C_{p,q} \subseteq S(\mathcal{M})$ which is defined by some $\psi_{p,q}(\overline{x};\overline{b}_{p,q})$ with $q \in C_{p,q}$ and $p \notin C_{p,q}$. By compactness, for each p the union, D_p, of some finite subset of the $C_{p,q}$ contains Y and is almost over B. By compactness again, some finite intersection of the D_p is defined by a formula $\psi(\overline{x};\overline{b})$ which is almost over B and such that $U_{\psi(\overline{x};\overline{b})} \cap N(\mathcal{M}, B) = Y$. Thus, $\psi(\overline{x};\overline{b}) \triangle \phi(\overline{x};\overline{m}) \in F_B$ and we finish.

1.30 Historical Notes. Most of these notions appeared first in [Shelah 1978]. The importance of calculating the number of conjugates was emphasized by Lascar and Poizat. The treatment here was greatly influenced by conversations with Ziegler. The finite equivalence relation theorem is due to Shelah [Shelah 1978]. The improvement of it in Theorem 1.23 was conjectured by Harnik and proved by Ziegler. Theorem 1.26 was proved by Lachlan [Lachlan 1974] using rank. The argument used here is due to Shelah. Definition 1.27 and Theorem 1.29 are due to John Vaughn.

2. *Definability and the Stability Hierarchy*

In this section we use the notions of definability of types developed in section 1 to provide another characterization of the stability hierarchy. To begin with we need more 'local' information about the possible extensions of a single type. Then we can proceed to the 'global' discussion of the relation between the number of types over each model and the extent to which these types are definable. We assume T is stable throughout this section.

In Definition III.4.28 we defined the ϕ-multiplicity of a type p to be the maximal number of nonforking mutually contradictory ϕ-types which are consistent with p. We now use the fact (Exercise III.3.16) that the collection of nonforking extensions of p to complete types over $\mathcal{M}$ is closed to show this number must always be finite. Then we will use that result to refine the assertion that $N(\mathcal{M}, A)$ is closed.

2.1 Theorem. *For any A, any $p \in S(A)$, and any formula ϕ, $\mu_\phi(p)$ (the ϕ-multiplicity of p) is finite.*

Proof. If not, there is an infinite family of ϕ-types p_i such that $p \cup p_i$ does not fork over A but for each i and j there is an $\overline{a}_{i,j}$ such that $\phi(\overline{x}; \overline{a}_{i,j}) \in p_i$ and $\neg\phi(\overline{x}; \overline{a}_{i,j}) \in p_j$. Without loss of generality for some M containing A we may extend each $p \cup p_i$ to a type $q_i \in S(M)$ which does not fork over A. Now for any κ construct κ distinct nonforking extensions of p to a model as follows. Add to L a new unary predicate symbol P, constants $\{\overline{c}_i : i < \kappa\}$ and $\{\overline{a}_{i,j} : i, j < \kappa\}$. Now let Γ contain $\phi(\overline{c}_i, \overline{a}_{i,j})$ if and only if $i = j$ and assert that P is the universe of an elementary submodel of the universe, $A \subseteq P$, all $\overline{a}_{i,j}$ are in P, and $t(\overline{c}_i; P)$ does not fork over A. The last clause is obtained by including in Γ, $(\forall \overline{y})[P(\overline{y}) \rightarrow \neg\phi(\overline{c}_i; \overline{y})]$ if $\phi(\overline{x}, \overline{y}) \notin \beta(p)$. These κ types contradict Lemma III.4.32 which showed every type has at most $\lambda(T)$ nonforking extensions.

2.2 Exercise. Conclude from Theorem 2.1 that for any A and $p \in S(A)$ there are only finitely many types $q_i \in S_\phi(\text{cl}(A))$ with $p \cup q_i$ consistent.

2.3 Exercise. Show that in Theorem 2.1 it is essential to assume p is complete.

Recall that Corollary III.3.15 asserted that if p is a type over B and p forks over $A \subseteq B$ then there is a formula $\phi(\overline{x}; \overline{y})$ and a type $q \in S(A)$ such that if $\overline{b}$ realizes q, then any type, r, containing $\phi(\overline{x}; \overline{b})$ forks over A. We will modify the result by removing the requirement that $\overline{b}$ realize a complete type but replacing the requirement r contains $\phi(\overline{x}; \overline{b})$ with the requirement that $p|A$ be complete and r contains $p|A \cup \phi(\overline{x}; \overline{b})$.

2.4 Corollary. *For every $p \in S(A)$ and every formula $\phi(\overline{x}; \overline{y})$ there is a formula $\theta(\overline{y}; \overline{a}) \in F(A)$ such that for any $\overline{b}$, $p \cup \phi(\overline{x}; \overline{b})$ does not fork over A if and only if $\models \theta(\overline{b}; \overline{a})$.*

Proof. Invoking Theorem 2.1, let $\{q_1, \ldots, q_n\} \subseteq S_\phi(\mathcal{M})$ with $n = \mu_\phi(p)$ be the nonforking complete ϕ-types over $\mathcal{M}$ such that $p \cup q_i$ does not fork over A. Let $\theta_l(\overline{y}; \overline{m}_l)$ define q_l for $l < n$ and let θ be the disjunction of the θ_l. Now $p \cup \phi(\overline{x}; \overline{b})$ does not fork over A if and only if for some $l < n$, $p \cup \phi(\overline{x}; \overline{b}) \subseteq q_l$ if and only if $\models \theta(\overline{b})$. It remains only to show that the formula θ is over A. But this follows from the fundamental definability lemma (Theorem I.2.30), since clearly it is fixed by every A-automorphism of $\mathcal{M}$.

We will now prove that in a countable theory the multiplicity of each complete type is either finite or $2^{\aleph_0}$. This yields as an immediate corollary that if T is ω-stable $u(p)$ is finite for each p. There are several ways to establish these results. Probably the most efficient is via ϕ-ω rank which we discuss briefly in Chapter VII (cf. II of [Shelah 1978]). We choose here to use techniques from Chapter III. The following lemma is the key step.

2.5 Lemma. *Let T be a stable theory and $p \in S(A)$ a complete type. Suppose that there are extensions p_i, p'_i of p for each $i < \omega$ such that $p_i \subseteq p_{i+1}$, $|\operatorname{dom} p_{i+1} - \operatorname{dom} p_i| < \omega$, $\operatorname{dom} p_{i+1} = \operatorname{dom} p'_{i+1}$, p_{i+1} and p'_{i+1} are distinct nonforking extensions of p_i. Then,*

i) $\mu(p) \geq 2^{\aleph_0}$.
ii) *It is impossible that all the p_i (p'_i) have the form $p \cup q_i$ ($p \cup q'_i$) where q_i (q'_i) is a ϕ-type.*

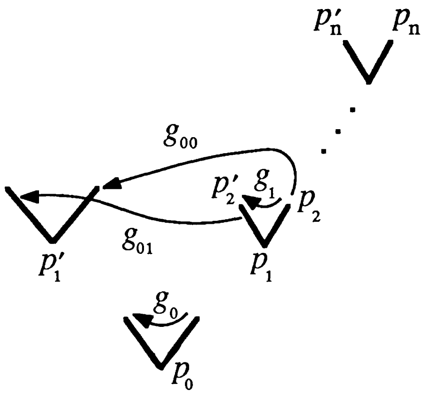

Fig. 3. Spreading a chain into a tree

Proof. (Fig. 3). i) Let $\hat{p}_i$, $\hat{p}'_i$ be nonforking extensions of p_i, p'_i to global types. Then for each i there is an automorphism g_i of M which fixes $\operatorname{dom} p_i$ and maps $\hat{p}_{i+1}$ to $\hat{p}'_{i+1}$. Now for $s \in 2^n$ define g_s by induction: $g_{\langle\,\rangle}$ is the identity, $g_{t\frown 0} = g_t$ and $g_{t\frown 1} = g_t \circ g_i$. Now let $p_s = g_s(p_i)$. For $\sigma \in 2^\omega$ let $p_\sigma = \bigcup_{n<\omega} p_{\sigma|n}$. The p_σ are $2^{\aleph_0}$ distinct nonforking extensions of p.

ii) If the p_i (p'_i) all have the form $p \cup q_i$ ($p \cup q'_i$) for ϕ-types q_i (q'_i) we have $2^{\aleph_0}$ ϕ-types over a countable set and contradict stability.

2.6 Theorem. *If T is a countable stable theory and p is a complete type $\mu(p)$ is finite or $2^{\aleph_0}$.*

Proof. Suppose $p \in S(A)$ and $A \subseteq M$. Since types over models are stationary it suffices to count the nonforking extensions of p to $S(M)$. Let Δ_i for $i < \omega$ be an increasing union of finite sets which exhaust $F(L)$. Now we define a finite branching tree of extensions of p. Let $\langle q^i_j : j < k_0 \rangle$ list the complete Δ_i types over M which are consistent with p. Write $p_{<j>}$ for $p \cup q^0_j$. If p_s with $\lg(s) = n$ has been defined let $p_{s\frown j} = p_s \cup q^n_j$ if this type is consistent and do not define $p_{s\frown j}$ otherwise. If $\mu(p)$ is infinite, this tree contains infinitely many t such that for some $i \neq j$ both $p_{t\frown i}$ and $p_{t\frown j}$ are defined. By König's Lemma we have a chain satisfying Lemma 2.5 i) so $\mu(p) = 2^{\aleph_0}$.

Since Theorem III.4.34 showed $\mu(p)$ is countable in an ω-stable theory we have

2.7 Corollary. *If T is a countable ω-stable theory then for every p, $\mu(p)$ is finite.*

There are two parts to the proof of Theorem 2.6. One is to describe a finite branching tree and prove it has an infinite path; the second is to convert the infinite path to a complete binary tree. Pillay [Pillay 1983a] (p. 61) uses the finite equivalence relation theorem to construct the finite branching tree and constructs the binary tree somewhat more concretely than we have.

Corollary 2.7 can be proved more directly with the machinery of Morley rank and degree; we will see in Chapter VII that $\mu(p)$ is the Morley degree of p.

2.8 Exercise. Give examples of types with each possible multiplicity.

Corollary 2.4 allows us to define for each $p \in S(A)$ and each formula $\phi(\overline{x};\overline{y})$ a formula $d_p\phi(\overline{y})$ with parameters from A such that for each nonforking extension $\hat{p}$ of p to a global type, $\phi(\overline{x};\overline{b}) \in \hat{p}$ if and only if $d_p\phi(\overline{b})$. We use this notation in the following exercise.

2.9 Exercise (Berline). Suppose $A_0 \subseteq A$ and each $d_p\phi$ is a formula over A_0. Show p is the unique nonforking extension of $p|A_0$ in $S(A)$.

The virtue of having every type over a set A definable over a finite subset of A may not be immediately clear. We will see many applications of this fact in Part D; even more appear in [Shelah 198?]. For the moment, we content ourselves with singling out sets with this property.

2.10 Definition. The set A is *good* if for every $p \in S(A)$ there is a subset A_0 of A with $|A_0| < \kappa(T)$ such that p does not fork over A_0 and $p|A_0$ is stationary inside A.

Note that if A is the universe of model, M, we can shorten the definition of good by saying that every $p \in S(M)$ is strongly based on some $A_0 \subseteq M$ with $|A_0| < \kappa(T)$.

Intuitively, if a type p is based on a set A then A 'controls' the properties of p. Thus, we would like to be able to base each type on a small subset of its domain. This is not always possible however.

2.11 Exercise. Find a model M of the theory REI_ω and a type $p \in S(M)$ which is not based on any finite subset of M.

2.12 Exercise. Find a model M of the theory REF_ω and a type $p \in S(M)$ which is based on some finite subset of M but not strongly based on any finite subset of M.

2.13 Exercise. Show the prototypical model of REF_ω is not good.

2.14 Lemma. *If $M \models T$ and for every $p \in S(M)$ there is an $A \subseteq M$ with $|A| < \kappa(T)$ such that p is based on A and $\mu(p|A) < \omega$ then M is good.*

Proof. The finitely many nonforking extensions of $p|A$ to complete types over M can be distinguished by finitely many finite equivalence relations over A. Since each class of each of these relations is realized in M it is easy to extend A to A' such that $p|A'$ is stationary inside M.

Good sets play an extremely important role in the structure theory. Much of the work needed in the extensions of the material in this book to completely solve the spectrum problem involves finding large classes of substructures of models of T which are good. We show now that every model of an ω-stable theory is good. We show the same for every strongly $\kappa(T)$-saturated model of a stable theory in Theorem 3.22.

2.15 Corollary. *If T is a countable ω-stable theory then every model of T is good.*

Proof. This is immediate from Theorem 2.7 and Lemma 2.14.

With this result in hand, we recast the stability hierarchy in terms of definability. We require one more technical remark.

2.16 Lemma. *If $p, q \in S(M)$ are definable over $A \subseteq M$ and $p \neq q$, then $p|A \neq q|A$.*

Proof. By Definition III.3.1 p and q do not fork over A; whence by the finite equivalence relation theorem (Theorem 1.24) if $p \neq q$ there is a finite equivalence relation E over A such that: $p(\overline{x}) \cup q(\overline{y}) \rightarrow \neg E(\overline{x};\overline{y})$. Now suppose that p is defined by d. Write $dE(\overline{v},\overline{w})(\overline{w})$ as $dE(\overline{w})$. Let $r(\overline{w})$ denote $p(\overline{w})|A = q(\overline{w})|A$. Then $\neg dE(\overline{w})$ is in $q|A$. But, $dE(\overline{w})$ is in $p|A$. For, if not, choosing a sequence $\overline{b}_i$ for $i \in \omega$ such that $\overline{b}_i$ realizes $d(p; \{A \cup B_j : j < i\})$, we have $\neg E(\overline{b}_i, \overline{b}_j)$ if $i \neq j$ contradicting the assumption that E has only a finite number of classes. Thus, $p|A \neq q|A$.

We distinguish one further class of theories which play an important role in the study of countable models of countable theories. Recall that $S^n(T)$ denotes the collection of all n-types over the empty set.

2.17 Definition. We say T is a *small theory* if for each n, $|S^n(T)| \leq |T|$.

This class of theories is important because any theory with fewer than $2^{\aleph_0}$ countable models is small. Thus, attempts to solve Vaught's conjecture for stable theories or to show there is no stable theory with finitely many (but more than one) countable models easily reduce to investigations of small stable theories.

2.18 Exercise. Show REF_ω is small but CEF_ω is not.

2.19 Theorem. The countable small theory T is ω-stable if and only if for every model M of T and every p in $S(M)$, there is a finite subset B of M such that p is definable over B.

Proof. Suppose first that T is ω-stable. Let M be a model of T and p be in $S(M)$. Then by Corollary 2.15 there is a finite subset B of M such that p is strongly based on B. By Theorem 1.20 ii) p is definable over B. Thus for any M and any p in $S(M)$, there is a finite subset B of M such that p is definable over B. Conversely, since there are only $|M|$ finite subsets of M and since T is small, there are only countably many types in each $S(B)$. Mapping each element of $S(M)$ to its restriction to a finite set over which it is definable shows (by Lemma 2.16) that there are only $|M|$ types in $S(M)$.

This theorem requires the assumption that T is small. Consider the theory CEF_ω of crosscutting equivalence relations. Now CEF_ω is superstable but not ω-stable yet every 1-type over any model is definable over a singleton.

The next exercise is an easy step towards the following improvement of Theorem 2.19. If for every pair of countable models $M \subseteq N$ of a small stable theory, T, there is an element $a \in N - M$ with $t(a; M)$ definable over a finite subset of M then T is ω-stable. The general result appears in [Pillay & Steinhorn 1985]

2.20 Exercise (Pillay-Steinhorn). Show that if T is a countable small superstable theory and every type over a finite set has finite multiplicity then T is ω-stable.

2.21 Theorem. *The countable theory T is superstable if and only if for every A contained in a model of T and every $p \in S(A)$, there is a finite B contained in A such that p is defined almost over B.*

Proof. Suppose first that T is superstable. Let M be a model of T and p be in $S(M)$. Then, since $\kappa(T) = \omega$, there is a finite B contained in M such that p does not fork over B. Thus p is definable almost over B. Now let M be a model of T and suppose that for each p there is a finite subset B of M such that p is definable almost over B. But then p does not fork over B so $\overline{\kappa}(T) = \omega$. Since T is countable $\overline{\kappa}(T) = \kappa(T)$ and we finish.

2.22 Historical Notes. The definability characterization of the stability hierarchy is implicit in III.4 of [Shelah 1978]. The proof of Theorem 2.1 is adapted from [Pillay 1983a]. The argument given here for Corollary 2.4 is taken from [Shelah 198?]. The other arguments at the beginning of this section are adapted from [Baldwin 1981].

3. Strong Types and Multiplicity

In Section 1 of this chapter we proved the finite equivalence relation theorem and immediately applied it to show the enormous significance of finite equivalence relations in investigating the nonforking extensions of a type. In this section we introduce the notion of the strong type of an element, which encodes all the information about it which is provided by finite equivalence relations. We investigate closely the relations among types over A, strong types over A, and types which are almost over A. In particular, we consider what it means for a type to be implied by a type which is almost over A. Then we analyze the relation between the strong type of a pair and the strong type of its projections. We conclude by discussing the relation between the multiplicity of a pair and that of its projections.

We begin with the fundamental definition for a rigorous discussion of the types which are almost over a set.

3.1 Definition. For any $\overline{c}$ and A, the *strong type* of $\overline{c}$ over A, denoted by $stp(\overline{c}; A)$, is $\{E(\overline{x}; \overline{c}) : E \in FE(A)\}$.

Note that $\mathrm{dom}(stp(\overline{c}; A))$ is not A but $A \cup \overline{c}$. However, if M is any model containing A then $t(c; M)$ implies $stp(c; A)$ and even equals $stp(c; M)$. The meaning of $stp(\overline{a}; A) = stp(\overline{b}; A)$ is slightly mysterious, since the two types have different domains. This equality does not assert the equality of two sets of formulas (which do not have the same parameters). Rather it asserts that for any $\overline{c} \in \mathcal{M}$, $\overline{c}$ realizes $stp(\overline{a}; A)$ iff $\overline{c}$ realizes $stp(\overline{b}; A)$. The situation is reminiscent of Cantor's definition of addition for cardinal numbers without specifying a representative for each equivalence class under equinumerousity.

Strangely, $stp(\overline{c}; A)$ is not preserved by A-automorphisms of $\mathcal{M}$ but equality of strong types is. For example, consider the theory of an equivalence relation with two infinite classes. Then any two elements of any model are conjugate over the empty set but inequivalent elements realize distinct strong types.

3.2 Exercise. Show that if $stp(\overline{a}; A) = stp(\overline{b}; A)$ and α is an automorphism which fixes A then $stp(\alpha\overline{a}; A) = stp(\alpha\overline{b}; A)$. The last example shows $stp(\alpha\overline{a}; A)$ need not equal $stp(\overline{a}; A)$.

3.3 Exercise. Show $stp(\overline{a}; A) = stp(\overline{b}; A)$ iff for each finite equivalence relation, E, over A (of appropriate arity) $\models E(\overline{a}, \overline{b})$.

3.4 Exercise. Suppose $stp(a; B) = stp(c; B)$. Show $t(a; \mathrm{cl}(B)) = t(c; \mathrm{cl}(B))$ and even that $stp(a; \mathrm{cl}(B)) = stp(c; \mathrm{cl}(B))$.

3.5 Exercise. Show that $t(\overline{a}; B)$ does not imply $t(\overline{a}; \mathrm{cl}(B))$.

3.6 Exercise. Show that if $M \models T$ and $\phi(\overline{x}; \overline{a})$ is almost over M then $\phi(\overline{x}; \overline{a})$ is equivalent to a formula which is over M.

3.7 Exercise. Show that for any $\overline{c}$ and A, $stp(\overline{c}, A)$ is stationary.

3.8 Exercise. With the aid of strong types, give a direct (rather than axiomatic) proof that the strong extension property (Exercise II.2.8) holds for nonforking.

3.9 Definition. Let $A \subseteq \mathcal{M}$. Then $\mathrm{Saut}_A(\mathcal{M})$ denotes the set of automorphisms of $\mathcal{M}$ which fix the strong types over A. We refer to an automorphism which fixes all strong types over A as a *strong A-automorphism.* Naturally, strong automorphisms which fix A preserve formulas which are almost over A. We write $\mathrm{Saut}(\mathcal{M})$ if A is empty.

3.10 Exercise. Show Saut($\mathcal{M}$) is a normal subgroup of Aut($\mathcal{M}$).

Let $G = Z_3 \oplus Z_2^{\aleph_0}$ and let b be one of the two elements of order three and $2b$ the other. Now in the language of Abelian groups the formula which asserts the order of x is six generates a complete type over the empty set. But this type has two nonforking extensions to types over G, both asserting $x \neq g$ for each $g \in G$ but one saying $2x = b$ and the other $2x = 2b$. Each of the latter two types is almost over the empty set. The formula $(2x = 2y)$ defines a finite equivalence relation, $E(x, y)$. If c is an element with $2c = b$ but c is not in G, $stp(c; \emptyset)$ is implied by the formula $E(x, c)$.

3.11 Exercise. Investigate the rather different situation when the roles of Z_2 and Z_3 are reversed in the definition of the group G. Note however that if Z_3 is replaced by Z_p for any prime other than 2 the situation remains exactly the same as in the example.

We saw after Definition 3.1 that the notion of equality of strong types is somewhat slippery. The notion of implication can be understood somewhat better by passing from strong types over A to types which are almost over A.

3.12 Lemma. *Let $B \subseteq A$ and $p = t(c; A)$. The following are equivalent.*

i) *For some $q \subseteq p$, $q \vdash p$ and q is almost over B.*
ii) *For some q which is almost over B, $q \vdash p$.*
iii) *$stp(c; B) \vdash p$.*
iv) *$stp(c; B) \vdash stp(c; A)$.*

Proof. Clearly i) implies ii). To see that ii) implies iii), let $\overline{c}'$ realize $stp(c; B)$. Then c' realizes q and so by ii) c' realizes p. Suppose iii) holds; choose c' with $stp(\overline{c}'; B) = stp(c; B)$. Then, c' realizes p so $t(\overline{c}'; A)$ does not fork over B as $stp(c; B)$ does not fork over B. Since $stp(c; B)$ is stationary this implies c' realizes $stp(c; A)$ so iii) implies iv).

Obviously, iv) implies iii) so we complete the proof by showing iii) implies i). For this, let $q = \{\phi(x, b) \in p : \phi(x, b) \text{ almost over } B\}$. We will show $q \vdash p$. Since $FE^m(B)$ is closed under conjunction, for each $\psi(x, b) \in p$ there is an $E(x, y) \in FE^m(B)$ such that $E(x, c) \rightarrow \psi(x, b)$. Let $\chi(x)$ be $(\forall y)[E(x, y) \rightarrow \psi(y, b)]$. Now, the solutions of χ are a union of equivalence classes of E so $\chi \in q$. But $E(x, x)$ holds so $\chi(x) \rightarrow \psi(x, b)$. Thus, $q \vdash p$ as required.

3.13 Exercise. Show that the (standing) hypothesis that T is stable is required only for iii) implies iv) of Lemma 3.12.

Note that just because $c \downarrow_A B$ it does not follow that $stp(c; B) \vdash stp(c; A)$. However the following weakening of this idea holds.

3.14 Exercise. Suppose T is superstable. Let $\overline{c}$ realize $t(\overline{a}; B)$ with $\overline{c} \downarrow_B A$. Show that for some finite B' with $B \subseteq B' \subseteq A$, $stp(\overline{c}; B') \vdash t(\overline{c}; A)$.

3.15 Exercise. Suppose $stp(\overline{d}; B) \vdash t(\overline{d}; A)$. Show $stp(\overline{d}; B) \vdash t(\overline{d}; \mathrm{cl}(A))$.

The next result provides a dual to the last exercise; it considers taking the algebraic closure of the elements realizing a type. In addition, it will will play an important role in the proof of the transitivity axiom for the notion of S-isolation in chapter VIII.

3.16 Lemma. *If $stp(C;B) \vdash t(C;A)$ then*

$$stp(\text{cl}(B \cup C);B) \vdash t(\text{cl}(B \cup C);A).$$

Proof. If $\overline{e} \in \text{cl}(B \cup C)$ then there is a $\overline{c} \in C$ such that $t(\overline{e}; B \cup \overline{c})$ is algebraic. We will prove $stp(\overline{e}^\frown\overline{c};B) \vdash t(\overline{e}^\frown\overline{c};A)$; from this the theorem is clear. It suffices to show all realizations of $t(\overline{c}^\frown\overline{e};B)$ are independent from A over B. Using only properties of their type over B we will show $\overline{c}^\frown\overline{e} \downarrow_B A$ which will then establish the result. Let $\overline{f} = \langle \overline{e}_i : i < m\rangle$ be a complete list of the realizations of $t(\overline{e}; B \cup \overline{c})$. By the strong extension property (Exercise II.2.8) we can choose an $\overline{f}'$ with $t(\overline{c}^\frown\overline{f}';B) = t(\overline{c}^\frown\overline{f};B)$ but $\overline{c}^\frown\overline{f}' \downarrow_B A$. Now $\overline{f}'$ is just a permutation of the sequence $\overline{f}$. Thus, since $\overline{c}'^\frown\overline{f}' \downarrow_B A$, $\overline{c}^\frown\overline{f} \downarrow_B A$ so $\overline{c}^\frown\overline{e} \downarrow_B A$ as required.

The precise result we have proved is: If $stp(\overline{c};B) \vdash t(\overline{c};A)$ and $\overline{e} \in \text{cl}(B \cup \overline{c})$ then $stp(\overline{c}^\frown\overline{e};B) \vdash t(\overline{c}^\frown\overline{e};A)$.

3.17 Example. A type p may be implied by a type q which is almost over B without p itself being almost over B. Let T be the theory of an equivalence relation, R, with two classes, one infinite and one with a single element a. Let $B = \emptyset$ and let A be an infinite set of equivalent elements. Now if $p = t(a; A)$, p is implied by the formula $(\exists! y)R(x,y)$ which is almost over (indeed over $\emptyset$) but p is not almost over $\emptyset$. For, p contains the formulas $x \neq c$ for each $c \in A$.

The description of implications from strong types in terms of automorphisms is sometimes helpful. It is easy to verify the following lemma.

3.18 Lemma. *Let $B \subseteq A$. The following are equivalent.*

i) $stp(\overline{d};B) \vdash t(\overline{d};A)$.
ii) *If there exists an $\alpha \in \text{Saut}_B(\mathcal{M})$ with $\alpha(\overline{d}) = \overline{d}'$ then there exists an automorphism $\hat{\alpha} \in \text{Saut}_A(\mathcal{M})$ with $\hat{\alpha}(\overline{d}) = \overline{d}'$.*

This trivially yields that if $stp(\overline{d};B) \vdash stp(\overline{d};A)$ and $\alpha \in \text{Aut}_B(\mathcal{M})$ then $stp(\alpha(\overline{d});B) \rightarrow stp(\alpha(\overline{d});\alpha(A))$.

The next topic is the relation between the multiplicity (or more specifically) the strong type of a pair and that of each of its projections. It is clear that $t(\overline{a}^\frown\overline{b};A) = t(\overline{a}'^\frown\overline{b}';A)$ if and only if $t(\overline{a};A) = t(\overline{a}';A)$ and for some automorphism α fixing A with $\alpha(\overline{a}') = \overline{a}$, $t(\overline{b};\overline{a} \cup A) = t(\alpha(\overline{b}');\overline{a} \cup A)$. (We may write $t(\overline{b};\overline{a} \cup A) = t(\overline{b}';\overline{a}' \cup A)$.) The analogous result obtained by replacing t by stp fails. The next three lemmas establish exactly which parts of it are correct and which fail.

3.19 Lemma. i) *$stp(\overline{a}^\frown\overline{b};A)$ implies $stp(\overline{a};A)$.*
ii) *$stp(\overline{a}^\frown\overline{b};A)$ need not imply $stp(\overline{b};A \cup \overline{a})$.*

Proof. i) Suppose $stp(\overline{a}^\frown\overline{b}; A) = stp(\overline{a}'^\frown\overline{b}'; A)$. Then $stp(\overline{a}; A) = stp(\overline{a}'; A)$. For, suppose for contradiction that $\phi(\overline{x}; \overline{y})$ defines a finite equivalence relation over A which separates $\overline{a}$ and $\overline{a}'$. Regard $\phi(\overline{x}_1; \overline{y}_1)$ as a formula $\phi'(\overline{x}_1{}^\frown\overline{x}_2; \overline{y}_1{}^\frown\overline{y}_2)$. Then $\phi'(\overline{x}_1{}^\frown\overline{x}_2; \overline{y}_1{}^\frown\overline{y}_2)$ defines a finite equivalence relation on pairs which separates $\overline{a}^\frown\overline{b}$ from $\overline{a}'^\frown\overline{b}'$.

For ii), consider the theory of the following structure. Let L contain a single binary relation, R, and let the universe of M be the set of two elements subsets of ω. Interpret R to hold of $\{a, b\}, \{c, d\}$ if and only if $\{a, b\} \cap \{c, d\} \neq \emptyset$.

Let $a = \{1, 2\}$, $a' = \{3, 4\}$ and $b = \{1, 3\}$. We will show $stp(a^\frown b; \emptyset) = stp(a'^\frown b; \emptyset)$ but $stp(a; b) \neq stp(a'; b)$.

For the first assertion we show that any pair $\langle c, d\rangle$ with $c \neq d$ and such that $c \cap d \neq \emptyset$ satisfies the same strong type over the empty set as $\langle a, b\rangle$. To see this choose such a pair $\langle e, f\rangle$ but with $e \cup f$ disjoint from both $a \cup b$ and $c \cup d$. Now consider any definable equivalence relation on pairs (from M), F. If F separates $\langle a, b\rangle$ and $\langle c, d\rangle$ then we must have $\neg F(a, b, e, f)$ or $\neg F(c, d, e, f)$. Since F is definable from R, its truth on a quadruple depends only on the intersections between elements of the quadruple. But then we have $\neg F(e, f, e', f')$ for any other pair with $e' \cap f' \neq \emptyset$ but $(e' \cup f') \cap (e \cup f) = \emptyset$. But then F has infinitely many classes; so $stp(a^\frown b; \emptyset) = stp(c^\frown d; \emptyset)$.

Finally, we see $stp(a; b) \neq stp(a'; b)$ since they differ on the following finite equivalence relation: $E(x, y)$ if $(\forall z)[R(b, z) \rightarrow (R(z, x) \leftrightarrow R(z, y))]$.

There exist a, b, a', b', such that $a^\frown b$ and $a'^\frown b'$ have the same strong type over $\emptyset$ but $stp(a; b) \neq stp(a'; b')$ (i.e. take $b = b'$ in Lemma 3.19 ii) and there is no ambiguity in the notation). Note this cannot happen (with $b = b'$) for types.

Suppose we follow the procedure from the paragraph before Lemma 3.19 and write $stp(a; b) = stp(a'; b')$ if there is an $\alpha \in \mathrm{Saut}_A(\mathcal{M})$ which takes b' to b and satisfies $stp(a; b) = stp(\alpha(a'); b)$. Then a simple composition of automorphisms argument shows that $stp(a; b) = stp(\alpha(a'); b)$ implies $stp(a^\frown b; \emptyset) = stp(a'^\frown b'; \emptyset)$. But, suppose we follow a more syntactic course and say $stp(a; b) = stp(a'; b')$ if $stp(a; b)$ is the set of formulas $\{E(x, \alpha(a'), b) : E(x, a', b') \in stp(b', a')\}$. The next two lemmas show we can still conclude $stp(a^\frown b; \emptyset) = stp(a'^\frown b'; \emptyset)$.

3.20 Lemma. *If $\phi(\overline{x}^\frown\overline{y}; \overline{c})$ is almost over A, then for any $\overline{b}$, $\phi(\overline{x}; \overline{b}^\frown\overline{c})$ is almost over $A \cup \overline{b}$.*

Proof. Suppose $\{\phi(\overline{x}; \overline{b}, \alpha_i(\overline{c})) : i \in I\}$ is an infinite family of distinct conjugates over $A \cup \overline{b}$ of $\phi(\overline{x}; \overline{b}^\frown\overline{c})$. Then for some fixed j and an infinite subset of the i, $\phi(\overline{x}, \overline{y}, \alpha_i(\overline{c})) \leftrightarrow \phi(\overline{x}, \overline{y}; \alpha_j(\overline{c}))$. But this contradicts the choice of distinct $\{\phi(\overline{x}; \overline{b}, \alpha_i(\overline{c})) : i \in I\}$.

The following lemma gives a more algebraic formulation of the previous one and thus simplifies its application.

3.21 Lemma. *Let* $stp(\overline{a}; A) = stp(\overline{b}; A)$ *and let* $\alpha \in \mathrm{Saut}_A(\mathcal{M})$ *map* $\overline{a}$ *to* $\overline{b}$. *For any strong type* q *over* $A \cup \overline{a}$ *let* $\alpha(q)$ *be* $\{E(x, \alpha(c); \overline{b}) : E(x, c; \overline{a}) \in q\}$. *If* d *realizes* $\alpha(q)$ *then* $stp(\overline{a}^\frown c; A) = stp(\overline{b}^\frown d; A)$.

Proof. We have $stp(\overline{a}^\frown c; A) = stp(\overline{b}^\frown \alpha(c); A)$ so it suffices to show

$$stp(\overline{b}^\frown d; A) = stp(\overline{b}^\frown \alpha(c); A).$$

If not, for some $E \in FE(A)$, $\models \neg E(\overline{b}^\frown d, \overline{b}^\frown \alpha(c))$. Thus, by Lemma 3.20, $E(\overline{b}, x, \overline{b}, y) \in FE(A \cup \overline{b})$ and so $E(\overline{a}, x, \overline{a}, y) \in FE(A \cup \overline{a})$. But, $E(\overline{a}, \overline{x}, \overline{a}, c)$ is in $q = stp(c; A \cup \overline{a})$ so d does not realize $\alpha(q)$ contrary to hypothesis.

The following result illustrates another way to apply strong types.

3.22 Theorem. *Any strongly* $\kappa(T)$*-saturated model* M *of a stable theory is good.*

Proof. Let $\overline{a}$ realize $p \in S(M)$ and choose $A \subseteq M$ with $|A| < \kappa(T)$ and such that p does not fork over A. Since $stp(\overline{a}; A)$ does not fork over A, it is finitely satisfied in M and hence realized in M by some $\overline{b}$. Then p is strongly based on $A \cup \overline{b}$.

The preceding material concerned the relation between the strong types of pairs and their projections. We now generalize that concern slightly and consider the relation between the multiplicity of a pair and that of its projections.

Let T be the theory REF_ω of infinitely many refining equivalence relations, each with finitely many classes. Now, if a, b realize the same strong type over the empty set and $p = t(a; \emptyset)$, the multiplicity of p is $2^{\aleph_0}$ but $\mu(t(a; b))$ is one.

The following lemma of Saffe shows that this phenomenon (of the multiplicity decreasing from infinite to finite) could only occur because $t(b; \emptyset)$ also had infinite multiplicity.

3.23 Lemma. *Let* T *be stable. If* $t(\overline{b}; A)$ *has infinite multiplicity and* $t(\overline{c}; A)$ *has finite multiplicity then* $t(\overline{b}; A \cup \overline{c})$ *has infinite multiplicity.*

Proof. Choose $\langle \overline{b}_i : i < \omega \rangle$ all realizing $t(\overline{b}; A)$ but realizing distinct strong types over A. For each $i < \omega$, choose $\overline{c}_i$ with $t(\overline{b}_i{}^\frown \overline{c}_i; A) = t(\overline{b}^\frown \overline{c}; A)$. Since $t(\overline{c}; A)$ has finite multiplicity, we may assume that all the $\overline{c}_i$ realize the same strong type over A. Thus for each $i < \omega$ there is an automorphism of $\mathcal{M}$ which fixes strong types over A and maps $\overline{c}_i$ to $\overline{c}$. Denote $f_i(\overline{b}_i)$ by $\overline{b}'_i$. Then, the $\overline{b}'_i$ realize distinct strong types over A and, a fortiori, over $A \cup \overline{c}$. But for each i, $t(\overline{b}_i; A \cup \overline{c}) = t(\overline{b}; A \cup \overline{c})$. Thus $t(\overline{b}; A \cup \overline{c})$ has infinite multiplicity as required.

3.24 Historical Notes. This section primarily clarifies some notions that are not spelled out in the literature. Lemma 3.12 is a variant on Lemma IV.2.1 of [Shelah 1978]. The example for Lemma 3.18 is primarily due to Steve Buechler. Matt Kaufmann supplied the proof that the example actually worked. Lemma 3.23 is due to Jürgen Saffe [Saffe 1981]. The proof here is taken from [Pillay & Steinhorn 1985].