Chapter III
Forking

In this chapter we introduce the concept of a stable theory and expound Shelah's notion of forking. We will show that in a stable theory nonforking obeys the axioms described in Chapter II. One intuition behind this notion is that if $t(\overline{a}; B)$ is not free over A then $\overline{a}$ must satisfy more relations over B than it does over A. Each of $e, e^{\sqrt{2}}$,and $e^{1/2}$ realize the same type over Q, the field of rational numbers, namely the type p of a transcendental element. Both $p_1 = t(e^{\sqrt{2}}; Q \cup \{e\})$ and $p_2 = t(e^{1/2}; Q \cup \{e\})$ extend p, but p_1 is clearly a more generic or freer extension than p_2. We will give an account of the distinction between p_1 and p_2 which applies to any stable theory. A first approximation, '$e^{1/2}$ is in the algebraic closure of e while $e^{\sqrt{2}}$ is not,' works in a few cases. Thinking of the theory of algebraically closed fields of characteristic zero as a prototypical ω-stable theory, one can notice that the Morley rank of p_1 equals the Morley rank of p while the Morley rank of p_2 is less than that of p. This version will apply to any ω-stable theory. The extension to arbitrary stable theories requires some effort.

The most naive statement of the leitmotif of stability theory reads, 'many types implies many models; few types implies few models.' To make this notion precise we must specify what is meant by 'many types' and we must refine the phrase 'few models.' The appropriate rendering of 'few models' is 'admits a structure theory', and more specifically in this chapter, 'admits a freeness relation satisfying the axioms described in Chapter II.' We show in this chapter that a theory which has few types admits a freeness relation satisfying our axioms. We show this by introducing the notion of 'definability of types' and showing that if a theory has few types then every type is definable. From this we derive the existence of an appropriate freeness relation.

Thus in Section III.1 we show the equivalence of the two main characterizations of a stable theory.

i) There are few types, in the sense made precise in Section III.1.
ii) Every type is definable.

Thereafter, we develop the positive structure theory for stable theories solely from this definability property without further recourse to the number of types. We outlined in Section I.5 the proof that 'many types' implies there are 'many models.'

From one standpoint, a freeness relation can be seen as a means of picking out certain distinguished extensions of a type. That is, when $C \subseteq B$, the property, '$t(\overline{a}; B)\mathcal{F}C$' distinguishes $t(\overline{a}; B)$ among the extensions of $t(\overline{a}; C)$. We restrict ourselves in Section III.2 to searching for such a distinguished extension when C is the universe of a model. We note that the definable extension of a type provides such an extension and then investigate several characterizations of this extension. We establish the symmetry lemma for types over models, an important precursor of the exchange lemma. One important tool for these investigations is the fundamental order of Lascar-Poizat; another is the notion of a strongly saturated model. We conclude Section III.2 by combining these two ideas to provide a characterization of conjugate types.

In Section III.3 we show that if T is stable we can define a notion of independence satisfying the axioms of Section II.1. We relate this notion, nonforking, to the fundamental order and show that all nonforking extensions of a complete type over A to the monster model M are conjugate over A. We return to the 'distinguished extension' theme and show that the nonforking extensions (to a global type) of a type $p \in S(A)$ are exactly those which have few conjugates over A.

In Section III.4 we investigate the effect of those axioms which are true, but trivial, in the vector space case and so had not been singled out: the finite character of freeness and the existence of stationary types. We connect these axioms with the function which computes the number of types over a set of power κ. This leads to the division of stable theories into three classes, depending on the values of this function for the theory. We provide a number of examples of theories in the various classes and compute the spectrum of saturation of a stable theory. Section III.5 is devoted to a survey of algebraic examples illustrating the material in this chapter. We explain the meaning of forking in the context of modules and discuss various definable chain conditions in algebra.

1. *Stable Theories: ϕ-Types, Rank, and Definability*

In [Shelah 1971], Shelah took the crucial step for the development of stability theory. He specialized Morley's discussion of complete types to a study of types which consist of instances of a single formula or its negation. This localization distinguishes stable from ω-stable theories. For, the concept of a stable theory is defined by a condition on instances of single formulas. It is important to deal with instances of one formula in order to make some crucial compactness arguments.

In this section we lay the basis for our study. We show the notion of a stable theory can be defined by a definability criterion, by conditions on the cardinalities of Stone spaces, or by the existence of local rank functions.

The basic intuition is a familiar one. If there are few possibilities for a given phenomenon then each possibility is definable. Beth's Theorem [Chang & Keisler 1973] is a prototype for such results and, in fact, the key Theorem 1.22 of this section can be proved using Beth's theorem [Lascar & Poizat 1979]. However, we will follow here the original route of Shelah and introduce rank as the tool used to find the definition of a ϕ-type over a set A.

Recall the following convention.

1.1 Separation of Variables. When we write the formula ϕ in the form $\phi(\overline{x};\overline{y})$, this indicates that $\overline{x}$ should be regarded as a sequence of free variables in the usual way, but $\overline{y}$ is a placeholder for a sequence of parameters. Thus, in forming $\phi(\overline{x};\overline{y})$-types over A as in the following definition, we know from the expression $\phi(\overline{x};\overline{y})$ that we are defining a subset of $S^n(A)$ (where $n = \lg(\overline{x})$) and that sequences from A will be substituted for $\overline{y}$.

1.2 Definition. Let $\phi(\overline{x};\overline{y})$ be a formula. A *ϕ-type* over A is a consistent set of formulas, each formula having the form $\phi(\overline{x};\overline{a})$ or $\neg\phi(\overline{x};\overline{a})$, where $\overline{a}$ is a $\lg(\overline{y})$-sequence from A. The ϕ-type p is a *complete ϕ-type* if for each $\lg(\overline{y})$ sequence $\overline{a}$ from A, either $\phi(\overline{x};\overline{a})$ or $\neg\phi(\overline{x};\overline{a})$ is in p. We denote by $S_\phi(A)$ the set of all complete ϕ-types over A. If $p \in S(A)$ we write p_ϕ for the restriction of p to instances of ϕ and $\neg\phi$.

For any set of formulas $\Delta = \{\phi_i(\overline{x};\overline{y}) : i \in I\}$ we define a Δ-type over A to be a consistent collection of A-instances of formulas in Δ or their negations. In the natural sense we may say an arbitrary type p is complete for ϕ if p_ϕ is a complete ϕ-type.

The notion of a Δ-type is fundamental for stability theory. As will be seen through this book, much of the material about stable theories can be seen as localizing results about types in ω-stable theories to results about Δ-types for arbitrary finite sets of formulas Δ. Shelah in [Shelah 1978] and earlier [Shelah 1971] shows how to code Δ-types for a finite Δ as ψ-types for a single formula ψ. The idea is to consider the conjunction of formulas of the form $(z_i = b_0 \rightarrow \phi_i(\overline{x};\overline{y})) \wedge (z_i = b_1 \rightarrow \neg\phi_i(\overline{x};\overline{y}))$ where the z_i are a collection of new variables and b_0, b_1 are a pair of points used as indicators.

1.3 Exercise. Show that for any set A with $|A| \geq 2$ and any finite set Δ of formulas there is a single formula ψ such that each Δ-type over A is equivalent to a ψ-type.

1.4 Exercise. Show that if A is subset of a model of any theory with $2 \leq |A| < \omega$ then for each ϕ, $|S_\phi(A)| < \omega$.

The following definition provides a technical framework for discussing freeness of types.

1.5 Definition. (Fig. 1). Let p be a type and let κ be a cardinal. Then $\Gamma_p(\phi, \kappa)$ is the following set of sentences in the language obtained by adding

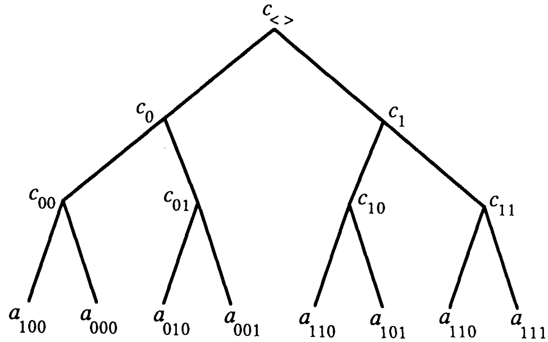

Fig. 1. $\Gamma_{\langle\,\rangle}(\phi, 2)$

to L additional constant symbols $\overline{a}_\tau, \tau \in 2^\kappa$ and $\overline{c}_s, s \in 2^{<\kappa}$.

$$\Gamma_p(\phi, \kappa) = \{p(\overline{a}_\tau) : \tau \in 2^\kappa\} \cup \{\phi(\overline{a}_\tau; \overline{c}_{\tau|i})^{\tau(i)} : i < \kappa\}.$$

Here $p(\overline{a}_\tau)$ denotes $\{\psi(\overline{a}_\tau) : \psi \in p\}$ and $\phi^{\tau(i)}$ denotes ϕ or $\neg\phi$ depending on whether $\tau(i)$ is 0 or 1.

The collection $\Gamma_p(\phi, \kappa)$ is consistent just if there is a complete binary tree of height κ of extensions of p by instances of ϕ such that each path is consistent but the paths are pairwise contradictory. Note we can easily extend our definition to $\Gamma_{\langle\,\rangle}(\phi, \kappa)$ by letting $\langle\,\rangle$ denote the empty type. The following diagram illustrates the definition of $\Gamma_p(\phi, \kappa)$. The $\overline{a}_\tau$ in the definition correspond to paths through the tree; the $\overline{c}_s$ correspond to nodes.

1.6 Theorem. *$\Gamma_p(\phi, n)$ is consistent for arbitrarily large finite n if and only if $\Gamma_p(\phi, \kappa)$ is consistent for all κ.*

Proof. The result is immediate by compactness noting that the consistency of an arbitrary n-element subset of $\Gamma_p(\phi, \kappa)$ is implied by the consistency of $\Gamma_p(\phi, n)$.

The key to this argument is the fact that all the nodes on the tree, except the base, are instances of ϕ or $\neg\phi$.

We now show that if for some infinite A, $|S_\phi(A)| > |A|$ then there is such an A of each infinite cardinality. The proof has two stages. The first is a variant on the proof that a closed uncountable set of reals has a perfect subset. While in that proof we choose a pair of disjoint open subsets which are each uncountable, here we show there is a single formula $\phi(\overline{x}; \overline{a})$ such that both ϕ and $\neg\phi$ are elements of more than $|A|$ distinct ϕ-types. Then, as completeness of the reals allows us to construct the perfect set, the compactness theorem allows us to find arbitrarily large A satisfying the condition.

1.7 Lemma. *Suppose that for some A and ϕ, $|S_\phi(A)| > |A| = \kappa \geq \omega$. Then for each infinite cardinal λ there exists a B with $|B| = \lambda$, $|S_\phi(B)| > \lambda$ and, a fortiori, $|S(B)| > \lambda$.*

Proof. Note first that for some $\overline{a} \in A$ both $X(\overline{a}) = \{p \in S_\phi(A) : \phi(\overline{x}; \overline{a}) \in p\}$ and $Y(\overline{a}) = \{p \in S_\phi(A) : \neg\phi(\overline{x}; \overline{a}) \in p\}$ have more than κ elements. If not, $S_\phi(A) = \bigcup_{\overline{a} \in A} Z(\overline{a}) \cup \{q\}$ where $Z(\overline{a})$ is whichever of $X(\overline{a})$, $Y(\overline{a})$ has $\leq \kappa$ elements, while q is the unique complete ϕ-type such that $\phi(\overline{x}; \overline{a}) \in q$ if and only if $|X(\overline{a})| > \kappa$. But then $|S_\phi(A)| \leq \kappa$. Relativizing this argument to $X(\overline{a})$ and $Y(\overline{a})$ and proceeding by induction, we see $\Gamma_{\langle\,\rangle}(\phi, \aleph_0)$ is consistent. Fix λ and let μ be the least cardinal with $2^\mu > \lambda$. Then, by compactness, $\Gamma_{\langle\,\rangle}(\phi, \mu)$ is consistent. Let B be a set of interpretations for $\{\overline{c}_s : s \in 2^{<\mu}\}$; we have the lemma. For, $2^{<\mu}$ has at most λ nodes while 2^μ has more than λ.

The preceding argument of Shelah localized to a single formula the following seminal result of Morley.

1.8 Exercise. Let T be a countable theory. Show that if for some infinite λ there is a set B with $|B| = \lambda$ and $|S(B)| > \lambda$, then there is a countable set A with $|S(A)| > \aleph_0$. In fact, $|S(A)| = 2^{\aleph_0}$.

We can now relate the cardinalities of $S(A)$ and $S_\phi(A)$.

1.9 Theorem. *Let T be an arbitrary complete first order theory. The following are equivalent:*

i) *For every A, $|S(A)| \leq |A|^{|T|}$.*
ii) *For every A, if $|A|^{|T|} = |A|$ then $|S(A)| \leq |A|$.*
iii) *For every countable A and each ϕ, $|S_\phi(A)| \leq \aleph_0$.*
iv) *For every infinite A and every ϕ, $|S_\phi(A)| \leq |A|$.*

Proof. Clearly, i) implies ii). By Lemma 1.7, ii) implies iii) and iii) implies iv). As $S(A)$ can be embedded into the Cartesian product of the $S_\phi(A)$ for $\phi \in F(\emptyset)$ by mapping p to $\langle p_\phi : \phi \in F(\emptyset)\rangle$, iv) implies i).

We now introduce a notion of local rank. While the entire subject of stability theory can be erected on the basis of this notion, we use it primarily as a tool to obtain the definability of types in this section and use definability as our basic notion. Our axioms for independence guide the choice of properties derived from the definability of all types in a stable theory. The relation between the formulation here and the various rank notions is explained in Chapter VII.

1.10 Definition. Let p be a type over A. The ϕ-rank of p is defined by induction as follows.

- $R(p, \phi) \geq 0$ for all p.
- $R(p, \phi) \geq \delta$ if $R(p, \phi) \geq \alpha$ for all $\alpha < \delta$ when δ is a limit ordinal.
- $R(p, \phi) \geq \beta + 1$ just if for each finite $q \subseteq p$ there is a sequence $\overline{a} \in \mathcal{M}$ such that:

$$R(q \cup \{\phi(\overline{x}; \overline{a})\}, \phi) \geq \beta \text{ and } R(q \cup \{\neg\phi(\overline{x}; \overline{a})\}, \phi) \geq \beta.$$

- $R(p, \phi) = \alpha$ if $R(p, \phi) \geq \alpha$ but $R(p, \phi) \not\geq \alpha + 1$.
- $R(p, \phi) = \infty$ if $R(p, \phi) > \alpha$ for every ordinal α.

1.11 Exercise. Show that for $n < \omega$, $R(p, \phi) \geq n$ iff $\Gamma_p(\phi, n)$ is consistent.

1.12 Exercise. For every p and ϕ, $R(p, \phi) < \omega$ or $R(p, \phi) = \infty$.

1.13 Exercise. For every formula ϕ

i) If $p \subseteq q$, $R(p, \phi) \geq R(q, \phi)$.
ii) For every p, there is a finite subtype $p_0 \subseteq p$ with the same rank.

These results are immediate from the definition. The following fundamental result is only slightly harder to prove.

1.14 Proposition. *Fix a formula ϕ. If for each countable A, $|S_\phi(A)| \leq \aleph_0$ then for every p, $R(p, \phi) < \omega$.*

Proof. By Exercise 1.11 unless $R(p, \phi)$ is bounded below ω, each $\Gamma_p(\phi, n)$ is consistent; so to, by Theorem 1.6, is $\Gamma_p(\phi, \aleph_0)$. But if $A = \{\overline{c}_s : s \in 2^{<\aleph_0}\}$, $|A| = \aleph_0$ while $|S_\phi(A)| = 2^{\aleph_0}$.

We come now to the critical notion of a definable type. We shall see that stable theories can be characterized by the definability of every type. Moreover, we will construct our freeness relation by generalizing the intuition that definable extensions of types are free.

1.15 Definition. Let $p(\overline{x})$ be an n-type over A. Let d be a map from formulas $\phi(\overline{x}; \overline{y})$ into formulas $d\phi(\overline{y})$ with parameters from B. Then d *defines* p over B if for each formula $\phi(\overline{x}; \overline{y})$ and each $\overline{a} \in A$:

$$\phi(\overline{x}; \overline{a}) \in p \text{ implies } \models d\phi(\overline{a}) \text{ and } \neg\phi(\overline{x}; \overline{a}) \in p \text{ implies } \models \neg d\phi(\overline{a}).$$

Note that if p is complete we can simplify this definition to $\phi(\overline{x}; \overline{a}) \in p$ if and only if $\models d\phi(\overline{a})$. In Corollary 1.23 and Lemma 1.26 we clarify the dependence of d on both p and ϕ and show that for A a model d is a Boolean homomorphism.

1.16 Exercise. Let M be a model of the theory of an equivalence relation with infinitely many infinite classes. Find a definition over M for each 1-type in $S(M)$.

1.17 Exercise. Show that if $\{\phi(x, \bar{a})\} \vdash p$ then p is definable over $\overline{a}$.

1.18 Exercise. Using the quantifier elimination results for modules discussed in Chapter I, verify that every 1-type, p, over an Abelian group A is definable as follows. Let $p = t(c; A)$. Every formula $\phi(x; \overline{y})$ is equivalent to a Boolean combination of formulas of the forms: $n|(kx + y)$ or $kx = y$. (Check that the replacement of $\overline{y}$ by y is legitimate.) In the first case, if for some $a \in A$, $n|kc + a$ let $d\phi$ be $n|y - a$; otherwise, let $d\phi$ be $y \neq y$. In the second case, if for some $a \in A$, $kc = a$, let $d\phi$ be $y = a$; otherwise, let $d\phi$ be $y \neq y$.

The following exercise uses the same trick explained before Exercise 1.3 to code information about a finite set of formulas in terms of one formula.

1.19 Exercise. Show that for any set A with $|A| \geq 2$ and any n formulas $\phi_0(\overline{x};\overline{y}), ..., \phi_{n-1}(\overline{x};\overline{y})$ there is a formula $\psi^*(\overline{x};\overline{y},\overline{z})$ such that for any χ, if a χ-type p is definable by one of the ϕ_i, then it is definable by an instance of ψ^*.

Now we can define the fundamental concept of this book.

1.20 Definition. The theory T is *stable* if for every $A \subseteq \mathcal{M}$ and every $p \in S(A)$, p is definable over A.

In accordance with our convention this means that for some unspecified n, every n-type is definable. After we relate the definability of n-types to the cardinality of $S^n(A)$, we will note (Exercise 1.36) that this condition holds for one n if and only if it holds for all n. In the meantime we do not hesitate, as in the next exercise, to verify that a theory is stable by showing only that all its 1-types are definable.

1.21 Exercise. Deduce from Exercise 1.18 that for any Abelian group A, $\mathrm{Th}(A)$ is stable.

Using the rank machinery we can immediately link the definability requirement with conditions on the cardinality of Stone space

1.22 Theorem. *The theory T is stable if and only if for each formula ϕ and for each set A, $|S_\phi(A)| \leq \sup(|A|, |T|)$.*

Proof. Clearly if T is stable the number of complete ϕ-types over A is bounded by the number of definitions over A. So, for each A, $|S_\phi(A)| \leq |A|$. If $|S_\phi(A)| \leq |A|$ for all A and ϕ, then, by Theorem 1.9 and Proposition 1.14, for every $p \in S(A)$, $R(p,\phi)$ is finite. Fix p with, say, $R(p,\phi) = n$. Choose by Proposition 1.13 a finite subset p_0 of p so that $R(p,\phi) = R(p_0,\phi)$. Note that $\phi(\overline{x};\overline{a}) \in p$ if and only if $R(p_0 \cup \{\phi(\overline{x};\overline{a})\},\phi) = n$. That is, $\phi(\overline{x};\overline{a})$ is in p if and only if $\Gamma_{p_0\cup\{\phi(\overline{x};\overline{a})\}}(\phi,n)$ is consistent. The consistency of $\Gamma_{p_0\cup\{\phi(\overline{x};\overline{a})\}}(\phi,n)$ can be expressed by a single first order formula $d\phi(\overline{y})$ whose parameters are those occurring in p_0. That is, $\models d\phi(\overline{a})$ just if $\Gamma_{p_0\cup\{\phi(\overline{x};\overline{a})\}}(\phi,n)$ is consistent.

We now explore several ways to uniformize the definability of types in a stable theory. The difference between Corollary 1.23 and Lemma 1.24 is clarified by the proof of Theorem 2.23. First we show that we can 'define' the collection of parameters which may serve to define p.

1.23 Corollary (Harnik). *If T is stable, for every formula $\phi(\overline{x};\overline{y})$ and each $p \in S(A)$ there exist L-formulas $\theta(\overline{x};\overline{z})$ and $\chi(\overline{y};\overline{z})$ such that:*

i) *For some $\overline{a} \in A$, $\theta(\overline{x};\overline{a}) \in p$.*
ii) *For every $\overline{a},\overline{b} \in A$, if $\theta(\overline{x};\overline{a}) \in p$ then $[\phi(\overline{x};\overline{b}) \in p$ iff $\models \chi(\overline{b};\overline{a})]$.*

Proof. We use the notation from the proof of the last theorem. Let $\theta_0(\overline{x};\overline{z})$ be the result of replacing the parameters in the conjunction of p_0 by the variables $\overline{z}$, let $\theta(\overline{x};\overline{z})$ be $\theta_0(\overline{x};\overline{z}) \wedge (\bigwedge \Gamma_{\theta_0(\overline{x};\overline{z})}(\phi,n))$ and let $\chi(\overline{y};\overline{z})$ express the consistency of $\Gamma_{\{\theta(\overline{x};\overline{z})\}\cup\{\phi(\overline{x};\overline{y})\}}(\phi,n)$.

Note that both θ and χ depend on p as well as ϕ. Yet our notation in Definition 1.15 indicates that d depends only on ϕ even though a formula $d\phi$ which satisfied the definition could depend on both ϕ and p. We now show that we can choose a formula $d\phi(\overline{y};\overline{z})$ (depending only on ϕ) and express the dependence on p by the choice of the parameter substituted for $\overline{z}$.

Technically, the following lemma only holds when any A under consideration has cardinality at least two. Since this is the only interesting case and can always be easily arranged we suppress that hypothesis to ease readability.

1.24 Lemma. *If for each A, every $p \in S(A)$ is definable over A then there is a map d taking formula $\phi(\overline{x};\overline{y})$ to formulas $d\phi(\overline{y};\overline{z})$ such that for each A and each $p \in S(A)$ there is a sequence $\overline{a}_{\phi,p} \in A$ such that $d\phi(\overline{y};\overline{a}_{\phi,p})$ defines p.*

Proof. If not, there is a formula ϕ, such that for each formula $\psi(\overline{y};\overline{z})$ there is a set A, and a type $p \in S(A)$ such that for each $\overline{a} \in A$, $\psi(\overline{y};\overline{a})$ does not define p_ϕ. Add to L a new unary predicate R and a new constant $\overline{c}$. Now the following set of sentences is consistent.

$$\{(\forall\overline{z})(R(\overline{z}) \rightarrow ((\exists\overline{y})(R(\overline{y}) \wedge \neg[\phi(\overline{c};\overline{y}) \leftrightarrow \psi(\overline{y};\overline{z})]))) : \psi(\overline{y};\overline{z}) \in L\}.$$

But, if B is the interpretation of R in a model of these sentences then $t(\overline{c};B)$ is not definable over B, contrary to the stability of T.

The argument for Lemma 1.24 seems rather nonconstructive. Note, however, that by the completeness theorem the function d is recursive (in T). Moreover, the most natural candidate for a constructive proof of the theorem fails. That is, we would like to argue that there is some uniform choice of the type p_0 which arises in the proof of Theorem 1.22. We couldn't hope to fix the type p_0 but since it is a conjunction of instances of ϕ and instances of $\neg\phi$, we might hope to fix the number of each. The following example shows that life is not that simple.

1.25 Exercise. If p is a finite ϕ-type, let $|p| = (n, k)$ assert that p contains n instances of ϕ and k instances of $\neg\phi$. Show that there is a stable theory and a formula ϕ such that for infinitely many distinct (n, k) there exist an A and a $p \in S(A)$ such that if p_0 is the least (where $|p|$ is ordered by the sum of n and k) subtype of p with $R(p, \phi) = R(p_0, \phi)$, then $|p_0| = (n, k)$. (Hint: Consider the theory of an equivalence relation with arbitrarily large finite classes.)

We will rely on the following easy but important fact about the map d which defines a type p.

1.26 Lemma. *Let T be stable. For any model M and $\overline{c}$, the map d which defines $t(\overline{c};M)$ is a Boolean homomorphism from $F(T)$ into $F(M)$.*

1.27 Exercise. Prove Lemma 1.26; give an example showing the necessity to assume that we work over a model M rather than an arbitrary set A.

The following definition provides an alternative view of definability. The duality between the roles of parameters and the sets they pick out which is exploited in the next few results recurs in the symmetry lemma in Section 3.

1.28 Definition. We say N is a *conservative extension* of M and write $M \prec_c N$ if for every formula $\phi(\overline{x};\overline{y})$ and every $\overline{n} \in N$, there is a formula $\phi^*(\overline{x};\overline{z})$ and an $\overline{m} \in M$ such that $\phi(N,\overline{n}) \cap M = \phi^*(M,\overline{m})$.

1.29 Exercise. Show that if $M \prec N$ are models of $\mathrm{Th}(Z,S)$ then N is a conservative extension of M.

1.30 Exercise. Show that as models of the theory of dense linear order without endpoints the reals are not a conservative extension of the rationals.

1.31 Lemma. *Let N be an elementary extension of M. Then N is a conservative elementary extension if and only if every type over M which is realized in N is definable over M.*

Proof. Suppose M is an elementary submodel of N and for each $\overline{n} \in N$, $t(\overline{n};M)$ is definable over M. Then for any formula $\phi(\overline{x};\overline{y})$,

$$\begin{aligned}\{\overline{m} \in M : N \models \phi(\overline{n};\overline{m})\} &= \{\overline{m} \in M : \phi(\overline{x};\overline{m}) \in t(\overline{n};M)\}\\ &= \{\overline{m} \in M : M \models d\phi(\overline{m})\}.\end{aligned}$$

Similarly, if N is a conservative extension of M,

$$\begin{aligned}\{\overline{m} : \phi(\overline{x};\overline{m}) \in t(\overline{n};M)\} &= \{\overline{m} : N \models \phi(\overline{n};\overline{m})\}\\ &= \{\overline{m} : M \models \phi^*(\overline{m})\}\end{aligned}$$

so we can use ϕ^* as $d\phi$ to define $t(\overline{n};M)$.

We can now easily prove the following proposition.

1.32 Proposition. *Suppose for every pair, M, N, of models of T, $M \prec N$ implies $M \prec_c N$. Then T is stable.*

1.33 Exercise. Prove Proposition 1.32.

1.34 Exercise. Show $\mathrm{Th}(Z,S)$ is stable. Show $\mathrm{Th}(Q,<)$ and $\mathrm{Th}(Z,<)$ are not.

The following theorem summarizes this section.

1.35 Theorem. *The theory T is stable iff it satisfies any of the following equivalent conditions.*

i) *For every A, $|S(A)| \le |A|^{|T|}$.*
ii) *For every A, if $|A|^{|T|} = |A|$ then $|S(A)| \le |A|$.*
iii) *For every A, if $|A| = \aleph_0$ then for each ϕ, $|S_\phi(A)| \le \aleph_0$.*
iv) *For every A and every ϕ $|S_\phi(A)| \le |A|$.*
v) *For every ϕ and p, $R(p,\phi) < \infty$.*
vi) *Every elementary extension of a model of T is a conservative elementary extension.*

The following exercise and condition ii) of Theorem 1.35 justify our failure to distinguish the various $S^n(A)$ when defining stability.

1.36 Exercise. Show that for each A with $|A| \leq \lambda$, $|S^1(A)| \leq \lambda$ if and only if for each A with $|A| \leq \lambda$ and each $n < \omega$, $|S^n(A)| \leq \lambda$. (It is essential that we quantify over A on both sides of the equivalence.)

1.37 Historical Notes. Morley began the investigation of cardinalities of families of Stone spaces $S(A)$ for A contained in a model of a countable complete theory T in [Morley 1965]. Ressayre [Ressayre 1969] and Rowbottom [Rowbottom 1964] extended this analysis to uncountable theories while working on the generalized Los conjecture: if T is categorical in some $\lambda > |T|$ then T is categorical in all powers greater than T. Shelah ([Shelah 1969a], [Shelah 1971]) discovered the importance of considering the restriction to instances of a single formula. Most of the results in this section come from those two papers. In particular, the definability lemma was first proved by Shelah [Shelah 1971]. A weaker version (for ω-stable T) was proved by Baldwin [Baldwin 1971] at about the same time. This formulation was applied directly in [Baldwin 1975] and, with more effect, in Lascar [Lascar 1973]. The definition of stability in terms of conservative extensions was first made explicitly in [Baldwin 1975]. It was suggested by analogy with the role of conservative extensions of models of Peano arithmetic. This analogy deserves further examination [Kirby & Pillay 1986]. The significance of varying Theorem 1.22 to obtain Corollary 1.23, by uniformizing the defining formula, was discovered by Harnik in 1980. Various applications of the strengthened form appear in [Harnik & Harrington 1984] and later in this book. In [Shelah 1974], Shelah provides another argument for Theorem 1.22 in the guise of a generalization of the Chang-Makkai Theorem. The examples of two 'obviously' independent transcendental numbers given in the first paragraph of this section require some acquaintance with number theory. (One cannot simply choose an apparently random pair of transcendental numbers. It is unknown whether e and π are algebraically independent or whether $e + \pi$ is rational; or even whether e and e^π are algebraically independent.) The example given follows from the Lindenbaum-Weierstrass theorem: If $b_1, \ldots, b_p$ are algebraic numbers which are linearly independent over the rationals then $e^{b_1}, \ldots, e^{b_p}$ are algebraically independent. Siegel [Siegel 1949] contains a clear introduction to this material.

2. Types Over Models

In this section we begin the proof that a freeness relation satisfying the axioms of Chapter II can be defined in any stable theory. We investigate the distinguished (i.e. definable) extensions of types over models. In the process, we consider the relation between definability and Poizat's fundamental order. We also introduce the notion of a strongly saturated model and prove an initial version of the exchange lemma. This chapter provides the groundwork for the extension to types over arbitrary sets in Section 3. For the remainder of this chapter the theory T is tacitly assumed to be stable.

We begin by noticing that if $p \in S(B)$ is definable over A, this definition induces a (possibly inconsistent) set of formulas over any set.

2.1 Definition. Let p be in $S(A)$ and B be arbitrary. If p is defined by d, the *d-extension of p* on B, denoted $d(p, B)$, is the collection of formulas $\phi(\overline{x}; \overline{b})$ with parameters from B which satisfy: $\phi(\overline{x}; \overline{b}) \in d(p, B)$ iff $\models d\phi(\overline{b})$.

We use the term d-extension because usually A will be a subset of B. The map d encodes all the information from p so we sometimes write just $d(B)$ for $d(p, B)$. This observation leads to the following definition.

2.2 Definition. i) A *preschema over A* is any map d taking formulas $\phi(\overline{x}; \overline{y}) \in F(\emptyset)$ to formulas $d\phi(\overline{y}) \in F(A)$.
ii) A *good schema* is a preschema which is also a Boolean algebra homomorphism.

Recall from Lemma 1.26 that if d defines $p \in S(M)$ then d is a good schema. Lascar [Lascar 1976] has an intermediate notion of schema.

2.3 Exercise. Let p be a type over a model M. Suppose d and d' are definitions of p. Show that for each formula $\phi(\overline{x}; \overline{y})$, $\models d\phi(\overline{y}) \leftrightarrow d'\phi(\overline{y})$.

2.4 Exercise. Show Exercise 2.3 fails if M is replaced by an arbitrary subset A.

2.5 Exercise. Give an example of $p \in S(A)$ such that $d(p, B)$ is not a (consistent) type for some B containing A.

In general, the d-extension of $p \in S(A)$ to a set may not even be consistent. However, if p is definable over a model, it is. We have the following slightly more general result.

2.6 Theorem. *Let M be a model and $p \in S(B)$. Suppose the preschema d defines over M an extension of p to an element of $S(M \cup B)$. Then for any A containing B, $d(p, A)$ is a consistent complete type.*

Proof. Consider any finite subset $\phi_1(\overline{x}; \overline{a}_1), \ldots \phi_n(\overline{x}; \overline{a}_n)$ of $d(p, A)$. Then $d\phi_i(\overline{a}_i)$ holds for each i. Now M satisfies the formula:

$$(\forall \overline{u}_1) \ldots (\forall \overline{u}_n)[\bigwedge_{i \in I} d\phi_i(\overline{u}_i) \rightarrow (\exists \overline{x}) \bigwedge_{i \in I} \phi_i(\overline{x}, \overline{u}_i)]$$

and M is a model of T so $(\exists \overline{x}) \bigwedge_{i \in I} \phi_i(\overline{x}; \overline{a}_i)$ holds in the monster model whence $d(p, A)$ is consistent. But clearly $d(p, A)$ is complete since for each $\overline{a} \in A$ and $\phi(\overline{x}; \overline{y})$, $d\phi(\overline{a}) \vee \neg d\phi(\overline{a})$ holds and $\neg d\phi$ is $d(\neg\phi)$.

2.7 Exercise. Show that if d is a good schema then for any A, $d(A)$ is a consistent complete type.

We introduce now some simple notions which turn out to be extremely useful in the context of stable theories. We begin with a simple but significant ordering relation on types.

2.8 Definition. The formula $\phi(\overline{x};\overline{y})$ is *represented* in the type p over A if for some $\overline{a}$ in A, $\phi(\overline{x};\overline{a})$ is in p.

Thus the formula $y = x^2$ is represented in $t(e^{1/2}; Q \cup \{e\})$ although not in $t(e^{1/2}; Q)$. Suppose E is an equivalence relation. If E has only finitely many classes then $E(x;y)$ is represented in every type whose domain is a model; this fails if E has infinitely many classes.

We now formalize the idea that one type satisfies more relations than another.

2.9 Definition (The Fundamental Order). Let p be in $S(A)$ and q in $S(B)$. Then $p \geq q$, if every formula which is represented in p is represented in q. We write $p \simeq q$ if $p \leq q$ and $q \leq p$. We denote the equivalence class of a type p under the resulting equivalence relation by $[p]$.

Note that the domain of this ordering relation is the collection of all types with domain a subset of the monster model. When $C \subseteq \operatorname{dom} p \cap \operatorname{dom} q$ we can naturally strengthen this notion to $p \geq_C q$ if every formula in $L(C)$ which is represented in p is represented in q. We call the resulting order the fundamental order over C. Observe that this ordering is 'upside down.' That is, $q \leq p$ if q represents more formulas than p. Nothing in this definition restricts us to types of finite sequences. Usually only a type whose domain is a model is considered in defining the fundamental order [Lascar & Poizat 1979].

2.10 Example. i) Consider the theory T of dense linear order without endpoints and suppose $a < b_1 < b_2 < \ldots$ are elements of a model of T. Let $B = \{b_1, b_2, \ldots\}$ and $C = \{b_2, b_3, \ldots\}$ and let $p = t(a, B)$ and $q = t(a, C)$. Recalling that T is quantifier eliminable, we see that (up to equivalence in T) the only formulas represented in p are equivalent to Boolean combination of $x < y_1, x < y_1 < y_2, \ldots$ and some equality formulas and these are all represented in q so $p \geq q$. Since $C \subseteq B$, $q \geq_C p$. However, the formula $x < y < b_2$ in $L(C)$ is represented in p but not in q thus $p \not\geq_C q$.

ii) Let T be a complete theory of Abelian groups which admits elimination of quantifiers (in the language of groups). Then every formula $\phi(\overline{x};\overline{y})$ is equivalent to a Boolean combination of equations of the form: $\sum_{i<n} p_i x_i + \sum_{j<m} q_j y_j = 0$. Now the position in the fundamental order of a 1-type $q \in S(A)$ is determined by the ideal, I_q, of those integers n such that if c realizes q, then nc is in A, the function f from I_q into A defined by $f(n) = nc$, and K_q, the ideal composed of integers which annihilate any realization of q. Note that K_q is determined from I_q and f by $K_q = ker(f)$.

2.11 Exercise. Let $p \in S(M)$. Show that if the type p is minimal in the fundamental order then p is realized in M. Show the converse is false. (Hint: Consider the formula $x = y$.)

2.12 Exercise. Construct the fundamental order for the theory T_1 of two refining equivalence relations and for the theory T_2 of two crosscutting equivalence relations. Assume each equivalence relation has infinitely many classes, all infinite. (These theories are considered in more detail in Section III.4.)

2.13 Exercise. Show that if p and q are comparable in the fundamental order, they have the same restriction to the empty set.

Certainly if q extends p, $p \geq q$. We distinguish those extensions q of p where the reverse inequality also holds.

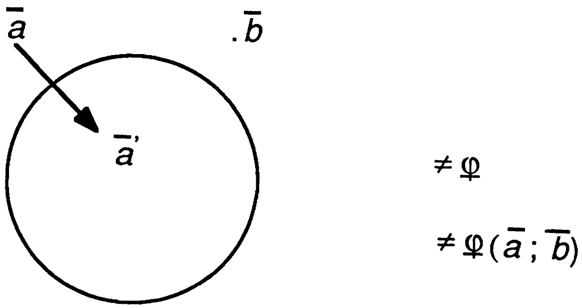

Fig. 2. $t(\bar{a}; M \cup \bar{b})$ is finitely satisfied in M.

2.14 Definition. (Fig. 2). Let M be a model of T and M be contained in A.

i) Let $p \in S(M)$; $q \in S(A)$ is an *heir* of p on A if and only if q extends p and $p \simeq_M q$.
ii) $t(\bar{a}; A)$ is a *coheir* of $t(\bar{a}; M)$ on A if and only if $t(\bar{a}; A)$ is finitely satisfiable in M.

The following exercise explains coheir in terms of heir.

2.15 Exercise. Suppose $M \subseteq A$. Let $\bar{a}$ realize $p \in S(M)$. Then $t(\bar{a}; A)$ is a coheir of p on A if $t(A; M \cup \bar{a})$ is an heir of $t(A; M)$.

Here is another characterization of $\bar{a}$ coheir.

2.16 Exercise. Show $t(\bar{a}; A)$ is a coheir of $t(\bar{a}; M)$ on A iff $t(\bar{a}; A)$ is an element of the topological closure of $\{t(\bar{c}; A) : \bar{c} \in M\}$.

We can derive the definability of types from the hypothesis that T is stable in some power λ using Beth's theorem as follows. From Beth's theorem one can deduce that each type which has a unique heir in $S(A)$ for every A containing $\operatorname{dom} p$ is definable. A non-trivial compactness argument shows that this result propagates to show that if $p \in S(M)$ is not definable (and thus has two heirs), then every heir $p_1 \in S(M_1)$ of p with $M \equiv M_1$ also has two heirs. Now a union of chains argument shows T is unstable in all sufficiently large λ. The details can be found in [Lascar & Poizat 1979] or [Pillay 1983a].

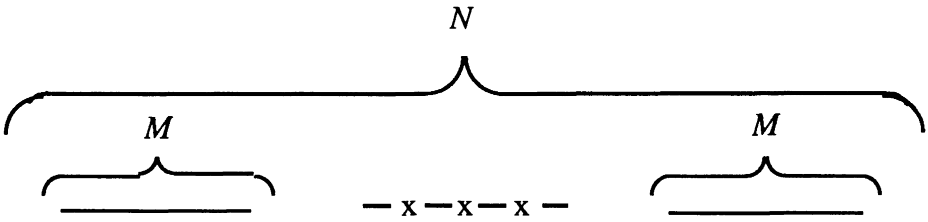

Fig. 3. Heirs

2.17 Example. Consider $(Q, <) = M$ as a dense linear order and let p be the 1-type of an irrational cut. Let N be an elementary extension of M with another copy of Q in the cut. Now p has continuum many heirs on N, namely one for each type over N extending p but not realized in N. (Fig. 3). However, p has only two coheirs on N, namely the types of points in the cuts between the models M and N. (Fig. 4). For, suppose there are elements a, b in $N - M$ such that $a < x < b$ is in q. Let c realize q. Then $x_1 < c < x_2$ is in $t(a^\frown b; Q \cup \{c\})$ and $x_1 < y < x_2$ is not represented in $t(a^\frown b; Q)$.

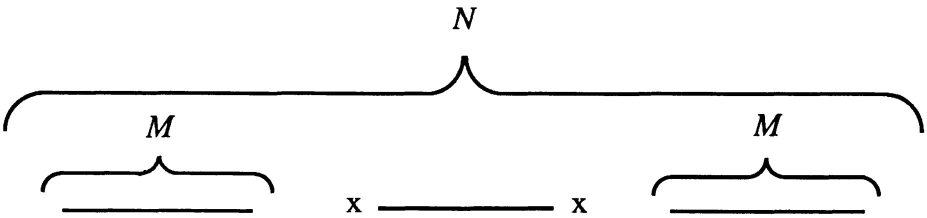

Fig. 4. Coheirs

2.18 Exercise. i) Let $A \subseteq B$ be Abelian groups and suppose $\mathrm{Th}(B)$ is quantifier eliminable, $p \in S(A)$, $q \in S(B)$. Show $p \geq q$ iff $K_p = K_q$,

$I_p \subseteq I_q$, $f_q|I_p = f_p$ and $f_q^{-1}(A) = I_p$. (K_p, I_p are defined in Exercise 2.10.)

ii) Let T be the theory of algebraically closed fields of characteristic zero. Let $k \models T$ and suppose L, K are superfields of k. Show $t(K; L)$ is an heir of $t(K; k)$ if and only if K and L are algebraically independent (equals linearly disjoint) over k ([Zariski & Samuel 1958]). What, if anything, can be said if the restriction that k is a model of T is removed?

We will consider the generalization of Exercise 2.18 i) to an arbitrary module in Section 5.

2.19 Definition. Let B be contained in A and $p \in S(A)$. Then p *splits* over B if for some $\overline{a}$, $\overline{b}$ in A, $\overline{a}$ and $\overline{b}$ realize the same type over B but for some formula $\phi(\overline{x}; \overline{y})$, $\phi(\overline{x}; \overline{a})$ is in p and $\neg\phi(\overline{x}; \overline{b})$ is in p.

2.20 Theorem. *Let M be contained in A, p an element of $S(M)$, and q a coheir of p on A. Then q does not split over M.*

Proof. Suppose $\overline{c}$ realizes q but $\phi(\overline{c}; \overline{a})$ holds while $\phi(\overline{c}; \overline{b})$ does not. Then, since $t(\overline{a}^\frown\overline{b}; M \cup \overline{c})$ is the heir of $t(\overline{a}^\frown\overline{b}; M)$ for some $\overline{m} \in M$, $\phi(\overline{m}; \overline{a})$ holds and $\phi(\overline{m}; \overline{b})$ does not. But then $\overline{a}$ and $\overline{b}$ do not realize the same type over M.

2.21 Corollary. *For any M a model of T and any p in $S(M)$ and any elementary extension N of M, the maximal number of coheirs of p on N depends only on the cardinality of M (not N).*

Note that we do not need to assume T is stable.

2.22 Exercise. Compute an upper bound in terms of $|\operatorname{dom} p|$ on the number of coheirs of p.

Now we examine the relation between the fundamental order and definability of types.

2.23 Theorem. *Let T be stable. If M is a model, $M \subseteq A$, and $p \in S(M)$ then $d(p, A)$ is the unique type q extending p such that $q \simeq p$. A fortiori, $d(p, A)$ is the unique heir of p on A.*

Proof. Note that $d(p, A)$ is an heir of p. For, $\phi(\overline{x}; \overline{a}, \overline{m}) \in d(p, A)$ (with $\overline{a} \in A - M$, $\overline{m} \in M$) implies $\models d\phi(\overline{a}, \overline{m})$ and thus $\models (\exists\overline{y})d\phi(\overline{y}, \overline{m})$. Since M is an elementary submodel of the monster model, this in turn implies $\models d\phi(\overline{m}', \overline{m})$ for some $\overline{m}' \in M$ which holds if and only if $\phi(\overline{x}, \overline{m}')$ is in p.

For the uniqueness, suppose $r \in S(A)$, $r \simeq p$ and $r \neq d(p, A)$. Fix $\theta(\overline{x}; \overline{z})$ and $\lambda(\overline{y}; \overline{z})$ as in Corollary 1.23. Since $r \neq d(p, A)$, the formula $\neg[\phi(\overline{x}; \overline{y}) \leftrightarrow \lambda(\overline{y}; \overline{z})] \wedge \theta(\overline{x}; \overline{z})$ is represented in r. But then it is represented in p, contrary to the definability of p.

If $M \subseteq A$ and $M \models T$, we can now speak of *the* definable extension of p without mentioning the particular definition d, since the last theorem asserts that all definitions of p yield the same extension to $S(A)$.

If $M \subseteq A$ and $p \in S(M)$ then p has a coheir on A. To see this, fix any $\bar{c}$ realizing p and let A' realize $d(t(A, M), M \cup \{\bar{c}\})$. Then, if f is an automorphism which fixes M and maps A' to A, $f(\bar{c})$ realizes a coheir of p on A. Unlike Theorem 2.23, this remark does not require T to be stable. With the stability assumption we could conclude the coheir is unique.

Note that Theorem 2.23 reduces the question of whether an extension of $p \in S(M)$ is an heir of p to properties of the fundamental order over the empty set rather than the fundamental order over M. More precisely, we have the following result.

2.24 Exercise. Let $M \subseteq N$, $p \in S(M)$ and $p \subseteq q \in S(N)$. Show $p \simeq q$ implies q is an heir of p.

2.25 Exercise. If $p \in S(M)$ and p is not realized in M then for any A containing M, $d(p, A)$ has infinitely many solutions in the monster model.

Now we will show the relation between heir and coheir in stable theories. This relationship leads ultimately to the exhange lemma for our notion of independence. We first indicate by a series of exercises how to prove the result using the basic definability theorem Theorem II.2.27. (This proof is due to Poizat.)

2.26 Exercise. Suppose $M \subseteq N$, N is $|M|^+$-saturated, $p \in S(M)$ and q is a coheir of p on N. Use Theorem 2.20 to show that q is definable not only over N but over M. Observe that this means that for any A with $M \subseteq A \subseteq N$, the coheir of p on A is definable over M.

2.27 Exercise. Use the previous exercise and Theorem 2.23 to show that if $p \in S(M)$ and $M \subseteq A$, the heir of p on A equals the coheir of p on A.

To introduce some useful techniques and show some further consequences of stability, we will give a different proof based on the following characterization of stability. This argument will be the only place in the development of the forking notion that we appeal to the characterization of a stable theory in terms of the cardinality of Stone spaces. The previous exercises show that this appeal is unnecessary.

2.28 Definition. The formula $\phi(\bar{x}; \bar{y})$ has the *order property* if in some model of T there exist infinite sequences $\{\bar{a}_i : i < \omega\}$ and $\{\bar{b}_i : i < \omega\}$ such that $\models \phi(\bar{a}_i; \bar{b}_j)$ iff $i < j$.

2.29 Theorem. *If some formula $\phi(\bar{x}; \bar{y})$ has the order property then T is not stable.*

Proof. Show that $\Gamma_{\langle\,\rangle}(\phi, n)$ is consistent for each n.

The converse to this theorem also holds ([Shelah 1978], Chapter II) but we will not need it here. In fact, the class of unstable theories can be divided into those which satisfy the strict order property and those which satisfy the independence property. These two notions mean that there is a formula $\phi(\bar{x}; \bar{y})$ and a sequence $\langle \bar{a}_i : i < \omega \rangle$ such that the sets $\{\phi(\mathcal{M}; \bar{a}_i) : i < \omega\}$ are, respectively, linearly ordered by inclusion or independent (i.e., every finite Boolean combination is nonempty.) We discuss these notions in more detail in Sections 4.42 and 4.43.

2.30 Theorem. *Let M be a model of the stable theory T. Suppose M is contained in C and p is in $S(M)$; then the heir of p on C equals the coheir of p on C.*

Proof. Let $\overline{a}$ realize q, a coheir of p on C. If q is not $d(p, C)$, the heir of p on C, then there is a formula $\phi(\overline{x}; \overline{y})$ and a sequence $\overline{c} \in C$ such that $\phi(\overline{x}; \overline{c}) \in q$ but $\neg d\phi(\overline{c})$. Let $r = t(\overline{c}; M)$. Since q is the coheir of p on C, $t(\overline{c}; M \cup \overline{a})$ is the heir, hence the definable extension, of r on $M \cup \overline{a}$. This means that, letting $\psi(\overline{y}; \overline{x}) = \phi(\overline{x}; \overline{y})$, $\models \phi(\overline{a}; \overline{c})$ iff $\models d'\psi(\overline{a})$ where d' defines $t(\overline{c}; M)$.

Now, choose $\overline{e}_i$, $\overline{f}_i$ for $i < \omega$, by induction, with $\overline{e}_0 = \overline{a}$ and $\overline{f}_0 = \overline{c}$ so that

i) $\overline{e}_i$ realizes the definable extension of p on $M \cup E_i \cup F_i$ and
ii) $\overline{f}_i$ realizes the definable extension of r on $M \cup E_{i+1} \cup F_i$.

Since $\overline{e}_i$ realizes p we have $d'\psi(\overline{e}_i)$ for all i. Similarly, since each $\overline{f}_i$ realizes r, we have $\neg d\phi(\overline{f}_i)$ for all i.

Now i) implies for $i > j, \phi(\overline{e}_i, \overline{f}_j)$ iff $d\phi(\overline{f}_j)$ while ii) implies for $i \leq j$ that $\models \phi(\overline{e}_i, \overline{f}_j)$ iff $\models d'\psi(\overline{e}_i)$. Thus $\models \phi(\overline{e}_i, \overline{f}_j)$ iff $i \leq j$ so T is unstable.

We can summarize the last few results in the following theorem.

2.31 Theorem. *Let M be a model of a stable theory T, $p \in S(M)$ and $M \subseteq C$; then p has a unique distinguished extension to a complete type over C which is its heir, coheir, and definable extension.*

In the next section we will define the notion of nonforking. Under this definition, for any model M and superset A, if $p \in S(A)$, p does not fork over M just if p is the definable extension of $p|M$. This notion satisfies the monotonicity and transitivity properties. ¿From Lemma II.2.10 and Theorem 2.31 we will deduce the symmetry theorem for nonforking.

We introduce now a strengthening of the notion of a saturated model which plays an important role in the investigation of stable theories. We show in Theorem XI.2.3 that this notion agrees with Shelah's notion of an F^a_κ-saturated model.

2.32 Definition. The model M is *strongly κ-saturated* if for each set A with $|A| < \kappa$, any type over A in fewer than κ free variables over A which is finitely satisfied in M is realized in M.

The crucial point in this definition is that A need not be a subset of M. The concept defined by replacing a type of less than κ elements by a type of any finite number of elements is apparently weaker. For, $t(\overline{b}_0 ^\frown \overline{b}_1; A)$ finitely satisfied in M does not imply that $t(\overline{b}_1; A \cup \overline{b}_0)$ is finitely satisfied in M. Thus, there is no obvious way to obtain the analog to the fact that if a model is saturated for types over finite sets then it is saturated for ω-types over finite sets.

2.33 Exercise. Let $p \in S(B)$ and $A \subseteq B$. Suppose there is a strongly $|B|^+$-saturated model M with $B \subseteq M$ and there is an extension of p to $p' \in S(M)$ such that p' is definable over every $N \models T$ with $A \subseteq N \subseteq M$. Then the definable extension (coheir) of p' to a global type is definable over every model containing A.

On first reading of the proof of the following lemma, take κ to be ω. We explain after the proof the, primarily notational, adjustments which must be made for arbitrary κ. The key to the proof is the recognition that in a formula $\phi(\overline{x}, \overline{y}, \overline{z})$, we are free to decide which variables are to stand for parameters and which to be viewed as the free variables of a type. We reflect this in our notation by writing $(d_r\overline{y})\phi(\overline{x}, \overline{y}, \overline{z})$ to indicate that $\overline{y}$ is the variable of the type r which is being defined. This marvelous notation was introduced in [Hrushovski 1986]. The quantifier $(d_r\overline{y})$ is read 'For generic $\overline{y}$ realizing r.'

2.34 Lemma. *If M is a κ-saturated model of a stable theory and $\kappa > |T|$ then M is strongly κ-saturated.*

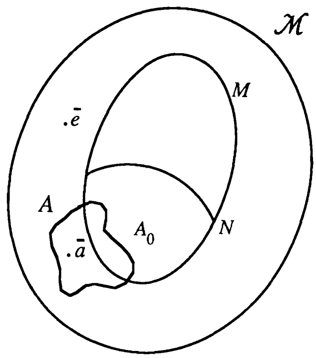

Fig. 5. Lemma 2.34

Proof. (Fig. 5). Suppose $|A| < \kappa$ and let $A \cap M = A_0$. Let $\overline{e}$ be a sequence of less than κ elements such that $p = t(\overline{e}; A)$ is finitely satisfiable in M. For any finite sequence $\overline{a}$ from $A - A_0$, we will show how to convert $t(\overline{e}; A_0 \cup \overline{a})$ into an equivalent type q over a subset N of M with $|N| < \kappa$ and $A_0 \subseteq N \subseteq M$. Given this conversion, the κ-saturation of M guarantees that $t(\overline{e}; A)$ is realized in M.

Let d define $r = t(\overline{a}; M)$ over M. By Lemma 1.24 we may suppose only $|T|$ parameters occur in the range of d. Let N be a submodel of M with A_0 and all these parameters contained in N and with $|N| < \kappa$. Now for any $\phi(\overline{x}, \overline{y}, \overline{z})$, $\overline{m} \in M$, $\overline{a} \in A - A_0$, and $\overline{b}$ from A_0,

$$\models \phi(\overline{m}, \overline{a}, \overline{b}) \text{ iff } \models (d_r\overline{y})\phi(\overline{m}, \overline{y}, \overline{b}).$$

So the required translation type is $\{(d_r\overline{y})\phi(\overline{m}, \overline{y}, \overline{b}) : \phi(\overline{x}; \overline{a}, \overline{b}) \in p\}$. To see this type is consistent, note that for any ϕ, $\models \phi(\overline{e}; \overline{a}, \overline{b})$ implies by finite satisfiability that there is an $\overline{e}' \in M$ such that $\models \phi(\overline{e}'; \overline{a}, \overline{b})$ and thus $\models (d_r\overline{y})\phi(\overline{e}', \overline{y}, \overline{b}')$.

If κ is uncountable, fix an enumeration of $\overline{e}$ (and thus of $\overline{x}$). Now the translation type is formed by applying the procedure above to each formula $\phi(\overline{x}', \overline{y}, \overline{z})$ where $\overline{x}'$ is a finite subsequence of $\overline{x}$.

2.35 Definition. Let $A \subseteq M$ and $p, q \in S(M)$. We say p is *conjugate* to q over A and write $p \sim_A q$ if there is an automorphism of M which fixes A and maps p to q.

We now relate this notion to the fundamental order. In the next argument we use the expression $t(B; A \cup \overline{c}) = t(B'; A \cup \overline{d})$. Recall from the discussion in Chapter I that this means there is an automorphism which fixes A, maps B to B' and $\overline{c}$ to $\overline{d}$.

2.36 Theorem. *Let T be stable. If M is saturated, $|M| = \kappa > |T|$, and $|A| < \kappa$ then for any $p, q \in S(M)$, if $p \simeq_A q$ then $p \sim_A q$.*

Proof. Let $\overline{c}$ realize p and $\overline{d}$ realize q. Choose for each formula $\phi(\overline{x}; \overline{y})$, formulas $\theta(\overline{x}; \overline{z})$ and $\lambda(\overline{y}; \overline{z})$ to define $t(\overline{c}; M)$ in the manner of Corollary 1.23. Choose $B \subseteq M$ such that $A \subseteq B$, $|B| \leq |A| + |T|$ and B contains all parameters necessary to define p. That is, for each θ chosen according to Corollary 1.23, there is a $\overline{b} \in B$ such that $\theta(\overline{x}; \overline{b}) \in p$ and for every $\overline{m} \in M$, $[\phi(\overline{x}; \overline{m}) \leftrightarrow \lambda(\overline{m}; \overline{b})] \in p$. Since $p \simeq_A q$, if $\beta \in \mathrm{Aut}(M)$ fixes A and maps $\overline{c}$ to $\overline{d}$, $t(\beta(B); A \cup \overline{d})$ is finitely satisfied in M. Now by strong saturation, choose $B' \subseteq M$ such that $t(B; A \cup \overline{c}) = t(B'; A \cup \overline{d})$. Let α be an automorphism of M fixing A and taking B to B'. Now for each ϕ, and each θ and λ depending on ϕ, if $\theta(\overline{x}; \overline{b}) \in p$ then $\theta(\overline{x}; \alpha\overline{b}) \in q$. Now towards a contradiction suppose that $\alpha p \neq q$. Therefore $\neg[\phi(\overline{x}; \overline{m}) \leftrightarrow \lambda(\overline{m}; \alpha\overline{b})] \in q$. Since $p \simeq_A q$, there exist $\overline{b}'$ and $\overline{m}'$ with $\theta(\overline{x}; \overline{b}') \in p$ and $\neg[\phi(\overline{x}'; \overline{m}') \leftrightarrow \lambda(\overline{m}'; \overline{b}')] \in p$, contrary to our choice of θ and λ. Thus $\alpha p = q$ as required.

We need to have M saturated to express our condition in terms of full automorphisms of the model M. A somewhat weaker version (expressing the result in terms of extending partial automorphisms) holds with only the hypothesis that M is $|A| + |T|$-saturated.

2.37 Exercise. Show the hypothesis of stability is necessary for Theorem 2.36. (Hint: Consider Example 2.10.)

Since $\mathcal{M}$ is extremely saturated and conjugate types certainly represent the same formulas we have

2.38 Corollary. *On $S(\mathcal{M})$, $\simeq_A$ and $\sim_A$ define the same equivalence relation.*

There is a further obvious corollary.

2.39 Corollary. *If T is stable in $|A|$ then the equivalence relation $\sim_A$ has at most $2^{|T|+|A|}$ equivalence classes.*

2.40 Historical Notes. There are several variants in the literature of the definition of a good schema. The definition here agrees with that in [Lascar 1976] and [Pillay 1983e] but not with the one in [Berline 1983]. The notions of coheir and fundamental order first appear in [Lascar & Poizat 1979]. The notion of splitting was introduced by Shelah. The order property characterization of stability is from [Shelah 1971]. The proof given for heir = coheir is adapted from [Lascar 1976]. The definition here of strong saturation is from [Baldwin 1984]. The relation to Shelah's notion of $F^a_{\kappa(T)}$-saturation is discussed in Chapter XI. This notion is entirely different from that called strong saturation by Shelah in [Shelah 198?c]. Theorem 2.36 is implicit in [Lascar & Poizat 1979]. There has been considerable work, [Lascar & Poizat 1979], [Harnik & Harrington 1984], [Pillay 198?] developing a local stability theory where one assumes only that a particular formula is stable rather than the entire theory. Since this project can only aid in the study of independence, as opposed to generation, we have not pursued it in this book.

3. Nonforking Types Over Sets

In Section 1 we saw that for $p \in S(M)$, M a model, the definable extension of p to a set A containing M is the kind of canonical extension we envisioned in Chapter II. Here, we find an analogous extension for types over arbitrary sets and verify that all the axioms of Section II.1 hold. This analog will be called a nonforking extension of p. Following a suggestion of M. Ziegler, we introduce nonforking by first defining the notion for types over the monster model (so called global types) and then extending it to types over arbitrary subsets in the natural manner to satisfy the monotonicity axioms.

3.1 Definition. i) Let $\hat{p} \in S(\mathcal{M})$. Then $\hat{p}$ *does not fork over* A iff for every $M \models T$ with $A \subseteq M$, $\hat{p}$ is definable over M.

ii) Let p be a type over B. Then p *does not fork over* A if for some $\hat{p} \in S(\mathcal{M})$ with $p \subseteq \hat{p}$, $\hat{p}$ does not fork over A. We say the formula $\phi(\overline{x}; \overline{y})$ *does not fork over* A if the type $\phi(\overline{x}; \overline{y})$ does not fork over A.

The following exercise gives a sufficient, but by no means necessary, condition for showing a type $\hat{p}$ does not fork over A.

3.2 Exercise. Show using Theorem 2.23 that if $\hat{p} \in S(\mathcal{M})$ and for some model $M \supseteq A$, $\hat{p}|M$ is definable over A then $\hat{p}$ does not fork over A.

There are actually three layers of complexity in defining nonforking: complete types over models, complete types over arbitrary sets, and incomplete types. For most purposes one can ignore the third case on a first reading. The cases where the extension to incomplete types gives the most trouble are pointed out.

3.3 Notation. We may write '$(p\downarrow B;A)$' for p does not fork over A where $B=\operatorname{dom} p$. We write $(\bar{c}\downarrow B;A)$ or $\bar{c}\downarrow_A B$ for $t(\bar{c};B\cup A)$ does not fork over A. This notation naturally extends to $(C\downarrow B;A)$ as explained in Section II.2. We often read this as C is independent from B over A. Note that we do not require $A\subseteq B$ to write $t(C;B)$ does not fork over A, although this will frequently be the case. If so, $(C\downarrow B;A)$ is equivalent to $t(C;B)$ does not fork over A.

If $A\subseteq B$ we write $N(B,A)$ for $\{p\in S(B): p$ does not fork over $A\}$.

If the type p with parameters from A is stationary and $A\subseteq B$ we write p^B for the, necessarily unique, nonforking extension of p to a complete type over B whose existence is guaranteed by Theorem 3.11. Other authors write $p|B$ for this extension; we reserve that notation for the restriction of a type.

3.4 Exercise. Show that $p\in S(B)$ does not fork over A iff for some strongly $(|T|+|A|+|B|)^+$-saturated model M_1 with $A,B\subseteq M_1$ there is an extension of p to $p'\in S(M_1)$ such that p' is definable over every $N\models T$ with $A\subseteq N\subseteq M_1$.

3.5 Exercise. If $\hat{p}$, $\hat{q}\in S(\mathcal{M})$, $\hat{p}$ does not fork over A and $\hat{p}\sim_A \hat{q}$ then $\hat{q}$ does not fork over A.

3.6 Exercise. Give an example of a theory T and elements a, b, c which are pairwise independent but not independent. (Hint: Almost any theory with an addition operation will do.)

We now want to show that the nonforking relation satisfies the axioms for freeness. Immediately from the definition, we see that nonforking satisfies the monotonicity axioms and the second extension axiom.

3.7 Lemma. *The nonforking relation satisfies the following axioms.*

M_1. *If $q\subseteq p$ and p does not fork over A then q does not fork over A.*
M_2. *If $A\subseteq B$ and p does not fork over A then p does not fork over B.*
E_2. *If p does not fork over A and* $\operatorname{dom} p\subseteq B$, *there is a $p_1\in S(B)$ which extends p such that p_1 does not fork over A.*

To verify our remaining existence axiom, E_1, we show that if $p\in S(A)$ then p does not fork over A. We derive this from the fact that heir and coheir are the same for types over models and the following lemma. The lemma is proved by a compactness argument. The next exercise spells out the method for showing the consistency of each finite subset of the infinite set of sentences considered in the lemma.

3.8 Exercise. Suppose $p_1,\ldots,p_k$ is an increasing sequence of types in the fundamental order with each $p_i\in S(M_i)$. Show there is a model M containing $\operatorname{dom} p_i$ for each i and a type $p\in S(M)$ such that $p\geq p_i$ for each i. (Hint: Let M contain $\bigcup_i M_i$ and let p be the definable extension of p_k to M.)

3.9 The Extension Lemma. Let $p\in S(A)$. There exists a $\hat{p}\in S(\mathcal{M})$ with $p\subseteq\hat{p}$ such that for every model M with $A\subseteq M$, $\hat{p}$ is the heir of $\hat{p}|M$.

Proof. Let $Z = \{q$: There exists a model M, $A \subseteq M$, $q \in S(M)$ and $p \subseteq q\}$. Z is partially ordered by $\leq_A$. If p^* is a maximal element of Z and $\hat{p}$ is the heir of p^* on $\mathcal{M}$ then by Theorem 2.23, $\hat{p}$ is as required. To show such a p^* exists, we need, invoking Zorn's lemma, show only that if $P = \{p_i : i \in I\}$ is an increasing chain in Z then P has an upper bound p'. For this, let N be a model of the following set of sentences, Γ, which can easily be shown consistent using the previous exercise. Γ is in the language obtained by adding to L new constant symbols $c_0, \ldots, c_{n-1}$, names for the members of $\operatorname{dom} p_i = M_i$, and a new unary predicate symbol R. Let Γ be:

$$T \cup T|R \cup \bigcup_{i \in I} \operatorname{Diag}(M_i) \cup \{R(m) : m \in M_i, i \in I\}$$
$$\cup \{(\forall \overline{y})(R(\overline{y}) \rightarrow \neg\phi(\overline{c}; \overline{y})) : (\exists i)\phi(\overline{x}; \overline{y}) \text{ is not represented in } p_i\}.$$

There is little difficulty in extending Lemma 3.9 to allow the type p to have infinitely many variables and we will use the extended version without comment.

3.10 Theorem. *If p is a type over A then p does not fork over A.*

Proof. In view of the first monotonicity axiom we can assume $p \in S(A)$. By the previous lemma, we can find $\hat{p} \in S(\mathcal{M})$ which extends p and is the heir of each of its restrictions to a model containing A. By Theorem 2.31, (heir = coheir) it is the definable extension of each of these restrictions. Thus $\hat{p}$ and, a fortiori, p does not fork over A.

Note that the use of (heir = coheir = definable extension) is vital here; a direct proof of the existence of a $\hat{p}$ which is the coheir of each of its restrictions can be given but is much longer (see Paragraph 3.30).

We can now establish a useful equivalent to nonforking.

3.11 Definition. (Fig. 6).

i) The formula $\phi(\overline{x}; \overline{c})$, where the parameters $\overline{c}$ come from anywhere in the monster model, is *almost satisfied* in A if for every model M containing A, $\phi(\mathcal{M}; \overline{c}) \cap M \neq \emptyset$.
ii) The type p is *almost satisfied* in A if every finite conjunction of formulas in p is almost satisfied in A.

3.12 Lemma. *Let $A \subseteq B$ and let p be a type over B; p does not fork over A iff p is almost satisfied in A.*

Proof. If p does not fork over A then $p \subseteq \hat{p}$ for some global type $\hat{p}$ which is definable over M for each model M containing A (and thus by Theorem 2.31 finitely satisfied in M). For the converse, suppose p is almost satisfied in A. Choose by the Extension Lemma a strongly $|B|^+$-saturated model M such that $t(M; B)$ does not fork over A. Then p is finitely satisfied in M and thus realized in M by some $\overline{c}$. Now $p \subseteq t(\overline{c}; B)$ which does not fork over A. By monotonicity, p does not fork over A.

3.13 Corollary. *If p is a type over B which is closed under conjunction and p forks over A then there is a formula $\phi(\overline{x}; \overline{b}) \in p$ such that every type containing $\phi(\overline{x}; \overline{b})$ forks over A.*

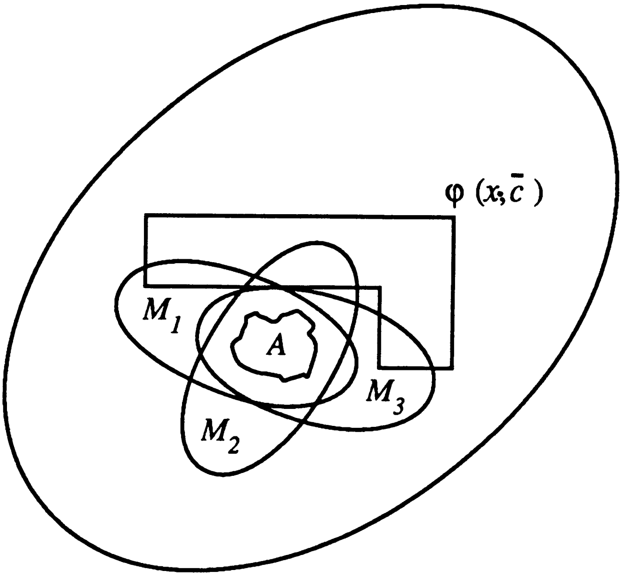

Fig. 6. $\phi(x;\bar{c})$ is almost satisfied in A.

Proof. The notion of almost satisfaction obviously has the required finite character.

The necessity of the assumption that p is closed under finite conjunction is demonstrated in Example V.3.10.

The notation disguises an important aspect of Corollary 3.13. The corollary does not say there is a formula $\phi(\bar{x};\bar{y})$ such that for any $\bar{b}'$, any type containing $\phi(\bar{x};\bar{b}')$ forks over A. This holds only for those '$\bar{b}'$' which realize $t(\bar{b};A)$. We will show in Corollary IV.2.2 a closely related result: if $A \subseteq B$, $q \in S(B)$ and q forks over A then there is a formula $\phi(\bar{x};\bar{y})$ such that for some $\bar{b}$, $\phi(\bar{x};\bar{b}) \in q$, and for any $\bar{c}$, $q|A \cup \phi(\bar{x};\bar{c})$ forks over A.

The following exercise gives Corollary 3.13 a useful topological form.

3.14 Exercise. Show that for any $A \subseteq B$, $N(B,A)$ (the collection of types in $S(B)$ which do not fork over A) is closed in $S(B)$.

The next few results connect the fundamental order with the notion of forking. Recall $[q]$ denotes an equivalence class in the fundamental order.

3.15 Definition. For any A and any $p \in S(A)$, the *bound of* p, $\beta(p)$ is the least upper bound of $W = \{[q] : p \subseteq q$ and q is a type over a model$\}$.

We justify this definition by deducing from the following lemma that W does have a unique least upper bound.

3.16 Exercise. Show, using both the fact that the class of nonforking types is closed under isomorphism and the Extension Lemma that for any B and A we can find an N such that $t(B;N)$ does not fork over A.

3.17 Lemma. *Let $A \subseteq B$, $p \in S(B)$, and suppose p extends to $\hat{p} \in S(\mathcal{M})$ which does not fork over A. Let M be any model containing B and let $p|A \subseteq q \in S(M)$. Then $[q] \leq_A [\hat{p}|M]$.*

Proof. Choose, by Exercise 3.16, an N which is strongly $|M|^+$-saturated and so that $t(M; N)$ does not fork over A. Since $\hat{p}$ does not fork over A, $\hat{p}|M$ is realized in N by some $\overline{c}$. Let $\overline{d}$ realize q and let α be an automorphism of $\mathcal{M}$ which fixes A and maps $\overline{d}$ to $\overline{c}$. Let $\hat{q}$ denote the definable extension of q to a global type. Then, $\alpha t(\overline{d}; M) = (\alpha\hat{q})|(\alpha M) = t(\overline{c}; \alpha M)$. Now suppose $\phi(\overline{x}; \overline{y}) \in F(A)$ is represented in $[\hat{p}]$. This implies that for some $\overline{m} \in M$, $\models \phi(\overline{c}; \overline{m})$. Since $t(\overline{m}; N)$ is almost satisfiable in A, for some $\overline{m}' \in \alpha M$, $\models \phi(\overline{c}; \overline{m}')$. That is, $\phi(\overline{x}; \overline{y})$ is represented in $\alpha\hat{q}$, hence in $\hat{q}$, hence in q.

Of course, in Lemma 3.17 we could replace $\leq_A$ by the weaker relation $\leq$. Now we can show rapidly that the nonforking relation satisfies the rest of the axioms from Section II.1. We note first that forking can now be described in terms of the fundamental order.

3.18 Corollary. *Fix any A and $p \in S(B)$ with $A \subseteq B$. If $q = p|A$ then p forks over A iff $\beta(p) < \beta(q)$.*

Proof. Let $\hat{p}$ and $\hat{q}$ be $\beta(p)$ and $\beta(p|A)$ respectively. Then $\hat{q}$ does not fork over A. If $\hat{p}$ does not fork over A then p does not fork over A so Lemma 3.17 implies $[\hat{p}] = [\hat{q}]$. For the converse, if p forks over A, then for every $\hat{p}$ extending p to S($\mathcal{M}$) there is an M containing A such that $\hat{p}$ is not the definable extension of $\hat{p}|M$ so by Theorem 2.23, $[\hat{p}] < [\hat{p}|M]$. But by Lemma 3.17, $[\hat{p}|M] \leq [\hat{q}|M]$ and hence $\beta(p) < \beta(p|A)$.

We now can deduce the transitivity axiom without effort. From Theorem III.2.31 and the reduction of the symmetry lemma to symmetry for types over models in Lemma II.2.12 we derive the symmetry lemma.

3.19 Corollary. i) (Transitivity) *Let $A \subseteq B \subseteq C$ and let $p \in S(C)$. If p does not fork over B and $p|B$ does not fork over A then p does not fork over A.*

ii) (Symmetry) *If $t(\overline{a}; A \cup \overline{b})$ does not fork over A then $t(\overline{b}; A \cup \overline{a})$ does not fork over A.*

3.20 Exercise. Show that $(\overline{a}^\frown\overline{b}\downarrow\overline{c}^\frown\overline{d}; A)$ implies $(\overline{a}\downarrow\overline{d}; A \cup \overline{b} \cup \overline{c})$.

Applying Lemma 3.17 once again we get the conjugacy lemma for nonforking extensions.

3.21 Corollary (The Conjugacy Lemma). *If $\hat{p}|A = \hat{q}|A$ and neither $\hat{p}$ nor $\hat{q}$ forks over A then $\hat{p} \sim_A \hat{q}$.*

Proof. By Lemma 3.17 we have $\hat{p} \simeq_A \hat{q}$ whence the result follows by Theorem 2.36.

The following exercise provides an extremely useful corollary to the conjugacy lemma.

3.22 Exercise. Let $\phi(\overline{x};\overline{b})$ be a formula. Show that the set of types q in $S(\mathcal{M})$ which do not fork over A and contain a conjugate of $\phi(\overline{x};\overline{b})$ is the same as the collection of q in $S(\mathcal{M})$ which do not fork over A and satisfy: $q|A$ has a nonforking extension to a type over $\mathcal{M}$ which contains $\phi(\overline{x};\overline{b})$. Note that the first of these collections is open in $N(\mathcal{M},A)$.

The next result, the open mapping theorem, has enormously important consequences. These consequences and some generalizations of the result will be discussed in Chapter X. It is an easy consequence of the results at hand.

3.23 Theorem (The Open Mapping Theorem). *The restriction map r from $N(B,A)$ onto $S(A)$ is an open map.*

Proof. First note it suffices to prove the result for $B = \mathcal{M}$, since the restriction map from $\mathcal{M}$ to A is the composition of the map from $\mathcal{M}$ to B with the one from B to A. We use the following easy topological fact: if f is a continuous map between compact spaces and W is a saturated (i.e., $W = f^{-1}(f(W))$), open subset of the domain, then $f(W)$ is open. Thus, to show r is an open map it suffices to show that if U is an open subset of $N(\mathcal{M},A)$ there is a saturated $\bar{U}$ with $r(U) = r(\bar{U})$. But this is exactly the content of the previous exercise.

Since it is clear that types over models are stationary we have established all the axioms except $\overline{\kappa}(T) < \infty$. This easily follows from Corollary 3.18 but we delay the derivation until Theorem 4.22 since Section 4 is devoted to an exhaustive analysis of $\kappa(T)$ and its variants.

The following exercise provides a useful reformulation of the open mapping theorem.

3.24 Exercise. Show that if $\phi(\overline{x};\overline{b})$ does not fork over M then there is a formula $\psi_\phi(\overline{x};\overline{a}) \in F(M)$ such that if $p \in N(M \cup \overline{b}, M)$ and $\psi_\phi(\overline{x};\overline{a}) \in p$ then $\phi(\overline{x};\overline{b}) \in p$. (Hint: Use the fact that *all* types over M are stationary.)

3.25 Exercise. Show the previous exercise fails if the model M is replaced by an arbitrary set A.

We can summarize the various characterizations of a nonforking extension as follows.

3.26 Theorem. *Let $A \subseteq B$ and $p \in S(B)$. The following are equivalent.*

i) *p does not fork over A.*
ii) *p is almost satisfied in A.*
iii) *There is a model M with $B \downarrow_A M$ and a good schema d over M with $p \subseteq d(\mathcal{M})$.*

Proof. By Lemma 3.12, i) implies ii). To see that ii) implies iii), note that if p is finitely satisfiable in M and M is strongly $|\operatorname{dom} p|^+$-saturated then p is contained in a type which is defined by a good schema d over M (taking $d\phi(\overline{x};\overline{y})$ to be $\phi(\overline{m};\overline{y})$ where $\overline{m}$ realizes p). Now choosing an appropriate

M we have ii) implies iii). Assuming iii), let $\overline{c}$ realize the extension of p to $M \cup B$ given by the schema d. Then $t(\overline{c}; M \cup B)$ does not fork over M. Since we have $t(B; M)$ does not fork over A, symmetry and transitivity yield $t(B; M \cup \overline{c})$ does not fork over A; by symmetry and monotonicity we conclude $t(\overline{c}; B)$ does not fork over A and we finish.

3.27 Exercise. Let $A \subseteq M$ and $p \in S(A)$. Show that $q \in S(M)$ does not fork over A if and only if $[q]$ is maximal among the $[r]$ for r an extension of p to a type over a model.

3.28 Exercise. Give an example of an $A \subset B$ and a $p \in S(B)$ such that p forks over A even though p is definable over A. (Hint: Consider an equivalence relation with infinitely many infinite classes. Let a be a member of one these classes and show the type over a which contains only the formula $E(x, a)$ is definable over $\emptyset$ but forks over $\emptyset$.)

3.29 Exercise. Show using the example in the previous exercise that the stronger form of Corollary 3.13 discussed in the paragraph after Corollary 3.13 is false.

3.30 Other Approaches to Forking. The definition here of forking is a variant on the definition in [Lascar & Poizat 1979]. We will briefly discuss another variation on the Lascar-Poizat approach due to Harnik and Harrington. Both the approach in the text above and the one outlined below differ from that in Lascar-Poizat primarily in the way they avoid certain arguments which depend on an extremely judicious choice of objects in general position. This theme is by no means eliminated from the theory, it is just played less often. In Section V.3 we discuss the original approach of Shelah and in Chapter VII we consider the definition of forking in terms of rank.

3.31 Definition. The type $p \in S(A)$ *needs* the formula $\phi(\overline{x}; \overline{y})$ if for each extension of p to a type p' over a model, $\phi(\overline{x}; \overline{y})$ is represented in p'. In particular, if $\phi(\overline{x}; \overline{a}) \in p$ then p needs $\phi(\overline{x}; \overline{y})$.

3.32 Exercise. The type p needs $\phi(\overline{x}; \overline{y})$ iff for some formula $\delta(\overline{v})$ over A and some finite set I with $\overline{v}_i \subseteq \overline{v}$ for $i \in I$:

$$(\exists \overline{v})\delta(\overline{v}) \wedge (\forall \overline{v})(\delta(\overline{v}) \rightarrow \bigvee_{i \in I} \phi(\overline{x}; \overline{v}_i)) \in p.$$

3.33 Definition. If $A \subseteq B$ and $p \in S(B)$ then p *does not fork* over A just if every formula needed by p is also needed by $p|A$.

With this definition it is easy to verify the transitivity axiom but the extension lemma requires a complicated compactness argument referring to the definition of stability in terms of the number of ϕ-types. The key step is to prove: If p needs $\phi \vee \psi$ then p needs ϕ or p needs ψ.

3.34 Exercise. Derive this assertion assuming that the two notions of forking are the same.

3.35 Exercise. Show that $p \in S(M)$ does not fork over A if and only if for every $\phi(\overline{x};\overline{y})$ which is represented in p, $p|A$ needs ϕ.

3.36 Exercise. Prove directly that if p needs $\phi \vee \psi$ then p needs ϕ or p needs ψ. (Hint: (Ziegler) The following five steps are one way to solve this exercise.)

1. Show using Exercise 3.32 that if p needs $\phi(\overline{x};\overline{u}) \vee \psi(\overline{x};\overline{u})$ then for some formula $\delta(\overline{v})$: $(\exists\overline{v})\delta(\overline{v}) \wedge (\forall\overline{v})[\delta(\overline{v}) \rightarrow \phi(\overline{x};\overline{v}) \vee \psi(\overline{x};\overline{v})] \in p$.
2. Now if p is over A and p needs $\phi \vee \psi$ but p needs neither ϕ nor ψ show that there is a $|T|^+$-saturated $M \supseteq A$ and sequences $\overline{b}_0$, $\overline{b}_1$ realizing p such that for every $\overline{m} \in M$, $\models \neg\phi(\overline{b}_1;\overline{m}) \wedge \neg\psi(\overline{b}_0;\overline{m})$.
3. By mapping $\overline{b}_0$ to $\overline{b}_1$ and considering both M and its image under this map, trade the picture of two realizations of p over one model, for a picture with one realization of p and two models. That is, construct an element $\overline{c}$ and two models M_0, M_1 contained in a third model N with all three saturated such that

   i) $\overline{c}$ realizes p.
   ii) $\phi(N;\overline{c}) \cap M_0 = \emptyset$; $\psi(N;\overline{c}) \cap M_1 = \emptyset$.
   iii) $\emptyset \neq \delta(N) \cap M_0 \subseteq \psi(N;\overline{c})$; $\emptyset \neq \delta(N) \cap M_1 \subseteq \phi(N;\overline{c})$.

4. Using the fact that $N \approx M_i$ for $i = 1, 2$, iterate this construction ω times to build a tree.
5. Deduce that $\phi(\overline{x};\overline{u})$ is unstable.

3.37 Historical Notes. The notion of forking was invented by Shelah [Shelah 1978]. The use of finite satisfiability as a characterization of a nonforking type over a model also appears in [Shelah 1978]. The fundamental order was discovered by Lascar and Poizat[Lascar & Poizat 1979]. They showed how to define forking in terms of the fundamental order, heirs, and coheirs. The approach in Definition 3.30 is described in [Harnik & Harrington 1984]. Some further refinements occur in Harnik [Harnik 1985]. Both of these approachs require an appeal to the characterization of stability in terms of cardinality of Stone spaces as well as the definability of types. Our approach, which avoids this appeal and which uses strong saturation to eliminate still more of the appeals to symmetry, first appeared inBaldwin [Baldwin 1984]. A further refinement of this approach is due to Rothmaler [Rothmaler 1983]. In fact, the precise definition of almost satisfaction (closing under conjunction) is taken from [Rothmaler 1983]. Another approach which avoids the appeal to uncountable cardinals was developed by Hodges [Hodges 1981].

4. $\kappa(T)$ and the Spectrum of Stability

One of the most remarkable features of the stability notion is that it provides a fruitful division of all first order theories into four classes: unstable, stable, superstable and ω-stable. Most of this book is devoted to developing the positive properties of the last three of these classes. In this section we define this classification and provide prototypic examples of stable and superstable theories. We also calculate the stability spectrum (the class of cardinals where T is stable) and the spectrum of saturation (the class of cardinals where T has a saturated model) of a theory T. We show, with the possible exception of a few small cardinals, these spectra are the same. We prove in fact that there are only four possible functions for the spectra of stability and these determine the classification.

We introduce three invariants of a theory T: $\kappa(T)$, $\lambda(T)$, and $\mu(T)$. The first two determine the stability classification. The main step in the classification is Theorem 4.25, whose proof provides a certain free tree to witness the non-superstability of a theory. This tree is used here to calculate the spectrum of saturation of T and will be used in Section IX.6 to show that a stable but not superstable theory has 2^λ models of power λ. We refine this tree to show the nonexistence of saturated models in certain cardinalities. We conclude by discussing unstable theories.

4.1 Definition. i) T is *stable in* μ if for every A with $|A| \leq \mu$, $|S(A)| \leq \mu$.

ii) T is *stable* if T is stable in some μ.

iii) The theory T is *superstable* if T is stable in all $\mu \geq 2^{|T|}$.

iv) The countable theory T is *ω-stable* (or totally transcendental) if T is stable in all infinite μ.

A theory T is *strictly stable* if it is stable but not superstable while T is *strictly superstable* if it is superstable but not ω-stable. Shelah [Shelah 1978] provides the appropriate generalization of ω-stability to uncountable theories.

One of the most natural ways of constructing theories which are stable but not ω-stable is through consideration of families of equivalence relations. There are several variations on this idea. These variations depend on three factors, how many equivalence relations there are, whether the different equivalence relations crosscut or refine, and whether there is any additive structure present. The following exercise is an important tool for investigating these examples.

4.2 Exercise. Show that if T is a quantifier eliminable theory of equivalence relations (i.e. T asserts each basic relation of T is an equivalence relation.) then T is stable.

4.3 Example (Refining Equivalence Relations with Finite Splitting). See Fig. 7. For each ordinal α let REI_α denote the theory of α equivalence relations E_i, $i < \alpha$, such that for $i < j$, E_j refines E_i and each E_i class is refined into infinitely many E_{i+1} classes.

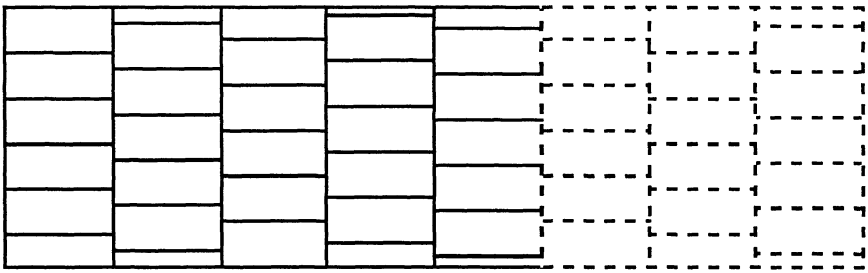

Fig. 7. REI_2

The following series of exercises outlines the crucial properties of a number of examples. Full solutions to these exercises, particularly Exercise 4.16, would be much longer than the solutions to the other exercises in this book.

4.4 Exercise. Show each REI_α is quantifier eliminable and stable.

4.5 Exercise. Show REI_α is not superstable if α is infinite.

4.6 Example (Refining Equivalence Relations with Infinite Splitting).

i) Let REF_α be the variant on Example 4.3 obtained by insisting that each E_i equivalence class is refined into two E_{i+1} equivalence classes.
ii) Let REF^+_α be the theory of the structure $(2^\alpha, +, E_i)_{i<\alpha}$ where the $E_i(\sigma, \tau)$ holds only if $\sigma|i = \tau|i$ The operation $+$ is interpreted as coordinatewise addition mod 2.

4.7 Exercise. Show that REF_α is stable. Show that if $\alpha < \omega$ then REF_α is superstable.

4.8 Exercise. Show that if $\alpha \geq \omega \cdot \omega$ then REF_α is not superstable.

4.9 Example (Crosscutting Equivalence Relations with Finite Splitting).

i) Let CEF_κ be the variant on Example 4.6 obtained by demanding that for $i < \kappa$ each equivalence class of E_{i+1} splits *each* equivalence class of E_i into two classes. Thus, each E_i has only two classes.
ii) Let CEF^+_ω be the theory obtained from CEF_ω as REF^+_ω is obtained from REF_ω, i.e. $Th(2^\omega, E_i, +)$ where $E_i(\sigma, \tau)$ holds if and only if $\sigma|n = \tau|n$ and $+$ denotes coordinatewise addition.

4.10 Exercise. Show that if $\kappa \leq \omega$ then CEF_κ is superstable.

4.11 Exercise. Show that if T is CEF_ω or REF_ω, and A is an arbitrary subset of a model of T, then for any unrealized $p \in S(A)$, p does not fork over $\emptyset$.

4.12 Example (Crosscutting Equivalence Relations with Infinite Splitting). (Fig. 8). Let CEI_κ be the variant of Example 4.9 obtained by insisting that for $i < j$ each E_j equivalence class intersects *each* E_i class in infinitely many classes.

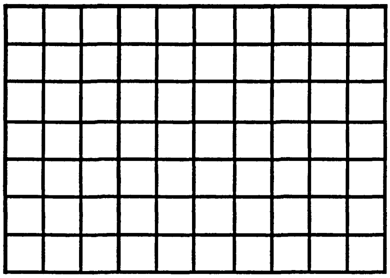

Fig. 8. CEI_2

Note that the theories of crosscutting equivalence relations are defined with respect to a cardinal while the refining equivalence relations are defined with respect to an ordinal.

4.13 Exercise. Formulate precisely the idea indicated in Example 4.12 and determine the spectrum of stability of the resulting theory.

4.14 Notation. When we refer to any of these examples without specifying κ, we mean κ to be ω.

4.15 Exercise. Show each of these examples is ω-stable if κ is finite.

4.16 Exercise. Compute the number of models of each of these theories in every power. (This is rather subtle for Example 4.6.)

4.17 Exercise. Find natural models for each of these theories. For $\kappa = \omega$, the universe of the natural model will be either 2^ω or ω^ω .

In Chapter II and Section III.2 we discussed the local (or finite) character of nonforking. That is, for any type p, there is a 'small' subset A of $\operatorname{dom} p$ such that p does not fork over A. We show here the size of this A can be uniformly bounded for all p by a cardinal $\kappa(T)$ and this cardinal largely determines the stability spectrum of T. There are several slight variations on the definition of the cardinal $\kappa(T)$. We begin by defining $\overline{\kappa}(T)$ as well and describing the relation between these two cardinals. If $|T|$ is singular we will have to consider still a third invariant $\kappa_r(T)$.

4.18 Definition. i) $\kappa(T)$ is the least infinite cardinal, if one exists, such that for any finite sequence $\overline{a}$ and each strictly ascending sequence of sets $\langle A_i : i < \kappa(T)\rangle$, for some i, $t(\overline{a}; A_{i+1})$ does not fork over A_i.

ii) $\overline{\kappa}(T)$ is the least infinite cardinal, if one exists, such that for every type p there is a set $A \subseteq \operatorname{dom} p$ with $|A| < \overline{\kappa}(T)$ and p does not fork over A.

iii) In the event that the cardinals described in i) and ii) do not exist we set $\overline{\kappa}(T)$, respectively $\kappa(T)$, equal to ∞.

iv) Let $\kappa_r(T) = \kappa(T)$ if $\kappa(T)$ is regular and $\kappa(T)^+$ if $\kappa(T)$ is singular.

While $\overline{\kappa}(T)$ is the bound that we will apply in practice; it is easier to evaluate $\kappa(T)$. Fortunately, the cardinals are almost the same. In particular, they are equal for countable T. The next two results showing the relations between these cardinals are of most interest for uncountable languages. We begin with a technical proposition.

4.19 Proposition. *For every $\lambda < \kappa(T)$, $\mathrm{cf}(\lambda) < \overline{\kappa}(T)$.*

Proof. Suppose $\langle A_i : i < \lambda \rangle$ and $\overline{a}$ are such that $t(\overline{a}; A_{i+1})$ forks over A_i for each $i < \lambda$ and the A_i are a continuous increasing chain. Then if $B \subseteq A_\lambda$ and $t(\overline{a}; A_\lambda)$ does not fork over B, $|B| \geq \mathrm{cf}(\lambda)$. So for every $\lambda < \kappa(T)$, $\mathrm{cf}(\lambda) < \overline{\kappa}(T)$.

4.20 Lemma. *For every T, $\overline{\kappa}(T) \leq \kappa(T) \leq \overline{\kappa}(T)^+$. If $\overline{\kappa}(T)$ is regular then $\kappa(T) = \overline{\kappa}(T)$.*

Proof. Choose $\lambda < \overline{\kappa}(T)$ and let p be a type such that for every $A \subseteq \mathrm{dom}\, p$ with $|A| < \lambda$, p forks over A. Define $\langle A_i : i < \lambda \rangle$ so that $A_{i+1} - A_i$ is finite and $p|A_{i+1}$ forks over A_i. (This is possible by the finite character of forking.) Then $\langle A_i : i < \lambda \rangle$ demonstrates $\kappa(T) > \lambda$. Thus, $\kappa(T) \geq \overline{\kappa}(T)$. The second inequality and the second assertion are immediate from the preceding proposition.

4.21 Examples. $\overline{\kappa}(\mathrm{REI}_{\aleph_\omega}) = \aleph_\omega$. $\kappa(\mathrm{REI}_{\aleph_\omega}) = \aleph_\omega^+$. Let T be the theory of a disjoint union of models of REI_α for $\alpha < \aleph_\omega$. Then $\overline{\kappa}(T) = \kappa(T) = \aleph_\omega$.

Now we bound $\kappa(T)$.

4.22 Theoren. *For every T, $\kappa(T) \leq |T|^+$.*

Proof. Suppose $\langle A_i : i < \lambda \rangle$ and $\overline{a}$ are such that $t(\overline{a}; A_{i+1})$ forks over A_i for each i and the A_i are a continuous increasing chain. Now, if $p_i = t(\overline{a}; A_i)$, by Corollary 3.20 we have $[\hat{p}_i] > [\hat{p}_j]$ if $i < j$. But it is clear that any decreasing sequence in the fundamental order has at most $|T|$ elements so $\lambda < |T|^+$ as required.

4.23 Exercise. What are the possible values of $\kappa(T)$ and $\overline{\kappa}(T)$ when T is countable?

4.24 Exercise. Define $\kappa^1(T)$ to be the cardinal one obtains in the definition of $\kappa(T)$ if the types p are required to be 1-types. Show $\kappa^1(T) = \kappa(T)$. (Hint: If $t(a^\frown b; A_{i+1})$ forks over A_i then either $t(a; A_{i+1})$ forks over A_i or $t(b; A_{i+1} \cup a)$ forks over $A_i \cup a$.)

We will now show that for stable T there is a cardinal $\lambda(T) \leq 2^{|T|}$ such that T is stable in μ if and only if $\mu = \mu^{<\kappa(T)}$ and $\mu \leq \lambda(T)$. Simple cardinal computations show that the conclusion of this assertion holds for exactly the same μ if $\kappa(T)$ is replaced by $\kappa_r(T)$.

This computation of the stability spectrum could be carried out for an arbitrary freeness relation satisfying the axioms discussed in Section II.1. Such an argument would show that any first order theory with a freeness relation satisfying the axioms is stable. We give such an abstract formulation in Section VII.1 and show the 'categoricity' of the freeness axioms. Note that in the next construction we use only the monotonicity, extension, symmetry, and transitivity axioms along with the definition of $\kappa(T)$. We could rephrase this theorem as saying that if $\kappa(T)$ is infinity then T is unstable.

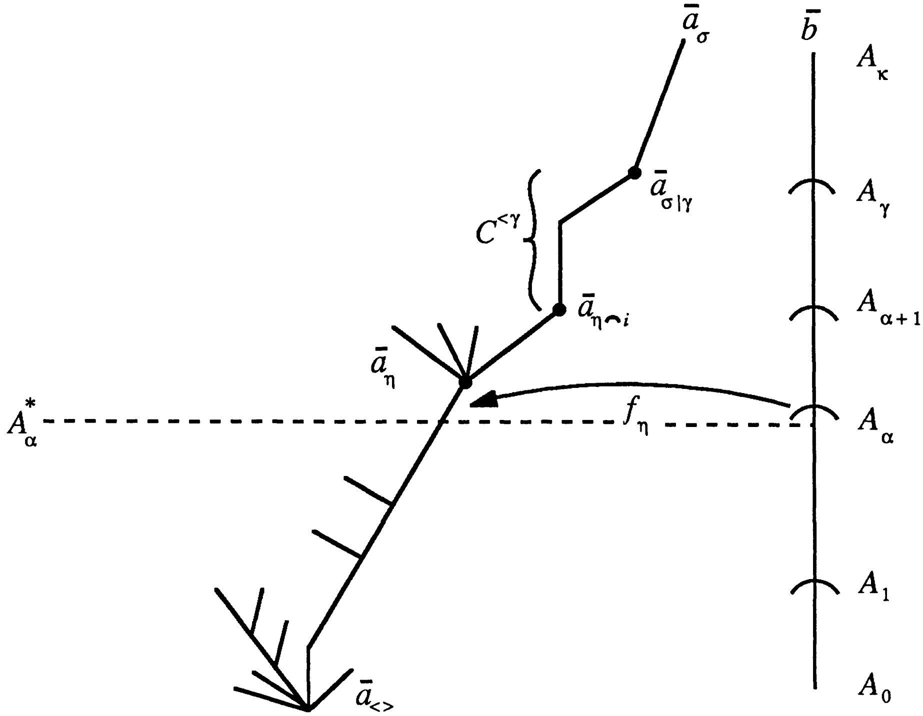

Fig. 9. Building a Shelah Tree

4.25 Theorem. *If T is stable and $\mu < \mu^{<\kappa(T)}$ then T is not stable in μ.*

Proof. (Fig. 9). Let κ be the least cardinal such that $\mu^\kappa > \mu$. Then there exists a sequence $\langle A_i : i < \kappa\rangle$ and a sequence $\bar{b}$ such that $t(\bar{b}; A_{i+1})$ forks over A_i (as κ is certainly less than $\kappa(T)$). Without loss of generality we may assume that $|A_i| \le |i| + \aleph_0$, the sequence is continuous and for each i, $A_{i+1} - A_i$ is a finite sequence $\bar{a}_i$. (This depends on the finite character of forking.) Now for each $\nu \in \mu^{\le\kappa}$ we will define an elementary map f_ν with $\operatorname{dom} f_\nu = A_{\lg(\nu)}$ and $\operatorname{rng} f_\nu \subseteq \mathcal{M}$. If $\lg(\nu) = \delta$, a limit ordinal, let f_ν be $\bigcup_{\alpha<\delta} f_{\nu|\alpha}$. If $\lg(\nu) = \alpha + 1$, first let $A^*_\alpha = \bigcup_{\eta\in\mu^\alpha} \operatorname{rng} f_\eta$. Now for $\nu = \eta\frown i$, define $f_{\eta\frown i}$ by induction on $i < \mu$ by choosing $\bar{a}_{\eta\frown i} = f_{\eta\frown i}(\bar{a}_\alpha)$ so that $t(\bar{a}_{\eta\frown i}; A^*_\alpha \cup \{\bar{a}_{\eta\frown j} : j < i\})$ does not fork over $A_\eta = f_\eta(A_\alpha)$ and extends $f_\eta(t(\bar{a}_\alpha; A_\alpha))$. For $\sigma \in \mu^\kappa$, let g_σ be an arbitrary extension to $A_\kappa \cup \{\bar{b}\}$ of f_σ and extend $t(g_\sigma(\bar{b}); A_\sigma)$ to a complete type over $A^* = \bigcup_{\alpha<\kappa} A^*_\alpha$ realized by $\bar{a}_\sigma$ such that $t(\bar{a}_\sigma; A^*)$ does not fork over A_σ.

Now, if $\sigma \ne \tau$, then $t(\bar{a}_\sigma; A^*) \ne t(\bar{a}_\tau; A^*)$. To see this, note first that since g_σ is an elementary map, $t(\bar{a}_\sigma; A_{\sigma|\beta} \cup \bar{a}_{\sigma|\beta+1})$ forks over $A_{\sigma|\beta}$ for any $\beta < \kappa$. Let $\alpha = \beta + 1$ be least such that $\bar{a}_{\sigma|\alpha} \ne \bar{a}_{\tau|\alpha}$; so $A_{\sigma|\beta} = A_{\tau|\beta}$. We will show $t(\bar{a}_\sigma; A_{\tau|\beta} \cup \bar{a}_{\tau|\alpha})$ does not fork over $A_{\tau|\beta}$. Let $C_\gamma = \{\bar{a}_{\sigma|\delta} : \beta < \delta \le \gamma\}$. We show by induction that for every γ, with $\beta < \gamma \le \kappa$:

$$(C_\gamma \downarrow \bar{a}_{\tau|\alpha}; A_{\sigma|\beta}). \qquad (*)$$

The induction step depends on the following exercise which is essentially Lemma II.2.11.

4.26 Exercise. Let $A \cap B = \emptyset$ and $A \cap C = \emptyset$. If $C \downarrow_A B$ and $D \downarrow_{A \cup C} B$ then $C \cup D \downarrow_A B$.

Returning to the proof of the theorem, for $\gamma = \beta + 1$, (*) holds by the choice of $\overline{a}_{\tau|\alpha}$ (and the symmetry lemma). Now fix γ with $\beta < \gamma \leq \kappa$; let $C^{<\gamma} = \{\overline{a}_{\sigma|\delta} : \beta < \delta < \gamma\}$ and suppose by induction: $C^{<\gamma} \downarrow_{A_{\sigma|\beta}} \overline{a}_{\tau|\alpha}$ To apply the exercise, take A as $A_{\sigma|\beta}$, B as $\overline{a}_{\tau|\alpha}$, C as $C^{<\gamma}$, and D as $\overline{a}_{\sigma|(\gamma+1)}$; then $C \cup D$ is C_γ. The second hypothesis of the exercise follows from the construction by monotonicity. Applying the exercise we finish the induction.

Consider the case $\gamma = \kappa$. It asserts $C_\kappa \downarrow_{A_{\sigma|\beta}} \overline{a}_{\tau|\alpha}$. By monotonicity, as $\overline{a}_{\sigma|\kappa} = \overline{a}_\sigma$, $\overline{a}_\sigma \downarrow_{A_{\sigma|\beta}} \overline{a}_{\tau|\alpha}$. Thus, $t(\overline{a}_\sigma; A_{\sigma|\beta} \cup \overline{a}_{\tau|\alpha}) \neq t(\overline{a}_\tau; A_{\sigma|\beta} \cup \overline{a}_{\tau|\alpha})$ and we finish.

In this construction we only had to assure:

$$\overline{a}_{\eta^\frown i} \underset{A_\eta}{\downarrow} A^*_\alpha \cup \{\overline{a}_{\eta^\frown j} : j < i\}.$$

The relation of $\overline{a}_{\eta^\frown i}$ to $\overline{a}_\nu$ for a ν with $\lg(\nu) = \alpha$ but $\nu|\alpha \neq \eta|\alpha$ is irrelevant. We can deduce a further consequence to the proof. In fact, as we see below, a_σ can realize at most $2^{\kappa(T)}$ distinct types of the form $t(a_\tau; A_\tau)$. But $|A_\sigma| \leq \kappa(T)$. This leads to a proof (see Exercise 4.40 below) of Corollary 4.27 ii) with the additional hypothesis that $\mu \leq 2^{(T)}$. However, we can exploit the indiscernibility of the tree constructed in Theorem 4.25 to obtain an even stronger result.

The tree described in Theorem 4.27 i) epitomises a strictly stable theory. It is sometimes referred to as a Shelah tree.

4.27 Corollary. *Let T be a countable stable, but not superstable, theory.*

i) *There exists a sequence of formulas $\phi_i(\overline{x};\overline{y})$ for $i < \omega$ and for each μ sequences $\overline{a}_\eta$ for each $\eta \in \mu^{<\omega}$ such that:*

   a) *For each $\eta \in \mu^{<\omega}$, $\{\overline{a}_{\eta^\frown\alpha} : \alpha < \mu\}$ is a set of indiscernibles over $A_\eta = \{a_\nu : \nu \not\subseteq \eta\}$.*
   b) *For each $\sigma \in \mu^\omega$, $p_\sigma = \{\phi_i(\overline{x};\overline{a}_{\sigma|i}) : i < \omega\}$ is consistent.*
   c) *For each $\eta \in \mu^{<\omega}$ and $\alpha \neq \beta < \mu$, $\phi_i(\overline{x}, a_{\eta^\frown\alpha}) \wedge \phi_i(\overline{x}; a_{\eta^\frown\beta})$ is inconsistent.*

ii) *If $\aleph_1 \leq \mu < \mu^\omega$ then T does not have a saturated model of power μ.*

Proof. To prove i) we will construct a sequence of at most countably many approximations to the ϕ_i and $\overline{a}_\eta$. The kth approximation will satisfy conditions a) and b) (for $\overline{a}_\eta$ depending on k) and the following weaker form of c).

c′) For each $i < \omega$ and each $\eta \in \mu^i$ there is an m^k_i such that if $X \subset \mu$ and $|X| \geq m^k_i$, $\{\phi^k_i(\overline{x};\overline{a}_{\eta^\frown\alpha}) : \alpha \in X\}$ is inconsistent.

The chief point of the construction is that for each $i < \omega$, $m_i^{k+1} \leq m_i^k$ and for some $i < \omega$, $m_i^{k+1} < m_i^k$. In fact, for each i, for all sufficiently large $k < \omega$, $m_i^k = 2$. Thus, we eventually satisfy condition c).

For $k = 0$, we take the $\overline{a}_\eta$ constructed in Theorem 4.25 and choose $\phi_i(\overline{x}; \overline{y})$ to witness that $t(\overline{b}; A_i \cup \overline{a}_i)$ forks over A_i. Thus $\phi(\overline{x}; \overline{a}_\eta)$ forks over A_η. Since the $\langle \overline{a}_{\eta \frown \alpha} : \alpha < \mu \rangle$ are independent over A_η, Theorem II.2.18 guarantees the existence of m_i^0 for each $i < \omega$.

We say that i requires attention at stage k if for each $j < i$, $m_j^k = 2$. We may suppose then that at a certain stage, i is the least ordinal which requires attention. Thus we have $\overline{a}_\eta$, ϕ_j^k and m_j^k for $j < \omega$ so that conditions a), b) and c$'$) are satisfied and $m_j^k = 2$ if $j < i$. We must choose new $\overline{a}_\eta$, ϕ_j^{k+1} and m_j^{k+1} so that $m_i^{k+1} < m_i^k$. Fix $\eta \in \mu^i$. There are two cases.

To describe the first case, let $\hat{j}$ denote a sequence of j 1's. Then we are in the first case if the following collection of formulas is consistent.

$$\Gamma = \{\phi_l^k(\overline{x}; \overline{a}_{\eta|l}) : l < i\} \cup \{\phi_{i+j}^k(\overline{x}; \overline{a}_{\eta \frown \hat{j}}) : j < \omega\} \cup \{\phi_i^k(\overline{x}; a_{\eta \frown 0})\}.$$

Let

$$\begin{aligned} \psi_j^{k+1} &= \psi_j^k \quad \text{if } j \neq i \\ \psi_i^{k+1}(\overline{x}; \overline{y}_1, \overline{y}_2) &= \psi_i^k(\overline{x}; \overline{y}_1) \wedge \psi_i^k(\overline{x}; \overline{y}_2) \\ m_j^{k+1} &= m_j^k \quad \text{if } j \neq i \\ m_i^{k+1} &= [(m_i^k + 1)/2]. \end{aligned}$$

The indiscernibility of $\{\overline{a}_{\eta \frown \alpha} : \alpha < \mu\}$ at the previous stage guarantees that we have satisfied the condition at $\overline{a}_\eta$ and thus at $\overline{a}_\nu$ for each $\nu \in \mu^i$.

If Γ is inconsistent, we are in the second case. That is, for some $n < \omega$

$$\Gamma_0 = \{\phi_l^k(\overline{x}; \overline{a}_{\eta|l}) : l < i\} \cup \{\phi_{i+j}^k(\overline{x}; \overline{a}_{\eta \frown \hat{j}}) : j < n\} \cup \{\phi_i^k(\overline{x}; a_{\eta \frown 0})\}$$

is inconsistent. In this case, let

$$\begin{aligned} \phi_l^{k+1} &= \phi_l^k \quad \text{if } l < i \\ \phi_i^{k+1} &= \textstyle\bigwedge_{l < i+n} \phi_l(\overline{x}; \overline{y}_l) \\ \phi_l^{k+1} &= \phi_{l+n}^k \quad \text{if } l > i \\ m_l^{k+1} &= m_l^k \quad \text{if } l < i \\ m_i^{k+1} &= 2 \\ m_l^{k+1} &= m_{l+n}^{k+1} \quad \text{if } l > i. \end{aligned}$$

For ii), let κ be the least cardinal such that $\mu^\kappa > \mu$. Suppose for contradiction that M is a saturated model of T with power μ. Since $\mu^{<\kappa} \leq \mu$ and M is μ-universal, the tree constructed in part i) can be embedded in M. Identify the original tree with its image in M. Now the $\{p_\sigma : \sigma \in \mu^\omega\}$ are pairwise inconsistent and $\mu^\omega < \mu$ so they cannot all be realised in M contrary to the assumption that M is saturated.

Now we introduce the invariant that was mentioned in the introduction to Part A, the bound on the number of free (i.e. nonforking) extensions of an independent set. We will denote this invariant by $\mu(T)$ but it will not figure prominently in the sequel since we will shortly (Lemma 4.32) bound it by a more visible invariant denoted by $\lambda(T)$. If we had proceeded totally abstractly we would have deduced stability of T from the assumption that $\kappa(T)$ and $\mu(T)$ are less than infinity (and our other axioms).

4.28 Definition. Let p be a type in finitely many variables.

i) Let $\mu(p)$, the *multiplicity* of p be the least cardinal μ such that p does not have μ^+ pairwise contradictory nonforking extensions.
ii) Let $\mu(T)$, the *global multiplicity of* T, be the least cardinal μ such that for any type p, $\mu(p) \leq \mu$.
iii) Let $\mu_\phi(p)$, the *ϕ-multiplicity of* p, be the least cardinal μ such that there do not exist μ^+ contradictory ϕ-types which do not fork over $\operatorname{dom} p$ but whose restriction to $\operatorname{dom} p$ is p_ϕ.

We can derive a sufficient condition for stability in χ in terms of the global multiplicity of T and one further invariant.

4.29 Definition. For a stable theory T, let $\lambda(T)$ denote the least cardinal in which T is stable.

4.30 Lemma. *For any cardinal χ, if $\mu(T)$ and $\lambda(T)$ are at most χ and $\chi = \chi^{<\kappa(T)}$ then T is stable in χ.*

Proof. Let $|B| = \chi$. For each $A \subseteq B$ with $|A| < \kappa(T)$, $|S(A)| \leq 2^{<\kappa(T)}$. Each element of $S(A)$ has at most $\mu(T)$ distinct extensions in $S(B)$ which do not fork over A. But if $p \in S(B)$, for some $A \subseteq B$ with $|A| < \overline{\kappa}(T) \leq \kappa(T)$, p does not fork over A. Thus, there are at most $\chi^{<\kappa(T)}$ (the number of subsets of B with power $\kappa(T)$) times $2^{<\kappa(T)}$ (the number of types over each A) times $\mu(T)$ (the number of nonforking extensions of a type over A) which equals χ types over B.

To discuss the properties of $\lambda(T)$ and $\mu(T)$ for uncountable theories we need the following lemma which is obvious when the language is countable or if $\chi > |T|$.

4.31 Lemma. *If T is stable in χ then every set of power χ is contained in a model of power χ.*

Proof. Let A be an arbitrary set of power χ. Define a sequence A_i for $i < \omega$ with $A_0 = A$ and such that $|A_i| = \chi$ for all i and A_{i+1} realizes all types over A_i. Then $A_\omega = \cup\{A_i : i < \omega\}$ is the required model of T.

Now we can bound $\mu(T)$ by $\lambda(T)$ and $\lambda(T)$ by $2^{|T|}$.

4.32 Lemma. *If T is stable (i.e. if $\lambda(T) < \infty$) then $\mu(T) \leq \lambda(T) \leq 2^{|T|}$.*

Proof. For the first inequality, let $p \in S(B)$. There exists A with $A \subseteq B$ and $|A| < \overline{\kappa}(T)$ such that p does not fork over A. We know from Lemma 4.25 (taking $\lambda(T)$ as μ) that $\lambda(T) \geq \kappa(T)$. So if $|A| < \kappa(T)$, we can choose by Lemma 4.31 a model, M, containing A with $|M| \leq \lambda(T)$ and thus $|S(M)| \leq \lambda(T)$. But types over models are stationary so the number of contradictory nonforking extensions of $p|A$ and, a fortiori, of p is bounded by $\lambda(T)$ as required.

Now we prove the second inequality. For any A and any $p \in S(A)$, p can be viewed as the product over $\phi \in L$ of p_ϕ. But, since T is stable for each ϕ, $|S_\phi(A)| \leq |A|$ so $|S(A)| \leq |A|^{|T|}$. In particular, T is stable in $2^{|T|}$.

4.33 Exercise. Show there is no relation between $\mu(T)$ and $\kappa(T)$. For example, if T is theory of countably many independent unary predicates $\kappa(T) = \omega$ and $\mu(T) = 1$. But, if T is CEF_ω, $\kappa(T) = \omega$ and $\mu(T) = 2^\omega$.

Now using Lemma 4.32 we calculate the class of cardinals in which a stable theory is stable.

4.34 Theorem (The Stability Spectrum Theorem). *If T is stable then T is stable in χ if and only if $\chi = \lambda(T) + \chi^{<\kappa(T)}$.*

Proof. By Theorem 4.25 and the definition of $\lambda(T)$, the equality holds if T is stable in χ. The converse is immediate from Lemma 4.29 and 4.32.

From Theorem 4.34, Exercise 1.8, and simple facts about cardinals (e.g. There are arbitrarily large cardinals of cofinality ω; $\chi^{\mathrm{cf}(\chi)} > \chi$.) one can easily deduce the following result.

4.35 Corollary. *T is superstable iff $\kappa(T)$ is ω.*

Theorem 4.34 gives a characterization of the stability spectrum above $2^{|T|}$. For countable T, $\lambda(T) = \aleph_0$ or $\lambda(T) = 2^{\aleph_0}$ and the following theorem describes the situation completely. If T is uncountable and $\lambda(T) < 2^{|T|}$ the more complicated situation is completely described in III.5 of [Shelah 1978].

4.36 Theorem. *For a countable theory T one of the following four mutually exclusive situtations holds.*

i) *T is stable in all χ.*
ii) *T is stable in all $\chi \geq 2^{\aleph_0}$.*
iii) *T is stable in χ iff $\chi^{\aleph_0} = \chi$.*
iv) *T is stable in no χ.*

Proof. This easily follows from our previous results noting that $\kappa(T)$ is either $\aleph_0$ or $\aleph_1$.

We can combine Theorems 4.25 and 4.34 to establish

4.37 Theorem. *Suppose T is a countable stable theory and $\chi \geq \kappa(T)$. Then T has a saturated model of power χ iff T is stable in χ.*

Proof. Suppose that T is not stable in χ. Then by Theorem 4.34, either $\lambda(T) > \chi$ (and certainly T does not have a saturated model of power χ) or $\chi < \chi^{<\kappa(T)}$. In the second case, T does not have a saturated model in power χ by Corollary 4.27.

We now prove that if T is stable in χ then T has a saturated model of power χ. Let M_0 be an arbitrary model of power χ and construct by induction a continuous chain of models $\{M_i : i < \chi\}$ such that $|M_i| = \chi$ for all i and every type over M_i is realized in M_{i+1}. Now let M be the union of the chain $\{M_i : i < \chi\}$. We will show M is saturated. Let A be a subset of M with $|A| < \chi$ and let p be in $S(A)$. If χ is regular, it is easy to see any such p is realized in M so for the rest of the proof we assume χ is singular. Extend p to r in $S(M)$. There exists A_0 contained in M such that r does not fork over A_0 and $|A_0| < \kappa(T)$. Thus, we may assume A_0 is contained in M_0 as, by Theorem 4.25, $\kappa(T) \leq \mathrm{cf}(\chi)$. Now, choose $E = \{\bar{e}_i : i < \chi\}$ such that $\bar{e}_i$ is in M_{i+1} and realizes $r|M_i$. Note that $r|M_0$ is stationary. As in the proof of Theorem II.1.30, choose $E_0 \cup M_0 \subseteq E$ such that $t(A; E \cup M_0)$ does not fork over $E_0 \cup M_0$ and $|E_0| \leq \kappa(T) + |A| < \chi$ (since χ is singular). Thus $E - E_0$ is not empty. Any element e of $E - E_0$ satisfies: $t(e; E_0 \cup A \cup M_0)$ does not fork over $E_0 \cup M_0$ (by the symmetry lemma). But also $t(e; E_0 \cup M_0)$ does not fork over M_0 so $t(e; E_0 \cup A \cup M_0)$ does not fork over M_0. Hence e realizes p.

This argument is very similar to that in Theorem II.1.30. We obtain the sharper result because we have computed $\kappa(T)$. Note that the existence of saturated models did not rely essentially on the hypothesis that T is countable.

The following exercises consider some variations on Corollary 4.27 ii) and Theorem 4.37.

4.38 Exercise. Show that if $\langle M_i : i < \delta \rangle$ is an increasing sequence of λ-saturated models and $\kappa(T) \leq \mathrm{cf}(\delta)$ then $\bigcup_{i<\delta} M_i$ is λ-saturated.

4.39 Exercise. Find examples of superstable countable theories which do and do not have countable saturated models.

4.40 Exercise. Prove the following weaker form of Corollary 4.27 ii) without relying on Theorem 4.27 i). If $2^{|T|} \leq \mu \leq \mu^{<\kappa(T)}$ then T has no saturated model of power μ. (Hint: Imbed the tree of Theorem 4.25 in a putatively saturated model and show no $\bar{b}$ can realize more than $2^{|T|}$ of the p_σ.)

4.41 Exercise. Prove the following generalization of Theorem 4.37 to uncountable languages. Suppose T is stable and $\chi > \kappa(T)$. Then T has a saturated model of power χ if and only if T is stable in χ. (Hint: Extend the proof of Theorem 4.27 i) to deal with trees contained in $\mu^{<\kappa(T)}$ for uncountable $\kappa(T)$. Then proceed as in Theorem 4.37.)

4.42 Unstable Theories. In this subsection we briefly summarise the properties of unstable theories. An introductory but more detailed account of this subject appears in [Pillay 1983a]; a full treatment appears in Chapter II of [Shelah 1978].

For any theory T let $\sigma(\kappa) = \sigma(T, \kappa) = \sup\{|S(A)| : |A| \leq \kappa\}$ (where A is a subset of the monster model of T). The stability spectrum theorem shows that for countable stable T there are only three possibilities for this function: $\sigma_1(\kappa) = \kappa$, $\sigma_2(\kappa) = \sup(\kappa, 2^\omega)$, $\sigma_3(\kappa) = \kappa^\omega$. In the presence of the generalized continuum hypothesis the only other possible such function is

$\tau(\kappa) = 2^\kappa$. With no hypothesis beyond ZFC it is provable that there are only three possible such functions for unstable theories. These functions depend on the value of $\text{ded}(\kappa)$, the supremum of the cardinals λ such that there is a linear ordering of power κ with λ Dedekind cuts. Shelah [Shelah 1978] and Keisler ([Keisler 1976], [Keisler 1978]) have shown that the only other possibilities for σ are $\sigma_4(\kappa) = \text{ded}(\kappa)$, $\sigma_5(\kappa) = \text{ded}(\kappa)^\omega$, and $\sigma_6(\kappa) = 2^\kappa$. Each of these functions is determined by a syntactic property of T.

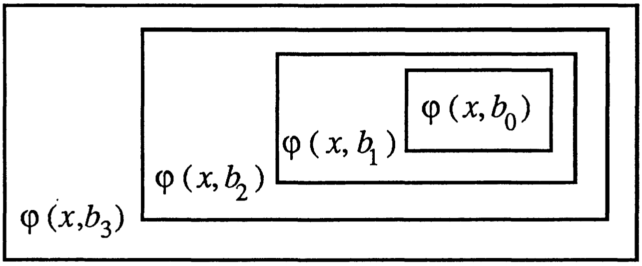

Fig. 10. $\phi(x; y)$ has the strict order property.

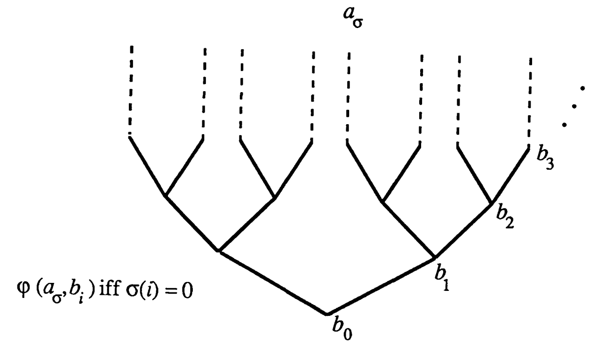

Fig. 11. $\phi(x; y)$ has the independence property.

4.43 Definition. i) T has the *strict order property* (Fig. 10) if there are a formula $\phi(\overline{x}; \overline{y})$ and sequences $\langle \overline{b}_i : i < \omega \rangle$, $\langle \overline{a}_i : i < \omega \rangle$ such that

$$\models \phi(\overline{a}_i; \overline{b}_j) \text{ iff } i < j.$$

ii) T is *multiply ordered* if there exist a family of formulas ϕ_i for $i < \omega$ and sequences $\overline{a}_{i,j}$ for i, $j < \omega$ and $\overline{c}_\sigma$ for $\sigma \in \omega^\omega$ such that

$$\models \phi_i(\overline{c}_\sigma; \overline{a}_{i,j}) \text{ iff } j < \sigma(i).$$

iii) T has the *independence property* (Fig. 11) if there are a formula $\phi(\overline{x}; \overline{y})$ and sequences $\langle \overline{a}_i : i < \omega \rangle$ and $\langle \overline{c}_\sigma : \sigma \in 2^\omega \rangle$ such that

$$\models \phi(\overline{a}_i; \overline{c}_\sigma) \text{ iff } \sigma(i) = 0.$$

Shelah [Shelah 1978] shows a theory has the order property (Fig. 12) if and only if it is unstable (Fig. 13) if and only if it has either the strict

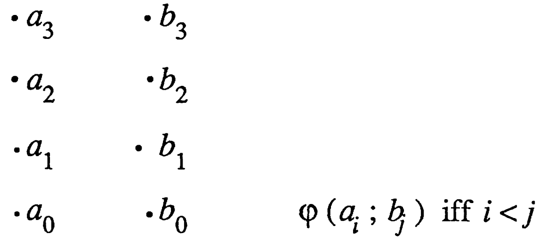

Fig. 12. $\phi(x;y)$ has the order property.

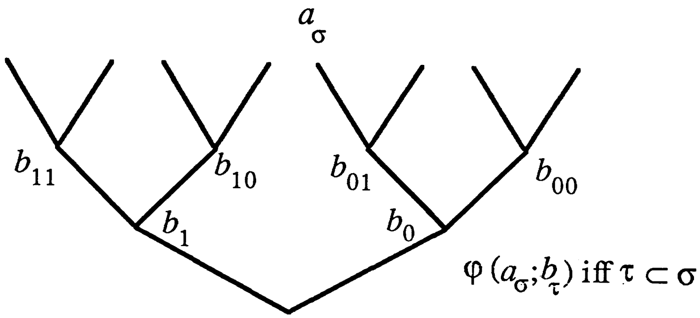

Fig. 13. $\phi(x;y)$ is unstable.

order property or the independence property ([Shelah 1978] II.2.25). Keisler shows the equivalence between $\sigma(T,\kappa) = \sigma_5$ and T is multiply ordered. It is not known whether it is consistent for σ_4 and σ_5 to be distinct functions. There has been considerable work on such questions as replacing $\phi(\overline{x};\overline{y})$ by $\phi(x;\overline{y})$ in the above definitions and in trying to extend results about stable theories to theories which are not too badly unstable.

The results discussed here illustrate a common phenomenon in stability theory which recurs for example in the solution of the spectrum problem for the number of models of an ω-stable theory. The spectrum functions that we have computed involve the exponential function and thus the computation of the functions is very much subject to axioms of set theory beyond those of ZFC. The properties which distinguish the cases, however, stability, strict order property etcetera are all very low in the Levy hierarchy (arithmetic in this case) and so are absolute between, e.g. V and L.

4.44 Historical Notes. Morley [Morley 1965] introduced the notion of an ω-stable countable theory and proved that an ω-stable theory is stable in all powers. Shelah, in considering uncountable languages, discovered the importance of considering single formulas and introduced the full stability hierarchy. The key Theorem 4.25 for establishing the stability spectrum theorem appears here with a new proof (also due to Shelah). This proof substitutes machinery from stability theory for the combinatorial argument in [Shelah 1971]. The particular argument given here for Theorem 4.37 was suggested to me by W.W. Tait. The theorem was first proved in various special cases by Harnik [Harnik 1975a] and in full generality by Shelah [Shelah 1978]. Poizat [Poizat 1985] gives a straightforward proof that an unstable theory has no saturated model of power λ if λ is singular.

5. *Definable Chain Conditions in Algebra*

In this section we apply the stability classification to the classification of such natural algebraic structures as groups, rings, modules and fields. Our primary interest is the identification of a family of examples for the general theory. Since the basic notions of stability theory are extremely general formulations of well known algebraic phenomena, they should and do acquire a concrete and natural interpretation when applied to the most common algebraic structures.

In general the calculation of the stability spectrum depends on finding the cardinality of the Stone space and in finding certain trees of formulas. But if a structure admits a multiplication then these trees can be replaced by chains.

5.1 Definition. i) A group G satisfies the *ω-stable descending chain condition* if there is no infinite properly descending chain of definable subgroups of G. (Fig. 14).

ii) A group G satisfies the *superstable descending chain condition* if there is no infinite properly descending chain of definable subgroups of G such that each subgroup has infinite index in its predecessor.

iii) A group G satisfies the *stable descending chain condition* if there is no infinite properly descending chain of definable subgroups of G each defined by an instance of a single formula $\phi(x;\overline{y})$.

iv) Let G be a group, $\phi(x;\overline{y})$ a formula, and Γ the collection of finite intersections of ϕ-definable subgroups of G. The group G satisfies the *full stable descending chain condition* if there is no infinite properly descending chain of elements of Γ.

There are two somewhat unusual usages in the statements of these definitions and the following theorem. First, when we say G is a group, we mean some formula in the language of G defines a binary operation under which G is a group. Thus, G may admit additional structure. For example, any field is a group in this sense. Secondly, in the following theorem we take properties such as stability, which are defined for theories, to be properties of algebraic structures, by saying, for example, G is stable when the theory of G is. With these caveats we can state the following theorem.

5.2 Theorem. i) *If G is ω-stable then G satisfies the ω-stable d.c.c.*

ii) *If G is superstable then G satisfies the superstable d.c.c.*

iii) *If G is stable then G satisfies the stable and even the full stable d.c.c.*

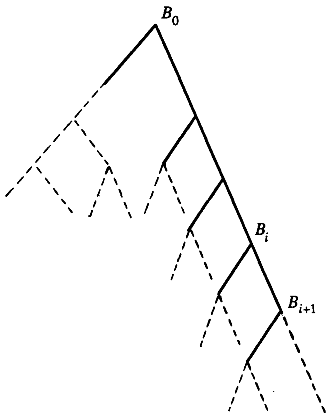

Fig. 14. The ω-stable d.c.c.

Proof. i) Let $\langle B_i : i < \omega \rangle$ be a properly decreasing chain of definable subgroups of G. Choose $b_i \in B_i - B_{i+1}$. For each $\eta \in 2^\omega$ define the type p_η which asserts that $x \in C_{\eta|i}$ where $C_{\eta|i}$ is the coset of B_{i+1} which contains $\prod_{j<i} \eta(j) b_j$. Now as the types p_η are pairwise inconsistent and can be extended to complete types over the countable set of elements necessary to define the B_i and their translates, there are $2^{\aleph_0}$ types over this set so G is not ω-stable.

The proof for ii) is similar, noting that since the index of B_{i+1} in B_i is infinite, one can construct an ω-branching tree and thus contradict superstability.

iii) It is immediate from the definition that a stable group satisfies the stable d.c.c. The following argument extends the result to the full stable descending chain condition. Let $\psi(x, y, \overline{z})$ denote $\phi(x \cdot y, \overline{z})$. Thus, each translate of a subgroup defined by an instance of ϕ is defined by an instance of ψ. So, if Γ contains an infinite descending chain the same type of argument as in i) allows us to constuct a full binary tree of instances of ψ. Thus, T is unstable.

This consideration of formulas which define the translations of sets within a group is the foundation for Poizat's *stratified order* [Poizat 1981]. This refinement of the notion of the fundamental order provides an elegant setting for much of the material discussed in this section. We survey this approach in Section VII.3.

5.3 Exercise. Show that if G is a stable group then G satisfies the descending chain condition on centralizers of elements. (Recall that the centralizer of $a \in G$ is $\{b \in G : ba = ab\}$.) We denote it by $C_G(a)$.)

5.4 Exercise. Show that if G is a stable group then for any subset X of G there is a finite $X_0 \subseteq X$ with $C_G(X) = C_G(X_0)$.

5.5 Exercise. Show that if G is an ω-stable Abelian group then G has the form $D \oplus H$ where D is a divisible group and every element of H

has order less than n for some integer n. In this situation H is said to have *bounded exponent.* (Hint: Consider the descending chain of definable subgroups $\langle n!G : n < \omega\rangle$.)

Macintyre [Macintyre 1971] proved the converse to the last exercise and noted that ω-stability is preserved by finite direct sums.

While the descending chain conditions are implied by the stability properties for any group, Garavaglia discovered the remarkable fact that for modules the converses hold. The statement of these converses is facilitated by the following useful notation.

5.6 Definition. Fix a theory T of modules. For any $p = t(\overline{a}; B)$, let p^+ denote $\{\phi(\overline{x};\overline{b}) \in p \colon \phi(\overline{x};\overline{y})$ a p.p. formula$\}$. Further, let p^- denote $\{\neg\phi(\overline{x};\overline{b}) \in p \colon \phi(\overline{x};\overline{y})$ a p.p. formula$\}$.

Note that p is axiomatized by $p^+ \cup p^-$ and p^- is determined by p^+. Thus the number of types over a set A is determined by the number of possible p^+ over A. The following exercise relies on the p.p. elimination theory for modules discussed in Section II.

5.7 Exercise. Prove that every theory of modules is stable.

5.8 Theorem. *For any module M, M is superstable, respectively, ω-stable just if M satisfies the superstable chain condition, respectively, the ω-stable chain condition on p.p.-definable subgroups.*

Proof. We treat the superstable case and leave the easier ω-stable version as an exercise. If M satisfies the superstable d.c.c. for p.p.-definable subgroups, then for every type p there is a formula $\phi(x;\overline{b}) \in p$ such that there is no $\psi(x,\overline{c}) \in p$ such that $\psi(\mathcal{M};\overline{0})$ has infinite index in $\phi(\mathcal{M};\overline{0})$. Now p^+ can be axiomatized by p.p. formulas $\psi(x;\overline{c})$ such that $\models \psi(x;\overline{c}) \rightarrow \phi(x;\overline{b})$. But since each $\psi(\mathcal{M};\overline{0})$ has finite index in $\phi(\mathcal{M};\overline{0})$, there are for each ψ only finitely many choices for inequivalent formulas $\psi(x;\overline{c})$. Thus we have only $|R| + |B| + \aleph_0$ choices for $\phi(x;\overline{b})$ and then only $2^{|R|+\aleph_0}$ choices for the sequence of $\psi(x;\overline{c})$. That is, the number of positive types over B and thus the number of types over B is bounded by $2^{|R|+\aleph_0} \times |B|$. Hence T is superstable.

The following argument provides a sufficient condition for a theory to be unstable.

5.9 Lemma. *Let M be a structure and $\phi(\overline{x};\overline{y})$ an atomic formula. If there exist $\overline{a},\overline{b} \in M$ such that*

$$\models \phi(\overline{a};\overline{a}) \wedge \phi(\overline{b};\overline{b}) \wedge \phi(\overline{a};\overline{b}) \wedge \neg\phi(\overline{b};\overline{a})$$

then M^ω is unstable.

Proof. Let $\langle \overline{c}_i : i < \omega\rangle$ be the sequence of elements of M^ω defined by: $\overline{c}_i(j) = \overline{a}$ if $j \geq i$ and $\overline{c}_i(j) = \overline{b}$ if $j < i$. Then $M^\omega \models \phi(\overline{c}_i;\overline{c}_k)$ if and only if $i \leq k$. Thus, ϕ has the order property (Definition 4.43) and so T is unstable.

This lemma has proved a key for applying stability theory to investigate varieties of algebras. [Baldwin & Lachlan 1973] contains a complete discussion of categoricity in power for varieties and more generally for universal Horn classes. [Baldwin &McKenzie 1982] shows that much of the spectrum problem for, at least, congruence modular varieties can be reduced to the study of varieties of modules. All of these studies are based on extending the definition of stability to include incomplete theories. An arbitrary theory T is said to be stable if every complete extension of T is stable. The following dichotomy summarises the value of this definition in the solution of the spectrum problem for incomplete theories. If some extension of T has the maximal number of models then so does T; the requirement that every complete extension of T be stable imposes very strong conditions on T.

5.10 Exercise. Show that if M is not ω-stable then M^{ω} is not superstable.

5.11 Exercise. Show that every non-Abelian variety of groups is unstable. (Hint: Apply Lemma 5.9 to the formula $\phi(x_0, x_1, y_0, y_1)$ which asserts $[x_0, y_1] = 1$. See [Baldwin & Lachlan 1971], [Baldwin & Saxl 1976].)

Many of the examples in the rest of the book come from the theory of modules. Thus, we need to develop some machinery for determining whether types in the theory of modules fork. The following definition is the first step.

5.12 Definition. Let T be a theory of modules, $A \subseteq \mathcal{M}$, and $p \in S(A)$.

i) Let $\mathcal{G}(p)$ be

$$\{\phi(\mathcal{M};\bar{0}) : \text{ for some } \bar{a} \in A,\ \phi(\bar{x};\bar{a}) \in p^{+}\}$$

and let $\mathcal{G}_0(p)$ be

$$\{\phi(\mathcal{M};\bar{0}) : \phi(\bar{x};\bar{y}) \text{ is p.p., } [\psi : \phi \wedge \psi] < \omega \text{ for some } \psi(\bar{x};\bar{0}) \in \mathcal{G}(p)\}.$$

ii) Let $G(p) = \bigcap \mathcal{G}(p)$ and $G_0(p) = \bigcap \mathcal{G}_0(p)$.

5.13 Exercise. Show that if $p \subseteq q$ then $G(p) \supseteq G(q) \supseteq G_0(q)$ and $G_0(p) \supseteq G_0(q)$.

We show that q is a nonforking extension of p just if the final containments of the last exercise can be reversed. Our argument relies on the characterization of nonforking extensions of a type in terms of the number of conjugates which is discussed in Section IV.1.

5.14 Theorem. *Let T be a theory of modules and $p \subseteq q$. Then q is a nonforking extension of p if and only if $G_0(q) = G_0(p)$.*

Proof. First observe that if $\hat{q}$ is a global type extending p and $G_0(\hat{q}) \supseteq G_0(p)$ then $\hat{q}$ is a nonforking extension of p. For, each conjugate of $\hat{q}$ is determined by a map which assigns to each p.p.-definable subgroup of $\mathcal{M}$, the coset of that subgroup which is in $\hat{q}$. But $G_0(\hat{q}) \supseteq G_0(p)$ implies that every such subgroup has finite index in some member of $\mathcal{G}(p)$. Thus, $\hat{q}$ has at most

$2^{|R|+|A|+\aleph_0}$ conjugates over A and thus does not fork over A (Lemma IV.1.14).

The following lemma shows that any q with $G_0(q) \subseteq G_0(p)$ can be extended to a $\hat{q}$ with the same property. By monotonicity, if $G_0(q) = G_0(p)$, q is a non-forking extension of p. But if q does not fork over p then q extends to a conjugate of a type $\hat{p}$ with $G_0(\hat{p}) = G_0(p)$. Since this last property is preserved by conjugacy over A we see $G_0(q) = G_0(p)$ and finish.

5.15 Lemma. *If $p \subseteq q$ and $G_0(q) \supseteq G_0(p)$ then there exists $\hat{q} \in S(\mathcal{M})$ with $p \subseteq \hat{q}$ and $G_0(\hat{q}) = G_0(p)$.*

Proof. It suffices to show that

$$p \cup \{\neg\psi(\overline{x};\overline{m}) : \overline{m} \in \mathcal{M}, \psi(\mathcal{M},\bar{0}) \notin \mathcal{G}_0(p)\}$$

is consistent. If not, there exist p.p. formulas ϕ, χ_i, ψ_i such that for some $\overline{a} \in A$, $\phi(\overline{x};\overline{a}) \in p$, $\neg\chi(\overline{x};\overline{a}) \in p$, and $\psi(\mathcal{M};\bar{0}) \notin G_0(p)$ but for some $\overline{m} \in \mathcal{M}$

$$\mathcal{M} \models \phi(\overline{x};\overline{a}) \rightarrow (\bigvee_{i<n} \chi_i(\overline{x};\overline{a}) \vee \bigvee_{j<m} \psi_j(\overline{x};\overline{m})).$$

Since $\mathcal{M} \not\models \phi(\overline{x};\overline{a}) \rightarrow (\bigvee_{i<n}\chi_i(\overline{x};\overline{a}))$, Lemma I.4.13 implies that for some j, $\phi(\mathcal{M};\bar{0}) \cap \psi_j(\overline{x};\bar{0})$ has finite index in $\phi(\mathcal{M};\bar{0})$. But this, in turn, implies that $\psi_j(\mathcal{M};\bar{0})$ has finite index in $\phi(\mathcal{M};\bar{0}) + \psi_j(\mathcal{M};\bar{0})$. Since the last group is in $\mathcal{G}(p)$ we have a contradiction.

5.16 Exercise. Show that if $G_0(p) = G(p)$ then p is stationary. Find a counterexample to the converse.

5.17 Exercise. Let T be the theory of $M = Z_4^{\aleph_0}$. For each p.p.-formula ϕ, let $\langle\phi\rangle$ denote the set of p.p. formulas which are deducible from ϕ. Let p be the type determined by $p^+ = \langle 2 \times v = 0\rangle$. Let $a \in M$ be an element of order 4 and let q be the type determined by $q^+ = \langle 2(x-a) = 0\rangle$. Then in $M \oplus Z_4$, p is realized by $\langle 0, 2\rangle$ and q is realized by $\langle a, 2\rangle$. Show that $G_0(p) = G_0(q) = \{x : 2 \times x = 0\}$ and thus that p does not fork over 0 but q does.

The definition of $G_0(p)$ reflects an idea that plays an important role in this area.

5.18 Definition. The group G is *connected* if it has no proper definable subgroup of finite index.

This name is suggested by analogy with the connected component of the identity in an algebraic group with the Zariski topology. The notion is important here both because of the relative ease with which connected groups can be found in this context and the consequences of assuming a group to be connected. In fact, if G is an affine algebraic group over an algebraically closed field F and we consider G in the language imposed by the imbedding of G in F^n, then G is connected in our sense exactly if G is connected in the sense of the Zariski topology.

The following terminology is convenient for discussing the structure of stable groups. The usage is standard in the study of infinite groups.

5.19 Definition. Let $\mathcal{H}$ and $\mathcal{K}$ be classes of groups. Then the group G is said to be $\mathcal{H}$ *by* $\mathcal{K}$ if there is a normal subgroup H of G which is in $\mathcal{H}$ such that $G/H \in \mathcal{K}$.

5.20 Exercise. Show that if G is an ω-stable group then G is connected by finite. (Hint: Apply the d.c.c.)

We prove a more delicate version of this result which is in [Baur, Cherlin & Macintyre 1979].

5.21 Theorem. *If G is $\aleph_0$-categorical and stable then G is connected by finite.*

Proof. Let Γ_n be the collection of subgroups of G which are definable with at most n parameters and have finite index in G. By the $\aleph_0$-categoricity and Ryll-Nardzewski's theorem (Theorem I.3.1), there are only a finite number of possible defining formulas for these groups. Thus by the full stable d.c.c. there is a minimal such group, H_n which is definable without parameters. If $\langle H_n : n < \omega \rangle$ forms an infinite decreasing chain, then G realizes infinitely many 1-types, again contradicting Ryll-Nardzewski's theorem.

The following results presage the proof in Chapter VIII that a superstable division ring is an algebraically closed field.

5.22 Theorem. *If F is an infinite stable division ring then the additive group F^+ of F is connected.*

Proof. Suppose for contradiction that A is a proper definable subgroup of F^+ with $[F^+ : A] < \omega$. Let Γ be the collection of subgroups aA for a a nonzero element of F. By the full stable d.c.c., there is a group $A_0 = \bigcap \Gamma$ with finite index in F. But then A_0 is an ideal in F and since F is infinite, $A_0 \neq (0)$; thus, $A_0 = F$.

A very similar argument solves the following exercise.

5.23 Exercise. Show that if N is a finite normal subgroup of a connected group G, then N is contained in the center of G. (Hint: Imbed G in the automorphism group of N via conjugation. Consider the kernel of this imbedding.)

5.24 Exercise. Show that if G is a group, f is an endomorphism of G such that $\ker f$ is finite, and $[G : f(G)]$ is infinite, then $[f(G) : f^2(G)]$ is infinite.

5.25 Theorem. *Let G be a connected superstable group and h a definable endomorphism of G with finite kernel. Then h is surjective.*

Proof. If f is not onto, G is connected implies $[G : f(G)]$ is infinite. Applying Exercise 5.24 and induction we see $[f^n(G) : f^{n+1}(G)]$ is infinite for each $n < \omega$. But this contradicts the superstable d.c.c.

5.26 Historical Notes. The reduction of stability to chains in structures admitting a multiplication is foreshadowed in [Macintyre 1971] and explicit in [Baldwin & Saxl 1976] and [Shelah 1975]. The proof that stable groups satisfy the full descending chain condition first appears in [Cherlin & Reineke 1976]. As remarked there the argument generalizes that for centralizers which Baldwin generalized from a lemma in [Baldwin & Saxl 1976]. The simple argument given here was pointed out by Ziegler. Poizat [Poizat 1981] gives still another proof.

There have been a number of powerful applications of the methods described in this section. It has been shown in varying degrees of generality in [Felgner 1975], [Cherlin & Reineke 1976], [Sabbagh 1975], and [Baldwin & Rose 1977] that any semisimple stable ring is a finite direct sum of matrix rings over division rings. This generalizes the classical proof of the Wedderburn-Artin theorem by showing that each use of the descending chain condition on all ideals can be replaced by a use of the stable descending chain condition.

The strongest known result concerning stable groups is that an $\aleph_0$-categorical stable group is nilpotent by finite; moreover, if the group is ω-stable then it is Abelian by finite. This profound result was proved in [Baur, Cherlin & Macintyre 1979].

In another direction, Cherlin [Cherlin 1979] has conjectured that the ω-stable simple groups with finite Morley rank are 'classical ' simple groups and verified this result for groups with low Morley rank. Simon Thomas [Thomas 1983] has verified this conjecture for locally finite groups. Further important work on this line is contained in [Nesin 1985], [Berline & Lascar 1986], [Hrushovski 198?d], and [Hrushovski 1986].