Part A

Independence

As we remarked in the introduction, to arrive at our structure theory we must develop the notions of independence and generation separately. Part A is devoted to the first of those tasks, Part B to the second. We begin by giving an axiomatic description of an independence or freeness relation. This description summarizes the properties of the nonforking relation just as the axioms of Whitney and van der Waerden summarize the properties of vector space independence. We adopt this axiomatic formulation for several reasons. First, it clarifies the principles applied in the various constructions and proofs later in the book. Second, it provides a general framework for the discussion of several of the main concepts of the book, notably nonforking and orthogonality. By allowing us to separate the arguments used to verify these axioms from the applications of the axioms, we take a step towards the generalization of this structure theory to other families of classes of structures. If $\mathcal{K}$ is the family of classes of models of first order theories, we show in Section III.4 and Chapter VII, that *all* the axioms are satisfied on a class $K \in \mathcal{K}$ only if the relation is nonforking and K is the class of models of a *stable* first order theory. However, some of the results proved here depend on proper subsets of the axioms listed and many of the axioms hold under less restrictive conditions ([Shelah 1980a], [Shelah 1986]). More importantly, many of the arguments from Shelah's extension of the theory to the nonelementary case [Shelah 1983a] can also be fit into this rubric. A unified account of the first order and infinitary case will undoubtedly require changes in the axioms proposed here; we regard this as simply a first step.

Chapter II

The Abstract Notion of Independence

In the 1930's van der Waerden [van der Waerden 1949] and Whitney [Whitney 1935] abstracted the following properties of linear independence in vector spaces and algebraic independence in fields and used them to define the general notion of an independence relation or matroid. One of the most important properties of the vector space notion is the immediate definition of dependent as 'not independent'. With this in mind, van der Waerden's notion is most easily described in terms of a point a depending on a set X. For all a,

i) (Reflexivity) a depends on $\{a\}$.
ii) (Monotonicity) If a depends on X and $X \subseteq Y$ then a depends on Y.
iii) (Transitivity of dependence) If a depends on X and each $x \in X$ depends on Y then a depends on Y.
iv) (Exchange axiom) If a depends on $X \cup \{b\}$ but a does not depend on X then b depends on $X \cup \{a\}$.
v) (Finite character) If a depends on X then a depends on a finite subset of X.

In this book we describe a similar notion of independence which also specializes to linear independence and algebraic independence. It is not, strictly speaking, a generalization of the usual notion since it is stronger in some respects, weaker in others. However, it retains the most important consequence of the usual theory, the ability to assign a dimension (or, more precisely in the general case, a family of dimensions) to each member of certain classes of models.

We deal with a ternary relation denoted $A \downarrow_C B$, which intuitively means, 'A is free from B relative to C' or 'A is no more restrained by B than by C'. This notion differs from the standard one in three fundamental respects. First, with C fixed the relation $A \downarrow_C B$ differs from normal independence relations only by obeying slightly different axioms. Second, we have replaced the element a in the usual definition by the set A. Finally, we deal with $t(A; B)$, that is, all B-automorphic images of A rather than just A itself.

The extension to families of independence relations depending on the parameter C is necessary to create the ranked systems of invariants described in the introduction. The theory of two equivalence relations, E_1 and E_2, with E_1 refining E_2, will illustrate the point. Under our definition, if $\langle a_\alpha : \alpha < \kappa \rangle$ is a sequence of E_1-inequivalent elements, they will be independent over $\emptyset$. For each α, a set $B_\alpha = \langle b_{\alpha,\beta} : \beta < \kappa_\alpha \rangle$ of E_2-inequivalent elements which are all equivalent to a_α will be independent over $\{a_\alpha\}$ but not over $\emptyset$. (With a_α we can impose the restraint 'xE_1a_α'.) To determine the isomorphism type of a model we must know the dimension of each set B_α, and, even more, we must know for each α, β, the dimension of the set $C_{\alpha,\beta}$ of distinct elements which are both E_1 and E_2 equivalent to $b_{\alpha,\beta}$. This set will be independent over $\{b_{\alpha,\beta}\}$ but not over $\{a_\alpha\}$.

The consideration of the type of an element rather than the element itself generalizes the passage in algebraic geometry from a point to the ideal of polynomials which are zero on that point. This observation is not merely an analogy. Let $\mathcal{M}$ denote the 'universal domain' of algebraic geometry (an algebraically closed field of immense transcendence degree), and k a subfield of $\mathcal{M}$ which need not be a model of $\mathrm{Th}(\mathcal{M})$. Then polynomials over k are special kinds of formulas over k, ideals in $k[\overline{x}]$ are (incomplete) types, and the spectrum of $k[\overline{x}]$, the set of maximal ideals in $k[\overline{x}]$, is in 1-1 correspondence with the Stone space of k. For $\overline{a}$, a sequence from $\mathcal{M}$, this correspondence sends the ideal of polynomials over k which vanish at $\overline{a}$ to $t(\overline{a}; k)$.

The extension from considering singletons to considering finite sequences parallels the extension from the study of solutions to equations in one variable to the study of higher dimensional surfaces. If one studies only single elements then the distinction between an element and its type is easily ignored, as the type is often algebraic. This is the case, for example, in both vector spaces and fields. As soon as the study of n-types is broached, the type is no longer algebraic (except in trivial cases) and this distinction becomes more important. In our study we posit certain symmetry and finite character properties which allow us to extend the basic notions from finite sequences to arbitrary sets.

We employ two notations for the notion of independence. One emphasizes the role of types; the other is more algebraic. Thus, we write $t(\overline{a}; B)\mathcal{F}C$ for the assertion that $t(\overline{a}; B)$ is abstractly free over C. From any freeness relation $\mathcal{F}$ one can derive a relation $A \downarrow^{\mathcal{F}}_C B$ among sets. We will never write the superscript $\mathcal{F}$, but in this chapter we use $A \downarrow_C B$ to refer to the relation on sets associated with an abstract freeness relation. In Chapter III we introduce the most useful example of an independence notion: $t(\overline{a}; B)$ does not fork over C. Thereafter, we write $A \downarrow_C B$ for the relation on sets associated with the nonforking relation. We introduce the axioms in terms of the type notation to emphasize that we consider all elements conjugate to $\overline{a}$ over B. This notation is convenient for establishing the basic properties of the nonforking relation; the set notation is more compact and flexible in applications. We now quickly summarize the axioms using the set notation. In Section II.1 we state the axioms carefully in the type notation and exhibit in Section II.2 the exact relation between the two notations.

The axioms fall naturally into three groups. The first contains those which arise from thinking of $A \downarrow_C B$ as an 'ordinary' independence relation with C fixed. Thus, we both require monotonicity and finite character of dependence in the second variable: if $A \downarrow_C B$ and $B' \subseteq B$ then $A \downarrow_C B'$; if $A \not\downarrow_C B$ then for some finite $B' \subseteq B$, $A \not\downarrow_C B'$. Similarly, we require a natural translation of the exchange axiom, the symmetry axiom: $A \downarrow_C B$ implies $B \downarrow_C A$. We also need a weak version of the reflexivity axiom. In slightly more generality than actually holds, we can state the reflexivity as: if $\overline{a} \not\subseteq C$ then $\overline{a} \not\downarrow_C \overline{a}$. However, we can not require the obvious translation of the transitivity axiom for dependence. It is easy to see that the full strength of the transitivity axiom yields a contradiction if independence is defined for n-types. For this, suppose $\{a,b,c,d\}$ are four elements with $\{a,b\} \downarrow_C \{c,d\}$. Then monotonicity and reflexivity yield first $\{a,b\} \not\downarrow_C \{b,c\}$ and then $\{b,c\} \not\downarrow_C \{c,d\}$. But then, from transitivity, we derive a contradiction.

A second group of axioms arises from the consideration of the relationship between dependence over different third coordinates. Here, we have upward monotonicity in the third coordinate: if $A \downarrow_C B$ and $C \subseteq C' \subseteq C \cup B$, then $A \downarrow_{C'} B$. We also invoke the following 'transitivity of independence': if $D \subseteq C \subseteq B$, $A \downarrow_C B$, and $A \downarrow_D C$ then $A \downarrow_D B$. We demand a generalization of local character in the third variable by defining a cardinal $\overline{\kappa}(T)$ such that, for any finite A and any B, there is a $B' \subseteq B$ such that $A \downarrow_{B'} B$ and $|B'| < \overline{\kappa}(T)$. Our dimension theory is best worked out when this local character becomes finite character, i.e. $\overline{\kappa}(T) = \omega$ (equivalently, T is superstable).

Finally, our axioms make explicit a property of vector space independence which does not follow from the usual axioms. In a vector space all maximal independent sets can be mapped to one another by automorphisms of the vector space. This will not be true in our situation but it will be true of certain kinds of independent sets: those which are infinite and based on a stationary type (cf. Definition 1.24). This property can be expressed in another way. In a vector space there is, up to isomorphism, only one way to freely extend an infinite independent set. We will demand a similar but slightly weaker property here, namely, up to isomorphism there are only a bounded number of ways to freely extend an independent set. (We can remove the requirement that the set be infinite by requiring, in addition, that the type be regular. But this is a second problem which will be discussed in Chapter XII.)

We close Section 1 with an illustration of the power of these axioms. We show that if T is any theory admitting a freeness relation satisfying our axioms then certain unions of chains of models of T preserve saturation. This foreshadows our proof in Section III.4 that if T is stable in μ then T has a saturated model of power μ.

In addition to describing the second notation for independence, a number of basic lemmas for calculating with the notion of independence are discussed in Section II.2. Moreover, the important notions of an independent sequence and, more generally, a set which is independent with respect to a partial order are introduced. Although these constructions are not exploited in full until much later in the book, we expound them now to emphasize that they can be easily derived from the axioms and do not depend on the more subtle considerations of Chapter III.

1. Axioms for Independence

In this section we formulate precisely the axioms described above and illustrate them with various examples which satisfy certain combinations of the axioms. Only the rather complicated Example 1.4 provides an algebraic example which satisfies all the axioms. This quick survey, along with the proof in Theorem 1.30 that a union of saturated models is saturated, give an overview of the environment in which we work and some of the arguments available there. The major purpose of this section is to provide a framework for the exposition of the nonforking relation in Chapter III. A second purpose is to extend a relation originally defined between the type of a finite sequence $\overline{a}$ over a set B and a set C to one defined between the type of a possibly infinite set A over B and the set C.

In our discussion here we will formalize the notion '$\overline{a}$ does not depend on B over A' by defining: $t(\overline{a}; B)$ is free over A. Since we are working inside the monster model, which is saturated, there is actually little difference between dealing with a complete type, $t(\overline{c}; B)$, and the pair $(\overline{c}; B)$. For, all realizations of the type can be mapped to $\overline{c}$ by an automorphism which fixes B. However, dealing with types has two advantages.

i) It allows us to compare two pairs $(\overline{c}; B)$ and $(\overline{d}; E)$ by considering $t(\overline{c}; B)$, $t(\overline{d}; E)$.
ii) It makes it convenient to speak of approximations to the pair $(\overline{c}; B)$ by discussing subtypes of $t(\overline{c}; B)$.

In Section II.2 we explore the relation between this notation and the set notation used in the introduction.

1.1 Free Extensions. A *notion of freeness*, $\mathcal{F}$, is a collection of pairs of the form (p, A). Each p is a (not necessarily complete) type which determines its domain, B. A is a subset of $\mathcal{M}$. If $(p, A) \in \mathcal{F}$ we say *p is free over A* and write $p\mathcal{F}A$. Alternatively, we may deal with a sequence $\overline{c}$ realizing p and write $t(\overline{c}; B)\mathcal{F}A$. We freely switch between these notations depending on which is most convenient. Note that the first is more general since it permits p to be an incomplete type. If $t(\overline{c}; B)$ is not free over A, we say $\overline{c}$ *depends on B over A*.

For an initial intuition into the meaning of the axioms one can assume that $A \subseteq B$. Those familiar with ω-stable theories can think of $t(a; B)$ is free over A as meaning that the Morley ranks (cf. Chapter VII) of $t(a; A)$ and $t(a; A \cup B)$ are the same. We will see in Chapter VII how to use rank on n-types to define a relation satisfying these axioms.

Each exercise or proposition in this section assumes the axioms that have already been introduced unless the contrary is given as a hypothesis.

1.2 Isomorphism Invariance. Like any other reasonable model theoretic notion, we require that $\mathcal{F}$ be closed under isomorphism; i.e., if f is an isomorphism and $(p, A) \in \mathcal{F}$ then $(f(p), f(A)) \in \mathcal{F}$.

We provide in Example 1.3 the prototypic algebraic examples of a freeness relation. Naturally, they satisfy some but not all of our axioms. Accordingly, they can provide some intuition about both the similarities and distinctions of our approach from the traditional one. In particular, these examples are only defined for 1-types.

1.3 Examples. i) Let V be a vector space over a field F and let $p \in S(B)$. Define $(p, A) \in \mathcal{F}$ if and only if for any a realizing p, a is not in the span of $A \cup B$ unless a is in the span of A.

ii) Let M be a model of a first order theory, A and B subsets of M, and $c \in M$. Define $(t(c; B), A) \in \mathcal{F}$ if $t(c; A \cup B)$ is not algebraic unless $t(c; A)$ is algebraic.

The dependence relation in the first example could be extended to tuples by declaring $\overline{a}$ to be independent from B over C if and only if for each $\overline{a}' \subseteq \overline{a}$, if $\overline{a}'$ is linearly independent over C then $\overline{a}'$ is linearly independent over $B \cup C$.

The following example, worked out by Chantal Berline [Berline 1983], is the algebraic interpretation of the nonforking relation for the case of algebraically closed fields. Naturally, one can check that it satisfies all the axioms. (Note that there is still one restriction. We deal only with complete types over subfields, not types over arbitrary subsets.)

1.4 Example. Let T be the theory of algebraically closed fields of characteristic 0. If k is a field then each n-type $p \in S(k)$ corresponds naturally to an ideal $I(p) = \{p(\overline{x}) : p(\overline{a}) = 0\}$ where $p \in k[\overline{x}]$, the polynomial ring in n variables over k and $\overline{a}$ is an arbitrary realization of p. Now suppose $k \subseteq K$ and p extends to $p' \in S(K)$. Call p' a free extension of p if $t.d.(k[\overline{x}]/I(p)) = t.d.(K[\overline{x}]/I(p'))$, where $t.d.$ abbreviates transcendence degree.

Thinking of $t(\overline{a}; B)\mathcal{F}A$ as meaning '$\overline{a}$ is no more constrained by $A \cup B$ than it is by A' suggests the following properties.

1.5 Monotonicity Axioms. M_1. If $q \subseteq p$ and p is free over A then q is free over A.

M_2. If $A_1 \subseteq A_2$ and p is free over A_1 then p is free over A_2.

In this formulation, M_2, of monotonicity, we do not assume that A_2 is a subset of $\operatorname{dom} p$. A more restricted form of monotonicity holds when this assumption is built into the notation (cf. Section II.1).

There are two aspects to the first monotonicity axiom. On the one hand, it asserts that if you take away information about a sequence $\overline{a}$ by considering a subset of $t(\overline{a}; B)$, then the resulting type is at least as likely to be free as the first. On the other, it asserts that every subsequence of a 'free sequence' is free. That is, we can conclude from M_1 that if $t(\overline{a}\frown\overline{b}; B)\mathcal{F}A$ then $t(\overline{a}; B)\mathcal{F}A$.

There is no assumption that p and q are complete types; in fact, incomplete types play an important role. Intuitively, the incomplete type p is free over A if some extension of p to a complete type over $\operatorname{dom} p$ is free over A. Indeed, this intuition is embodied in Axiom M_1.

1.6 Exercise. Verify that the examples in Example 1.3 satisfy the monotonicity axioms if we restrict their application to 1-types. For M_1, assume $A \subseteq \operatorname{dom} p$ and for M_2 assume that $A_2 \subseteq \operatorname{dom} p$.

The following can be viewed as a converse to the monotonicity axioms.

1.7 Transitivity Axiom for Independence. Let $C \subseteq B \subseteq \operatorname{dom} p$. If p is free over B and $p|B$ is free over C then p is free over C.

We will discuss later the much stronger requirement that dependence be transitive. We will be unable to establish this stronger axiom in general but rather will show in Chapter XII that a variant of it (which suffices to define dimension) holds on the realizations of stationary regular types.

1.8 Exercise. Show that Axiom 1.7 holds for Example 1.3 i).

The following combination of Axioms 1.5 and 1.7 is frequently used.

1.9 Exercise. Suppose $A \subseteq B \subseteq C$, $p \in S(B)$, and $p \subseteq q \in S(C)$. If $q\mathcal{F}B$ then $q\mathcal{F}A$ iff $p\mathcal{F}A$.

The next axiom corresponds to the exchange axiom in the van der Waerden formulation.

1.10 Symmetry Axiom. If $t(\overline{a}; B \cup \{\overline{b}\})$ is free over B then $t(\overline{b}; B \cup \{\overline{a}\})$ is free over B.

1.11 Exercise. Show this axiom fails for Example 1.3 ii).

Using transitivity it is easy to extend the symmetry axiom as follows:

1.12 Generalized Symmetry Lemma. *If both $t(\overline{a}; B)$ is free over A and $t(\overline{b}; B \cup \{\overline{a}\})$ is free over B then $t(\overline{a}; B \cup \{\overline{b}\})$ is free over A.*

In set notation the generalized symmetry lemma reads: if $D \downarrow_A B \cup C$ and $B \downarrow_A C$ then $D \cup B \downarrow_A C$. We can translate this as a symmetry relation for dependency. Note that now the auxilliary hypothesis applies to elements on opposite sides of the dependency sign. If $D \not\downarrow_A B \cup C$ and $B \downarrow_A D$ then $D \cup B \not\downarrow_A C$.

The following exercise shows the connection between the symmetry propery and the exchange principle.

1.13 Exercise. Show that the special case of Axiom 1.10 where $\overline{a}$ and $\overline{b}$ are singletons holds for Example 1.3 i). That is, show that the symmetry axiom generalizes the usual 'exchange axiom' for vector spaces.

The following axioms guarantee that our notion is nontrivial.

1.14 Existence and Extension Axioms. E_1. $t(\bar{b}; A)$ is free over A for any $\bar{b}$ and A.

E_2. If p is free over A and $\operatorname{dom} p \subseteq B$, there is a $p_1 \in S(B)$ which extends p such that p_1 is free over A.

We next consider the generalization of the finite character of dependence to our context. This also allows us to extend our relation to a relation among three arbitrary sets.

1.15 Axiom: The Finite Character of Nonfreeness. If $t(\bar{c}; B)$ is not free over A then there is a formula $\phi(\bar{x}; \bar{b}) \in t(\bar{c}; B)$ such that no type containing $\phi(\bar{x}; \bar{b})$ is free over A.

This axiom has an important logical, as opposed to algebraic, character because we insist that not only is there a finite subset B_0 of B such that every extension of $t(\bar{c}; B_0)$ is not free over A but that there is a specific formula which forces nonfreeness. The formulation is somewhat subtle because there is a hidden reference to $t(\bar{b}; A)$. The assertion does not imply that for any $\bar{b}'$, if $\phi(\bar{x}; \bar{b}')$ is in q then q depends on $\bar{b}'$ over A, but only for those $\bar{b}'$ realizing $t(\bar{b}; A)$.

Our definition of the freeness relation, $t(\bar{a}; B)\mathcal{F}A$, deals with the relation between a sequence $\bar{a}$ and sets A and B. The symmetry axiom moves the elements of the sequence $\bar{a}$ to the domain of a type where the order of the sequence does not matter. Thus, freeness becomes a property not of the sequence $\bar{a}$ but of the set $\{a_0, \ldots, a_{n-1}\}$. More formally, we have the following lemma whose proof would be obvious using the symmetry axiom if we assumed $B \subseteq C$. Without this assumption it requires a use of the extension axiom. In the proof of this lemma we emphasize the distinction between a sequence $\bar{b}$ and its range, $\operatorname{rng}(\bar{b})$. The lemma itself justifies our usual abuse of this distinction.

1.16 Lemma. *Let $\bar{a}$ be a finite sequence and $\bar{a}'$ a permutation of $\bar{a}$. Then for any B and C, $t(\bar{a}; B)\mathcal{F}C$ iff $t(\bar{a}'; B)\mathcal{F}C$.*

Proof. Choose $\bar{d}$ with $t(\bar{d}; B) = t(\bar{a}; B)$ and such that $t(\bar{d}; C \cup B)$ is free over C. If $\bar{a}'$ is the result of permuting the elements of $\bar{a}$, let $\bar{d}'$ be the result of applying the same permutation to the elements of $\bar{d}$. Since $t(\bar{d}; B \cup C)\mathcal{F}C$, for any $\bar{b} \in B$ monotonicity and symmetry yield $t(\bar{b}; \bar{d} \cup C)\mathcal{F}C$. Since in the second position the order of $\bar{d}$ makes no difference this implies, $t(\bar{b}; \bar{d}' \cup C)\mathcal{F}C$. Applying symmetry again we have $t(\bar{d}'; \bar{b} \cup C)\mathcal{F}C$. Since this holds for each $\bar{b} \in B$, $t(\bar{d}'; B \cup C)\mathcal{F}C$ as required.

We have justified the following definition.

1.17 Definition. For arbitrary sets C, B, and a finite set A, $t(A; B)\mathcal{F}C$ iff for some ordering $\bar{a}$ of A, $t(\bar{a}; B)\mathcal{F}C$.

Since formulas can contain only finitely many variables, finite character implies that if every finite subtype of p is free over A then p is free over A. There are two equivalent ways to extend the definition of freeness to allow the first coordinate to be infinite. We could extend the domain of the freeness relation $\mathcal{F}$ to pairs $(p; C)$ where p is an α-type over B for any ordinal α and then define A to be independent from B over C if for some ordering $\overline{a}$ of A, $t(\overline{a}; B)\mathcal{F}C$. It is easy to see, using the monotonicity and symmetry axioms and the axiom of finite character, that an equivalent notion is obtained by retaining the convention that $(p; A) \in \mathcal{F}$ implies p is a finite type and making the following definition.

1.18 Definition. For arbitrary sets A, B, C, $t(A; B)\mathcal{F}C$ iff for each finite sequence $\overline{a}$ from A, $t(\overline{a}; B)\mathcal{F}C$.

The finite character allows us to reformulate the symmetry axiom as follows.

1.19 Exercise. If $t(A; B \cup C)\mathcal{F}C$ then $t(B; A \cup C)\mathcal{F}C$.

Now, we can rephrase Definition 1.18 in somewhat more generality: $t(A; B)\mathcal{F}C$ iff for each finite $\overline{a} \in A$, $\overline{b} \in B$, $t(\overline{a}; C \cup \overline{b})\mathcal{F}C$.

Let X be a subset of a vector space V and $a \in V$. If a depends on X then, as in Axiom 1.15, a depends on a finite subset X_0 of X. Thus a depends no more on X than on X_0; in our notation, $t(a; X)\mathcal{F}X_0$. For vector spaces this is simply the assertion that a depends on X_0. But in the general case the assertion that a does not depend on X over X_0 has more content than this trivial assertion. We shall always be able to find such an X_0 and to bound its size uniformly over all types, thus providing a local character for freeness. More formally.

1.20 Definition. For any theory T, let $\overline{\kappa}(T)$ denote the least cardinal κ, if one exists, such that for any finite sequence $\overline{a}$ and any set A there is an $A_0 \subseteq A$ with $|A_0| < \kappa$ such that $t(\overline{a}; A)\mathcal{F}A_0$. If no such cardinal, κ, exists $\overline{\kappa}(T)$ is ∞.

In Chapter III, especially Definition III.4.18, we consider several variants on $\overline{\kappa}(T)$ and calculate bounds on its value for a stable theory T. For now, we simply require that it exist.

1.21 Axiom: The Local Character of Freeness. $\overline{\kappa}(T) < \infty$.

Note that if $\overline{\kappa}(T) = \omega$, we have full local, that is finite, character in the third coordinate. This additional condition corresponds to T being a superstable theory.

1.22 Exercise. For a sequence $\overline{a}$ with $\lg(\overline{a}) = \mu$ and for an arbitrary set A there is a subset A_0 of A with $|A_0| \leq \mu \times \overline{\kappa}(T)$ such that $t(\overline{a}; A)\mathcal{F}A_0$.

We have defined the monotonicity, transitivity (of independence), symmetry, existence, extension and finite character axioms. In fact, if we declare every type to be free over every set, all of these axioms are satisfied. The reflexivity axiom implies the existence of types which are not free.

1.23 Reflexivity Axiom. If $A \subseteq B$, $b \in B - A$ and $t(b; A)$ is not algebraic then $t(b; B)$ is not free over A.

The final axiom generalizes the fact that in vector spaces there is a unique extension of an independent set. We require one definition to state the axiom.

1.24 Definition. Let $\mathcal{F}$ be a notion of freeness. An n-type p is *stationary over* A if $p\mathcal{F}A$ and if for every B containing A, if $q_1, q_2 \in S^n(B)$ extend p and are free over A then $q_1 = q_2$. We say p is *stationary* if p is stationary over $\operatorname{dom} p$.

This concept would be easier to formulate if we could insist that p is a type over A. However, in a very important case (strong types, Chapter IV) p is not a type over A.

In any complete theory of vector spaces and more generally in a strongly minimal theory every non-algebraic type is stationary. When every type is stationary an independent set can be freely extended in only one way (up to isomorphism). In our generalization we will weaken this requirement somewhat by allowing more than one such extension but bounding the number of contradictory free extensions of a type. This bound is a cardinal number which depends only on T, not on the particular type p. This aim can be realized by demanding that types over models be stationary.

The existence of stationary types guarantees that the freeness relation is nontrivial. For, any non-algebraic type $p \in S(A)$ has more than one extension to any B properly containing A so there are nonfree extensions. In the situation in this book we have the following strong existence axiom for stationary types.

1.25 Boundedness Axiom. Every type over a model is stationary.

The following simple fact plays an important role.

1.26 Proposition. *If $p \in S(B)$ extends $q \in S(A)$, q is stationary over A, and $p\mathcal{F}A$ then p is stationary over B.*

1.27 Exercise. Prove Proposition 1.26.

1.28 Exercise. Show that if $\mathcal{F}$ is a notion of freeness which satisfies all the axioms from this section and $\mathcal{F}' = \{(p, A) : p \text{ is } \mathcal{F}\text{-stationary over } A\}$ then $\mathcal{F}'$ satisfies all the axioms except E_1, M_1 and symmetry.

We now illustrate the strength of these axioms by considering the problem of constructing saturated models. The natural way to try to construct a saturated model of power λ is to start with an arbitrary model of power λ, realize all types over subsets with power less than λ and iterate this procedure, hoping that it will close off. If T is stable and λ is regular, this procedure yields a saturated model of power λ. When λ is not regular we can build a chain of models M_i where M_i is μ_i-saturated and the μ_i have limit λ. If $M = \bigcup_{i<\lambda} M_i$ is μ_i-saturated for each i, then we could conclude that M is λ-saturated. We show now that this kind of preservation of saturation under unions of chains is implied by the axioms we have discussed in this section.

1.29 Notation. We frequently construct sequences denoted $\langle \bar{e}_i : i < \alpha \rangle$. We let $E_i = \{\bar{e}_j : j < i\}$. Technically, in the definition of E_i we should replace $\bar{e}_j$ by $\mathrm{rng}(\bar{e}_j)$. As usual, we dispense with such pedantry.

1.30 Theorem. *Let T be a theory which admits a freeness relation satisfying the axioms listed in this section. Suppose λ and $\mathrm{cf}(\delta)$ are cardinals strictly greater than $|T|^+ + \bar{\kappa}(T)$. The union of an increasing chain of length δ of models of T, each of which is λ-saturated, is itself λ-saturated.*

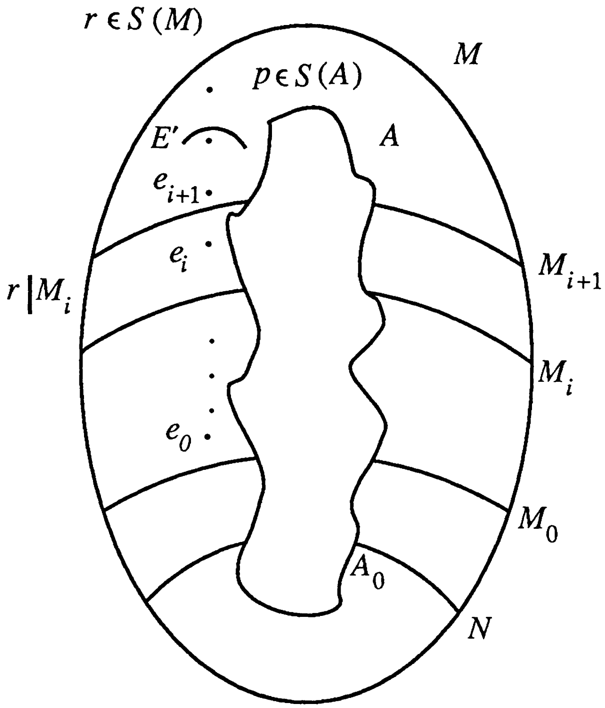

Fig. 1. Theorem 1.30

Proof. (Fig. 1). Let $M = \bigcup_{i<\delta} M_i$ and let $A \subseteq M$ with $|A| < \lambda$. Fix $p \in S(A)$; we must show p is realized in M. Extend p to $r \in S(M)$ and choose $A_0 \subseteq M$ with $|A_0| < \overline{\kappa}(T)$ such that $r \mathcal{F} A_0$. Choose N with $A_0 \subseteq N$ and $|N| \leq |T| + \overline{\kappa}(T)$. Without loss of generality, as $\mathrm{cf}(\delta) > |T| + \overline{\kappa}(T)$, N can be chosen inside M_0. Now choose by induction a sequence $E = \langle e_i : i < \lambda \rangle$ of elements of M_0 such that $t(e_i; E_i \cup N) = r|(E_i \cup N)$. The λ-saturation of M_0 guarantees the existence of this sequence. Now, by Exercise 1.22 we can choose an $E' \subseteq E$ such that $|E' \cup N| < (\overline{\kappa}(T) + |T|) \times |A| < \lambda$ and $t(A; E \cup N) \mathcal{F} (N \cup E')$. Thus $E - E' \neq \emptyset$. But, for $e \in E - E'$, the symmetry axiom implies that if q denotes $t(e; E' \cup N \cup A)$ then $q \mathcal{F} (E' \cup N)$. Now q and $r|(E' \cup N \cup A)$ are two free extensions of the stationary (since it is a free extension of a type over a model) type $r|(N \cup E')$. Thus they are equal and e realizes p.

This is a step towards proving that if T is stable in λ then T has a saturated model of power λ. A very similar proof suffices for this result after we compute in Section III.4 the relation between $\overline{\kappa}(T)$ and the set of cardinals in which T is stable.

1.31 Exercise. Suppose the countable theory T satisfies the axioms in this section with $\overline{\kappa}(T) = \omega$. Show that for every uncountable λ and any cardinal δ with $\mathrm{cf}(\delta) > \omega$, the union of a chain of length δ of λ-saturated models of T is λ-saturated.

1.32 Historical Notes. The abstract treatment of dependence relations began in the 1930's with the work of van der Waerden [van der Waerden 1949] and Whitney [Whitney 1935]. Our development here fuses these ideas with the axiomatic consideration of rank in [Baldwin & Blass 1974], [Lascar 1976] and Shelah's exposition of forking [Shelah 1978]. Theorem 1.30 is abstracted from a proof by Shelah. We show in Section III.4 that a proper subset of these axioms imply the rest. Our purpose here is to expound a set of principles which are often called on in the development of the theory.

Another approach to the study of independence was taken by universal algebraists in the 1960's. They attempted to axiomatize the properties of a dependence relation in a single algebra (which is free for some associated variety). This approach is summarized in Chapter 5 of [Grätzer 1979] and several survey articles [Marczewski 1966] and [Urbanik 1966]. This approach yielded a number of strong representation theorems ([Narkiewicz 1964], [Urbanik 1959/60], [Urbanik 1965]) which are very similar to those obtained later for strictly minimal sets by Zilber [Zilber 1981], [Zilber 1980b] and Cherlin [Cherlin, Harrington, & Lachlan 1985] (cf. Chapter VIII.3). These authors investigated the so-called v, v^* and v^{**} algebras. Urbanik proved that any v-algebra with a properly three-ary polynomial could be represented as an affine algebra. The connection between these results and work on categoricity is emphasized by the classification of categorical quasivarieties ([Givant 1973], [Givant 1976], [Palyutin 1973], [Palyutin 1976]) which relies on the representation theorems of Urbanik.

2. *Further Properties of Independence*

In this section we reformulate the freeness relation as a relation among three sets and then deduce some of the most important working tools of the theory. We wish to emphasize that these crucial tools depend only on the axiomatic properties of the dependence notions, and not on the intricacies of the proof in Chapter III that such a relation exists.

The notation is most compact if we adopt the following convention.

2.1 Definition. $C \downarrow_A B$ means $t(C; B \cup A) \mathcal{F} A$.

In light of the discussion after Definition 1.17 we can rephrase Definition 2.1 to read: for any sets A, B, and C

$$C \downarrow_A B \text{ iff for every finite } C_0 \subseteq C,\ C_0 \downarrow_A B$$

2.2 Exercise. Show $A \downarrow_C B$ iff for every D with $C \subseteq D \subseteq B \cup C, A \downarrow_C D$.

Note that in Definition 2.1, putting A into the domain of the type introduces an asymmetry between the treatment of the left and right side of the $\mathcal{F}$. This asymmetry is real; by burying it in the notation we can simplify our description of many situations. The price is a complicating of the monotonicity axiom (cf. Lemma III.3.7). The notation $A \downarrow_C B$ suggests two variants on the monotonicity axioms. We can restrict the extensions D of the base set C.

$$\text{If } A \downarrow_C B \text{ then for any } D \text{ with } C \subseteq D \subseteq B \cup C,\ A \downarrow_D B.$$

Alternatively, we can replace A by an A' which realizes the same type over B.

$$\text{If } A \downarrow_C B \text{ then for any } D \text{ with } D \supset C, \text{ there is an } A' \text{ with } t(A'; B) = t(A; B) \text{ and } A' \downarrow_D B.$$

The first of these assertions is an obvious consequence of the second monotonicity axiom. The second requires an application of the extension axiom (E_2) as well.

The following exercise spells out in a rather simple case the problems that can arise with monotonicity.

2.3 Exercise. Show that the 'freeness relation' of Example 1.3 i) does not satisfy the general form of the second monotonicity axiom. That is, the restriction in Exercise 1.6 that $A_2 \subseteq \operatorname{dom} p$ is necessary. (Hint: Let e_1, e_2, e_3 be independent elements and let $A_1 = \{e_1\}$, $A_2 = \{e_1, e_2\}$, $B = \{e_1, e_3\}$ and let $c = e_1 + e_2 + e_3$. Show that $t(e_2; B) = t(c; B)$ is free over A_1 but not over A_2. Thus, although e_2 and c realize the same type over A_1, one depends on B over A_2 and the other does not.)

It is natural to read $C \downarrow_A B$ as C is free from B over A. However, since we work primarily in contexts where the symmetry lemma holds, we will frequently write C and B are free over A or C and B are independent over A.

One value of this notation is indicated by the following simple statement of the symmetry axiom.

2.4 Exercise. Show that the symmetry axiom is equivalent to the following proposition: $B \downarrow_A C$ iff $C \downarrow_A B$.

The proofs of the next few propositions require the translation between the 'type' and the 'set' formulation of the freeness relation. This is an appropriate place for such a translation since these results are constantly used in the structure theory but in fact do follow from the properties listed in Section 1, rather than from special properties of the forking relation. Consider the relation between the assertion, '$t(\overline{a}; B)$ depends on A' and the assertion 'For each i, $t(a_i; B)$ depends on A.' where $\overline{a} = \langle a_0, \ldots, a_{k-1} \rangle$. It is perfectly possible to find c and d such that $t(c; B) \mathcal{F} A$ and $t(d; B) \mathcal{F} A$ but $t(c^\frown d; B)$ is not free over A. (e.g. Consider $e, e^{\sqrt{2}}$ as c, d, and $e + e^{\sqrt{2}}$ as B.) The situation is completely described by the following vital lemma.

2.5 Lemma. *The following are equivalent.*

i) $\overline{c}^\frown \overline{d} \downarrow_A B$.
ii) $\overline{c} \downarrow_A B$ *and* $\overline{d} \downarrow_{A \cup \overline{c}} B \cup \overline{c}$.
iii) $\overline{d} \downarrow_A B$ *and* $\overline{c} \downarrow_{A \cup \overline{d}} B \cup \overline{d}$.

Proof. We prove the equivalence of i) and ii); a permutation of the variables yields a proof of i) $\leftrightarrow$ iii). To facilitate the derivation from the axioms, we first restate the equivalence in the notation of Section 1.

i)′ $t(\overline{c}^\frown \overline{d}; B)$ is free over $A \subseteq B$.
ii)′ $t(\overline{c}; B) \mathcal{F} A$ and $t(\overline{d}; B \cup \{\overline{c}\}) \mathcal{F} (A \cup \{\overline{c}\})$.

To simplify notation, we apply the finite character of dependence and assume $B - A$ is a finite sequence $\overline{b}$. By the symmetry axiom $t(\overline{c}^\frown \overline{d}; A \cup \overline{b}) \mathcal{F} A$ iff $t(\overline{b}; A \cup \overline{c}^\frown \overline{d}) \; \mathcal{F} A$ which, by monotonicity and transitivity, is equivalent to $t(\overline{b}; A \cup \overline{c}^\frown \overline{d}) \mathcal{F} A \cup \overline{c}$ and $t(\overline{b}; A \cup \overline{c}) \mathcal{F} A$. By the symmetry axiom again these are equivalent to $t(\overline{d}; A \cup \overline{b}^\frown \overline{c}) \; \mathcal{F}(A \cup \overline{c})$ and $t(\overline{c}; A \cup \overline{b}) \mathcal{F} A$ as required.

There are a number of important corollaries to this result. The first is best stated in the type notation.

2.6 Corollary. *If $p \mathcal{F} A$ and $q \mathcal{F} A$, where $p(\overline{x})$ and $q(\overline{y})$ are complete types over B, then there exists $r(\overline{x}; \overline{y}) \in S(B)$ such that $r(\overline{x}; \overline{y}) \vdash p(\overline{x}) \cup q(\overline{y})$ and $r \mathcal{F} A$.*

2.7 Exercise. Prove Corollary 2.6.

We can now derive a stronger version of the extension property.

2.8 Exercise. Prove the following Strong Extension Property: If $t(\overline{a}; B)$ is free over the subset A of B then for any $\overline{c}$ there is a $\overline{c}'$ such that $t(\overline{a}^\frown \overline{c}'; B)$ is free over A and $t(\overline{a}^\frown \overline{c}'; A) = t(\overline{a}^\frown \overline{c}; A)$.

It is easy using the extension property to find $\overline{a}'^\frown \overline{c}'$ free from B over A realizing $t(\overline{a}^\frown \overline{c}; A)$. The greater strength of this assertion arises from the demand that $\overline{a}' = \overline{a}$.

2.9 Exercise. Deduce from the extension axiom (E_2) that if $p \vdash q$ and $p \mathcal{F} A$ then $q \mathcal{F} A$.

The next corollary is stated using both notations in parallel. The type notation makes explicit the fact that the theorem deals not just with sequences $\overline{a}$ and $\overline{b}$ but with all realizations of the appropriate types. This corollary shows that free extensions preserve freeness.

2.10 Corollary. *Suppose $A \subseteq B$.*

i) *If $t(\overline{a}{}^\frown\overline{b}; B)$ is free over A then $t(\overline{a}; B \cup \overline{b})$ is free over B iff $t(\overline{a}; A \cup \overline{b})$ is free over A.*
ii) *If $\overline{a}{}^\frown\overline{b} \downarrow_A B$ then $\overline{a} \downarrow_B B \cup \overline{b}$ iff $\overline{a} \downarrow_A A \cup \overline{b}$.*

Proof. We prove the type version and leave the translation to the reader. By Lemma 2.5, $t(\overline{a}; B \cup \overline{b})$ is free over $A \cup \overline{b}$. Thus by transitivity and monotonicity $t(\overline{a}; B \cup \overline{b})$ is free over A iff $t(\overline{a}; A \cup \overline{b})$ is free over A. So if $t(\overline{a}; A \cup \overline{b})$ is free over A, we have $t(\overline{a}; B \cup \overline{b})$ is free over B by monotonicity. Conversely, again by Lemma 2.5 $t(\overline{a}; B)$ is free over A. So, if $t(\overline{a}; B \cup \overline{b})$ is free over B we conclude $t(\overline{a}; A \cup \overline{b})$ is free over A by transitivity.

Corollary 2.10 easily yields that if $A = \operatorname{rng} \overline{a}_1 \cup \operatorname{rng} \overline{a}_2 \cup \cdots \cup \operatorname{rng} \overline{a}_n$ then, for any B and C, $A \downarrow_C B$ iff $\{\overline{a}_1, \ldots, \overline{a}_n\} \downarrow_C B$.

The proofs of the last three results require only the monotonicity, transitivity, and symmetry axioms. Using only the first two we can reduce the third to a question of 'symmetry over models'. Thus when verifying in Chapter III that nonforking satisfies the symmetry axiom we will only have to check monotonicity, transitivity and 'symmetry over models'. The following technical lemma which we use to establish this reduction illustrates the power of the extension axiom combined with a judicious use of automorphisms.

2.11 Lemma. *Suppose $t(\overline{a}; A \cup \overline{b})$ is free over A. There exists a model M with $A \subseteq M$ such that $t(\overline{a}; M \cup \overline{b})$ is free over A and $t(\overline{b}; M)$ is free over A.*

Proof. We first guarantee that $t(\overline{b}; M) \mathcal{F} A$. We provide this guarantee in a devious way by working with models which are isomorphic over A to our eventual M. First, let M'' be an arbitrary model containing A. By the extension axiom choose $\overline{b}'$ with $t(\overline{b}'; A) = t(\overline{b}; A)$ and $t(\overline{b}'; M'') \mathcal{F} A$. Now let M' be the image of M'' under an automorphism which fixes A and maps $\overline{b}'$ to $\overline{b}$. Certainly, $t(\overline{b}; M') \mathcal{F} A$. Since $t(\overline{a}; A \cup \overline{b})$ is free over A, by the extension axiom there is an $\overline{a}'$ such that $t(\overline{a}'; A \cup \overline{b}) = t(\overline{a}; A \cup \overline{b})$ but $t(\overline{a}'; M' \cup \overline{b})$ is free over $A \cup \overline{b}$. Thus, by transitivity $t(\overline{a}'; M' \cup \overline{b})$ is free over A. Now choose an automorphism f of $\mathcal{M}$ which fixes $A \cup \overline{b}$ and maps $\overline{a}'$ to $\overline{a}$. Let M be $f(M')$.

There is an easier argument for the conclusion of Lemma 2.11 using the symmetry axiom. The point of this version is to simplify checking the symmetry axiom.

2.12 Lemma. *If a notion of freeness $\mathcal{F}$ satisfies the transitivity and monotonicity axioms and, in addition, for each model M*

$$t(\overline{a}; M \cup \overline{b})\mathcal{F}M \text{ iff } t(\overline{b}; M \cup \overline{a})\mathcal{F}M$$

then $\mathcal{F}$ satisfies the symmetry axiom.

Proof. Suppose $t(\overline{a}; A \cup \overline{b})$ is free over A. Applying Lemma 2.11 choose an M such that $t(\overline{b}; M)$ is free over A and $t(\overline{a}; M \cup \overline{b})$ is free over A. By monotonicity, $t(\overline{a}; M \cup \overline{b})$ is free over M. Now, by symmetry for types over models, $t(\overline{b}; M \cup \overline{a})$ is free over M. Since $t(\overline{b}; M)$ is free over A, we have by transitivity $t(\overline{b}; M \cup \overline{a})$ is free over A and by monotonicity $t(\overline{b}; A \cup \overline{a})$ is free over A.

We next describe a construction which will appear repeatedly in this book, the construction of an independent sequence. We emphasize here that it depends only the axioms we have enunciated. The straightforward exercises following the definition show that this construction yields more than is at first apparent. Recall from Notation 1.29 our notation for initial segments of a sequence of sequences. The most common use of this notation is in the construction of a sequence by repeatedly extending a type. We formally label the result of such a construction.

2.13 Definition. The sequence $\langle \overline{e}_i : i \in I \rangle$ is *coherent* over A if $k < j$ implies $t(\overline{e}_j; E_k \cup A) = t(\overline{e}_k; E_k \cup A)$.

2.14 Definition. The sequence E of sequences $\langle \overline{e}_i : i \in I \rangle$ is *independent* over A if for each i, $t(\overline{e}_i; E_i)$ is free over A.

Note that this definition only makes sense in the presence of the convention identifying the finite sequence $\overline{e}_i$ with its range. That is, the definition concerns the set of elements in the range of the function $\overline{e}_i$, not the finite function itself.

2.15 Exercise. Show the coherent sequence of sequences $E = \langle \overline{e}_i : i \in I \rangle$ is independent if and only if for each i, $t(\overline{e}_{i+1}; E_{i+1})$ is free over E_i.

The proof of the following result is given as an exercise because it is an entirely routine application of the symmetry and monotonicity axioms. Nevertheless, the result is vitally important. For example, it justifies our frequent reference to a *sequence* E satisfying Definition 2.14 as an independent *set* of sequences.

2.16 Lemma. *Let $E = \{\overline{e}_i : i < \omega\}$ be constructed so that $t(\overline{e}_i; E_i)\mathcal{F}A$ for each i. Then*

i) $t(\overline{e}_i {}^\frown \overline{e}_{i+1}; E_i)\mathcal{F}A$ *for each i.*
ii) $t(\overline{e}_i; E - \{\overline{e}_i\})\mathcal{F}A$ *for each i.*

2.17 Exercise. Prove Lemma 2.16.

The following theorem plays a central role throughout this book. We introduced $\overline{\kappa}(T)$ in Definition 1.20. The theorem asserts that no finite sequence $\overline{b}$ can depend on each of $\overline{\kappa}(T)$ independent elements.

2.18 Theorem. *If $\langle \bar{e}_i : i < \bar{\kappa}(T) \rangle$ is an independent sequence over A then for any $\bar{b}$ there is an i with $t(\bar{b}; A \cup \bar{e}_i) \not\!\!f A$.*

Proof. By the definition of $\bar{\kappa}(T)$ there exists a $B \subseteq E$ with $|B| < \bar{\kappa}(T)$ such that $t(\bar{b}; A \cup E) \not\!\!f A \cup B$ and, a fortiori, $t(\bar{b}; A \cup B \cup \bar{e}) \not\!\!f A \cup B$ for any $\bar{e} \in E - B$. Since E is an independent sequence, $t(\bar{e}; B \cup A) \not\!\!f A$. By symmetry and transitivity this yields $t(\bar{e}; A \cup \bar{b}) \not\!\!f A$ which by symmetry again yields the theorem.

(This argument presages the discussions of the relation between $\kappa(T)$ and $\bar{\kappa}(T)$ in Section III.4.)

Unlike the situation in vector spaces, there are different kinds of maximal independent sets. The most useful are those which are strongly independent in the sense described in Theorem 2.18.

2.19 Definition. The sequence $E = \langle \bar{e}_i : i \in I \rangle$ of sequences is *strongly independent* over A if E is independent over A and all the $\bar{e}_i$ realize the same stationary type over A.

2.20 Exercise. Show that if E is a strongly independent sequence then E is coherent.

The following exercises follow by formal manipulation from the principles we have described. They will prove to be extremely important in some of our later constructions.

2.21 Exercise. Suppose $\bar{a}_1 \downarrow_A \bar{a}_2$ and for each i, $t(a_i; A)$ is stationary. Show $t(a_1{}^\frown a_2; A)$ is stationary.

2.22 Exercise. If a, b and c are independent show $a^\frown b \downarrow_b b^\frown c$.

2.23 Exercise. Show $B \cup C \downarrow_A D$ implies $B \downarrow_{A \cup C} C \cup D$.

The next few definitions and results will illustrate the complicated constructions that can be made with just the machinery we have discussed in this chapter. We require the following special notions about partial orders. We will not make substantial use of many of these notions until Part D. We place them here to emphasize that this type of construction depends only on the abstract properties of independence and not on the detailed development of forking in Chapter III.

2.24 Definition. i) A subset B of a partial order $(A, <)$ is an *ideal* if $a \in B$ and $b < a$ implies $b \in B$.

ii) A partial order is *well-founded* if it contains no infinite descending chains.

iii) For an element a in a partially ordered set $(A, <)$, we denote by $a_{\#}$ the set of elements which are incomparable with a and write $a_{<}$ for $\{b : b < a\}$, and $a_{\geq}$ for $\{b : b \geq a\}$.

Note that $<$ is a free variable in this definition. Thus, in considering a partial order $\prec$, we may write, e.g., $a_{\prec}$ for the set of $\prec$-predecessors of a.

Actually, workers in partially ordered sets call what we have called an ideal a semi-ideal. But, since we never call on joins in this context, such pedantry seems inappropriate. The following notion is the natural extension of the concept of an independent set.

2.25 Definition. The set A is *independent with respect to the partial order* $<$ if for any pair of ideals B, C contained in A, $B \downarrow_{B \cap C} C$.

Note that we will often use this notation when A is a set of finite sequences.

As a device to simplify the construction of sets independent with respect to a given well-founded partial order, we extend it to a well-order and then construct free sequences relative to that well order. The next lemma shows that the resulting construction fulfills our intention.

2.26 Lemma. *Suppose the partial order* $(A, <)$ *is extended by the well-order* $(A, \prec)$ *and for each* $a \in A$, $a \downarrow_{a_<} a_\prec$. *Then* A *is independent relative to* $<$.

Proof. Let B, C be ideals of $(A, <)$. We must show $B \downarrow_{B \cap C} C$. We induct on the order type of $(A, \prec)$. By the finite character of dependence, we can assume that $(A, \prec)$ has a greatest element b. The result holds by induction unless $b \in B \cup C$. Suppose first that $b \in B - C$ and let $B' = B - \{b\}$. Since b is last with respect to $\prec$, $b \downarrow_{b_<} (B' \cup C)$. Two applications of the monotonicity axioms yield $((b \cup B') \downarrow C; b_< \cup B')$. Since $b \in B$ and B is an ideal with respect to $<$, it follows that $b_< \subseteq B'$. Thus, $B \downarrow_{B'} C$. By induction, $B' \downarrow_{B' \cap C} C$, so by transitivity of independence $B \downarrow_{B \cap C} C$ as required. The case $b \in C - B$ is completely symmetric so we are left with the case $b \in B \cap C$. Let $C' = C - \{b\}$. By the first case $B \downarrow_{B \cap C'} C'$. Applying monotonicity twice, we get $B \downarrow_{(B \cap C') \cup \{b\}} C' \cup \{b\}$. That is, $B \downarrow_{B \cap C} C$.

After solving the next exercise the second one is an easy application of Lemma 2.26.

2.27 Exercise. Show that any well-founded partial order $<$ on a set A can be extended to a well-ordering $\prec$ of A.

2.28 Exercise. Suppose $(A, <)$ is a well-founded partial order such that for each $a \in A$, $a \downarrow_{a_<} a_\#$. Show A is independent with respect to $<$.

The following theorem is the key to the construction of sets which are independent relative to some prescribed partial order.

2.29 Theorem. *Let* $(A, <)$ *be a well-founded partial order,* $\mathcal{B}$ *a collection of ideals of* A, *and for each* $B \in \mathcal{B}$, p_B *a nonalgebraic type over* B. *Then we can choose for each* $B \in \mathcal{B}$ *a* c_B *realizing* p_B *such that, letting* C *be* $\{c_B : B \in \mathcal{B}\}$, *if we extend* $<$ *to an ordering on* $A \cup C$ *so that* $(c_B)_< = B$ *then* $A \cup C$ *is independent relative to the extended order.*

Proof. Well order $\mathcal{B}$ by $\prec\prec$. By induction on this order choose c_B so that $c_B \downarrow_B A \cup \{c_{B'} : B' \prec\prec B\}$. Let $\prec$ be a well order of $A \cup C$ which satisfies the following conditions.

i) $\prec |A$ extends $<$.
ii) All elements of A precede all elements of C.
iii) $c_{B'} \prec c_B$ if and only if $B' \prec\!\!\prec B$.

Let $<$ be the partial order of $A \cup C$ obtained by extending the original order on A only by adding $b < c_B$ if $b \in B$. It is easy to check that, applying Lemma 2.26, we have the result.

The remaining exercise in this section describes a more concrete but also more complicated notation for the last several results. This notation appears in [Shelah 1982], [Makkai 1984], and [Harrington & Makkai 1985].

2.30 Exercise. Let $\mathcal{S}$ be the collection of finite subsets of a set X and $\mathcal{A} = \{A_s : s \in \mathcal{S}\}$ a collection of subsets of a structure M such that if $t \subseteq s$ then $A_t \subseteq A_s$. Show that if $(A'_s \downarrow \bigcup\{A'_t : s \not\subseteq t\}; \bigcup\{A'_t : t \subseteq s \wedge t \neq s\})$ then under the partial ordering determined by the natural partial order of subsets of X, $\mathcal{A}$ is an independent partial order in the sense of Definition 2.25.

2.31 Historical Notes. An independent sequence is often called a Morley sequence. This notion, generalizes Morley's [Morley 1965] idea of finding indiscernibles by constructing sequences of elements such that the type of each element over its predecessors has a fixed rank. The notations $A \downarrow_C B$ and $(A\downarrow B; C)$ are variants of a notation introduced by Makkai. The development of 2.19 through 2.26 is buried in [Shelah 1978]. Except for simplifying the notation, we have followed the considerably clearer version of [Makkai 1984].