MIXING AND SPECTRAL GAP RELATIVE TO PINSKER FACTORS FOR SOFIC GROUPS

BEN HAYES

ABSTRACT. Motivated by our previous results, we investigate structural properties of probability measure-preserving actions of sofic groups relative to their Pinsker factor. We also consider the same properties relative to the Outer Pinsker factor, which is another generalization of the Pinsker factor in the nonamenable case. The Outer Pinsker factor is motivated by entropy in the presence, which fixes some of the "pathological" behavior of sofic entropy: namely increase of entropy under factor maps. We show that an arbitrary probability measure-preserving action of a sofic group is mixing relative to its Pinsker and Outer Pinsker factors and, if the group is nonamenable, it has spectral gap relative to its Pinsker and Outer Pinsker factors. Our methods are similar to those we developed in "Polish models and sofic entropy" and based on representation-theoretic techniques. One crucial difference is that instead of considering unitary representations of a group Γ, we must consider $*$-representations of algebraic crossed products of L^∞ spaces by Γ.

CONTENTS

1. INTRODUCTION

The goal of this paper is to further the investigation set out in [10] exploring the connections between representation theory and entropy of probability measure-preserving actions of groups, particularly for nonamenable groups. Entropy for probability measure-preserving actions of $\mathbb{Z}$ is classical and goes back to the work of Kolmogorov and Sinaĭ. Entropy is roughly a measurement of how "chaotic" the action of $\mathbb{Z}$ is (we refer the reader to [23] Chapters 4,7,8,9 for more information on Kolmogorov-Sinaĭ entropy). It was realized by Kieffer in [15] that one could define entropy for actions of amenable groups instead of $\mathbb{Z}$. Entropy theory for amenable groups has been well-studied and parallels the case of the integers quite well (see e.g. the seminal [16] which reproves some of the fundamental isomorphism results in the amenable case).

Fundamental examples in [16] led many to believe that it was not possible to define entropy in a reasonable way for actions of nonamenable groups. In stunning and landmark work, Bowen in [2] developed a reasonable notion of entropy for the class of *sofic* groups. Sofic groups are a class of groups vastly larger than amenable groups: they contain all amenable groups, all residually finite groups, all linear groups and are closed under free products with amalgamation over amenable subgroups (see [9],[8],[17],[20]). Roughly speaking, a group is sofic if it has "almost actions" which are "almost free" on finite sets (a sequence of such almost actions which behaves more and more like a free action is called a sofic approximation). Sofic entropy of a probability measure-preserving action $\Gamma \curvearrowright (X, \mu)$ then

---

measures the exponential growth of the number of "finitary simulations" there are of the space which are compatible with the sofic approximation.

We remark here that defining entropy for nonamenable groups is not merely generalization for generalizations' sake: the application of results in orbit equivalence and von Neumann algebras require showing that actions of nonamenable groups are not isomorphic. Recall that if (X, μ) is a standard probability space, then the *entropy* of (X, μ), denoted $H(X, \mu)$, is defined to be infinite if (X, μ) is diffuse and is given by

$$H(X, \mu) = -\sum_{x \in X} \mu(\{x\}) \log \mu(\{x\})$$

if (X, μ) is atomic (with the convention that $0 \log 0$ is 0). In Bowen's first paper on the subject he was able to use fundamental work of Popa [18],[19], to show that if Γ is an infinite conjugacy class property (T) sofic group (e.g. $PSL_n(\mathbb{Z})$ for $n \geq 3$) and $(X, \mu), (Y, \nu)$ are standard probability spaces with $H(X, \mu) \neq H(Y, \nu)$, then $L^\infty((X, \mu)^\Gamma) \rtimes \Gamma \not\cong L^\infty((Y, \nu)^\Gamma) \rtimes \Gamma$. Here $L^\infty((X, \mu)^\Gamma) \rtimes \Gamma$ is the von Neumann algebra crossed product, a natural von Neumann algebra associated to any probability measure-preserving action. Bowen also gave similar applications to orbit equivalence rigidity. The use of sofic entropy is completely unavoidable for this result: it is known that if Γ is a sofic group which contains $\mathbb{Z}$ as a subgroup, then sofic entropy is a *complete* invariant for isomorphisms of Bernoulli shifts. Thus one cannot deduce nonisomorphism of crossed product von Neumann algebras or failure of orbit equivalence for Bernoulli actions of such groups without using sofic entropy. Of course one cannot prove such rigidity for actions of amenable groups as the crossed product von Neumann algebras they produce are always the same, by Connes' Theorem on the uniqueness of the amenable II_1-factor (see [6]). Similar remarks apply to orbit equivalence by work of Ornstein-Weiss and Connes-Feldman-Weiss (see [16], [7]).

In [10], we expanded on connections to orbit equivalence theory. To summarize the results we need some terminology. If $\Gamma \curvearrowright (X, \mu)$ is a probability measure-preserving action, a *factor* of the action is another probability measure-preserving action $\Gamma \curvearrowright (Y, \nu)$ so that there is an almost everywhere Γ-equivariant measurable map $\pi \colon X \to Y$ so that $\pi_* \mu = \nu$. We call π a factor map. We sometimes say that $(X, \mu) \to (Y, \nu)$ is an extension and if we wish to specify the group we will say that $\Gamma \curvearrowright (X, \mu) \to \Gamma \curvearrowright (Y, \nu)$ is an extension. An action of a sofic group is said to have completely positive entropy if every nontrivial (i.e. not a one-point space) factor has positive entropy (see [13],[5] for examples of completely positive entropy actions of sofic groups). Lastly, a probability measure-preserving action $\Gamma \curvearrowright (X, \mu)$ is said to be strongly ergodic if for every sequence A_n of measurable subsets of X with $\mu(gA_n \Delta A_n) \to 0$ for all $g \in \Gamma$ we have $\mu(A_n)(1 - \mu(A_n)) \to 0$. In Corollary 1.2 of [10] we showed that every probability measure-preserving action of a nonamenable sofic group with positive entropy is strongly ergodic. We remark that strong ergodicity is an invariant of the orbit equivalence class of the action. Thus, a particular consequence of our results is that if a probability measure-preserving action of a nonamenable group is not strongly ergodic, then no action orbit equivalent to it has completely positive entropy. This results stands in stark contrast to the celebrated fact that all ergodic probability measure-preserving actions of amenable groups are orbit equivalent. The applications sofic entropy has to von Neumann algebra and orbit equivalence rigidity make it clear that generalizing entropy to the nonamenable realm is a useful endeavor, as such rigidity phenomena *never* occur for actions of amenable groups.

In this note we expand on some of the results in [10]. Because these results apply only in the case of completely positive entropy actions, and there are few known examples of such actions, we wish to generalize our results in [10] so that they give structural properties for arbitrary actions. In the amenable case it is well known how to do this: given any probability measure-preserving action of an amenable group there is a maximal factor, called the Pinsker factor, which has entropy zero. We can thus say that any action has completely positive entropy relative to its Pinsker factor and much of what is known for

completely positive entropy actions is known for a general action "relative to the Pinsker factor." For example, any action of an amenable group is mixing relative to its Pinsker factor. Given a sofic group Γ, a sofic approximation $\sigma_i\colon \Gamma \to S_{d_i}$ and a probability measure-preserving action $\Gamma \curvearrowright (X, \mu)$, there is a unique largest factor $\Gamma \curvearrowright (Z, \zeta)$ of $\Gamma \curvearrowright (X, \mu)$ which has entropy zero with respect to the sofic approximation $(\sigma_i)_i$. We can also call this the *Pinsker factor* (we remark that it depends, a priori, on the sofic approximation). However, because entropy can decrease under factors this factor does not, in our opinion, have the right monotonicity properties and so we wish to also investigate another generalization of the Pinsker factor, called the *Outer Pinsker factor*.

There is a way to fix the "pathological" behavior that entropy can increase under factor maps. Implicit in an alternate formulation of entropy due to Kerr in [12], given a countable discrete sofic group Γ, a sofic approximation $\sigma_i\colon \Gamma \to S_{d_i}$, a probability measure-preserving action $\Gamma \curvearrowright (X, \mu)$ and a factor $\Gamma \curvearrowright (Y, \nu)$ of $\Gamma \curvearrowright (X, \mu)$ we can define the " entropy of Y in the presence of X", which we denote by $h_{(\sigma_i)_i, \mu}(Y : X, \Gamma)$. Entropy in the presence measures how many "finitary simulations" of $\Gamma \curvearrowright (Y, \nu)$ there are which "lift" to "finitary simulations" of $\Gamma \curvearrowright (X, \mu)$. At this stage, we caution the reader that, when Γ is amenable, entropy in the presence is *not* the same as relative entropy. We show in the Appendix that when Γ is amenable the entropy of $\Gamma \curvearrowright (Y, \nu)$ in the presence of $\Gamma \curvearrowright (X, \mu)$ is just the entropy of $\Gamma \curvearrowright (Y, \nu)$. Thus while it is very tempting to think of entropy in the presence as some form of relative entropy, this is *not* the correct intuition. We remark that we have given a version of relative entropy for actions of sofic groups in [11]. It is easy to see that entropy in the presence has the right monotonicity properties: $h_{(\sigma_i)_i, \mu}(Y : X, \Gamma)$ is decreasing under factors of Y if we keep X fixed, it is increasing under intermediate factors between X and Y if we keep Y fixed, and it is subadditive under joins of factors in the first variable. Because of these monotonicity properties there is a canonical maximal factor $\Gamma \curvearrowright (Y, \nu)$, called the *Outer Pinsker Factor*, so that the entropy of Y in the presence of X (with respect to $(\sigma_i)_i$) is zero. Once again we remark that the Outer Pinsker factor depends, a priori, on the sofic approximation. We do not claim any originality on the definition of the Outer Pinsker factor, as its definition is quite natural (and appears to be folklore) and through private communication we know it has been observed at least by L.Bowen, Kerr, Seward (each independently of the other). The goal of this paper is merely to extend what representation-theoretic properties we know for completely positive entropy actions to spectral properties of an *arbitrary* probability measure-preserving action relative to the Pinsker and Outer Pinsker factors. In the appendix, we observe that when Γ is amenable we have $h_{(\sigma_i)_i, \mu}(Y : X, \Gamma) = h_{(\sigma_i)_i, \nu}(Y, \Gamma)$. The proof is essentially a combination of [3],[14] and Corollary 5.2 of [20] (which was first proved under the assumption that the action is free in Proposition 1.20 of [17]).

For entropy for single actions, instead of entropy in the presence, to deduce the desired spectral properties from assumptions of positive entropy it was enough to use just the unitary representation theory of the group. It turns out that in order to deduce our desired results for extensions

$$\Gamma \curvearrowright (X, \mu) \to \Gamma \curvearrowright (Y, \nu),$$

we will need to know how both how Γ and how Y (or more precisely $L^\infty(Y)$) "acts" on X. The right way to do this is to replace unitary representations of Γ with $*$-representations of the algebraic crossed product: $L^\infty(Y) \rtimes_{\text{alg}} \Gamma$. Recall that the algebraic crossed product is the algebra of all finite formal sums:

$$\sum_{g \in \Gamma} f_g u_g, \ f_g \in L^\infty(Y) \text{ and all but finitely many } f_g \text{ are zero,}$$

with the imposed relation

$$u_g f = (f \circ g^{-1}) u_g, \ g \in \Gamma, f \in L^\infty(Y).$$

Defining

$$\left(\sum_{g\in\Gamma} f_g u_g\right)^* = \sum_{g\in\Gamma} (\overline{f_g}\circ g)u_{g^{-1}},$$

the algebraic crossed product becomes a $*$-algebra. If $\pi\colon X\to Y$ is a factor map we have a $*$-representation ρ of $L^\infty(Y)\rtimes_{\mathrm{alg}}\Gamma$ on $L^2(X)$ given by:

$$(\rho(u_g)\xi)(x) = \xi(g^{-1}x),\ \text{for } g\in\Gamma,\ \xi\in L^2(X,\mu),$$
$$(\rho(f)\xi)(x) = f(\pi(x))\xi(x),\ \text{for } f\in L^\infty(Y,\nu),\ \xi\in L^2(X,\mu).$$

In order to properly formulate our generalization of Theorem 1.1 in [10], we will need a $*$-representation of $L^\infty(Y,\nu)\rtimes_{\mathrm{alg}}\Gamma$ which may be regarded as an analogue of the left regular representation of a group. Von Neumann algebra theory provides us with the correct analogue: consider the $*$-representation $\lambda\colon L^\infty(Y,\nu)\rtimes_{\mathrm{alg}}\Gamma\to B(L^2(Y,\nu,\ell^2(\Gamma)))$ uniquely determined by

$$(\lambda(f)\xi)(y) = f(y)\xi(y)\ \text{for } y\in Y,\ f\in L^\infty(Y),\ \xi\in L^2(Y,\nu,\ell^2(\Gamma)),$$
$$(\lambda(u_g)\xi)(y)(h) = \xi(g^{-1}y)(g^{-1}h)\ \text{for } y\in Y,\ g,h\in\Gamma.$$

We will see that this is the correct analogue of the left regular representation. We note this $*$-representation of $L^\infty(Y)\rtimes_{\mathrm{alg}}\Gamma$ is precisely the one obtained from the action of the von Neumann algebra crossed product $L^\infty(Y)\rtimes\Gamma$ on its L^2-space. Recall that if $\rho_j\colon A\to B(\mathcal{H}_j),\ j=1,2$ are two $*$-representations of a $*$-algebra on Hilbert spaces $\mathcal{H}_j,\ j=1,2$ then ρ_1,ρ_2 are *singular*, written $\rho_1\perp\rho_2$, if and only if no subrepresentation of ρ_1 is embeddable into ρ_2. Suppose we are given probability measure-preserving actions $\Gamma\curvearrowright(X,\mu),\Gamma\curvearrowright(Y,\nu),\Gamma\curvearrowright(Z,\zeta)$ of Γ and factor maps $\pi\colon X\to Y,\rho\colon X\to Z$. We say $\Gamma\curvearrowright(Z,\zeta)$ is an intermediate factor between X and Y if there is a factor map $\phi\colon Z\to Y$ so that $\pi=\phi\circ\rho$. We say that a set $\mathcal{F}$ of measurable functions $X\to\mathbb{C}$ generates Z if the smallest complete Γ-invariant sigma algebra of sets containing

$$\{f^{-1}(A): f\in\mathcal{F}, A\subseteq\mathbb{C}\text{ is Borel}\}$$

is

$$\{\rho^{-1}(E): E\subseteq Z\text{ is }\zeta\text{-measurable}\}.$$

Note that since we take complete sigma-algebras, this does not depend upon the elements of $\mathcal{F}$ mod null sets, so we can make sense of what it means for a subset of $L^2(X,\mu)$ to generate a factor. We are now ready to state the following analogue of the main theorem of [10] for entropy in the presence.

Theorem 1.1. *Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i\colon\Gamma\to S_{d_i}$. Let $\Gamma\curvearrowright(X,\mu)$ be a measure-preserving action of Γ with (X,μ) a standard probability space. Let $\Gamma\curvearrowright(Y,\nu)$ be a factor of $\Gamma\curvearrowright(X,\mu)$ and let $\Gamma\curvearrowright(Z,\zeta)$ be a intermediate factor between X and Y. Suppose there exists a $L^\infty(Y)\rtimes_{\mathrm{alg}}\Gamma$-subrepresentation $\mathcal{H}$ of $L^2(X,\mu)$ which generates Z and so that $\mathcal{H}$ is singular with respect to $L^2(Y,\nu,\ell^2(\Gamma))$ as a representation of $L^\infty(Y,\nu)\rtimes_{\mathrm{alg}}\Gamma$. Then,*

$$h_{(\sigma_i)_i,\mu}(Z:X,\Gamma) = h_{(\sigma_i)_i,\mu}(Y:X,\Gamma).$$

In particular,

$$h_{(\sigma_i)_i,\zeta}(Z,\Gamma) \le h_{(\sigma_i)_i,\nu}(Y,\Gamma).$$

As in [10], the following description of $L^2(X,\mu)$ as a representation of its Pinsker factor crossed product is automatic. Suppose $\pi\colon X\to Y$ is a factor map. Then we can view $L^2(Y)$ as a subspace of $L^2(X)$ via the embedding $f\mapsto f\circ\pi$ for $f\in L^2(Y)$.

Corollary 1.2. *Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i\colon \Gamma \to S_{d_i}$. Let $\Gamma \curvearrowright (X,\mu)$ be an arbitrary measure-preserving action where (X,μ) is a standard probability space. Let $\Gamma \curvearrowright (Y_0,\nu_0), \Gamma \curvearrowright (Y,\nu)$ be the Outer Pinsker factor and Pinsker factor of $\Gamma \curvearrowright (X,\mu)$ respectively. Then, as a representation of $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$, we have that $L^2(X) \ominus L^2(Y)$ embeds into $L^2(Y,\ell^2(\Gamma))^{\oplus\infty}$. Similarly, as a representation of $L^\infty(Y_0) \rtimes_{\mathrm{alg}} \Gamma$, we have that $L^2(X) \ominus L^2(Y)$ embeds into $L^2(Y_0,\ell^2(\Gamma))^{\oplus\infty}$.*

Since this formulation in terms of algebraic crossed product is somewhat abstract and far from the ergodic theoretic roots of sofic entropy, we mention a purely ergodic theory corollary of Theorem 1.1. We say that an extension
$$\Gamma \curvearrowright (X,\mu) \to \Gamma \curvearrowright (Y,\nu)$$
is *mixing* if for all $\xi,\eta \in L^\infty(X)$ with $\mathbb{E}_Y(\xi) = 0 = \mathbb{E}_Y(\eta)$ we have
$$\lim_{g\to\infty} \|\mathbb{E}_Y((\xi \circ g^{-1})\eta)\|_{L^2(Y)} = 0.$$
Here $\mathbb{E}_Y(f)$ is the conditional expectation of $f \in L^1(X,\mu)$ onto Y. The extension is said to have *spectral gap* if for every sequence $\xi_n \in L^2(X)$ with
$$\|\xi_n \circ g^{-1} - \xi_n\|_2 \to_{n\to\infty} 0 \text{ for all } g \in \Gamma,$$
we have
$$\|\xi_n - \mathbb{E}_Y(\xi_n)\|_2 \to 0.$$
To make sense of $\xi_n - \mathbb{E}_Y(\xi_n)$ we are using the embedding of $L^2(Y)$ into $L^2(X)$ defined before via the factor map.

Corollary 1.3. *Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i\colon \Gamma \to S_{d_i}$. Let $\Gamma \curvearrowright (X,\mu)$ be an arbitrary measure-preserving action where (X,μ) is a standard probability space. Let $\Gamma \curvearrowright (Y_0,\nu_0), \Gamma \curvearrowright (Y,\nu)$ be the Pinsker factor and Outer Pinsker factor of $\Gamma \curvearrowright (X,\mu)$ respectively.*

(i): If Γ is infinite, then $\Gamma \curvearrowright (X,\mu)$ is mixing relative to $\Gamma \curvearrowright (Y_0,\nu_0)$. In particular, $\Gamma \curvearrowright (X,\mu)$ is mixing relative to $\Gamma \curvearrowright (Y,\nu)$.

(ii) If Λ is any nonamenable subgroup of Γ, then $\Lambda \curvearrowright (X,\mu)$ has spectral gap relative to $\Lambda \curvearrowright (Y,\nu)$ and $\Gamma \curvearrowright (Y_0,\nu_0)$. In particular, $\Gamma \curvearrowright (X,\mu)$ is strongly ergodic relative to $\Gamma \curvearrowright (Y_0,\nu_0)$ and $\Gamma \curvearrowright (X,\mu)$ is strongly ergodic relative to $\Gamma \curvearrowright (Y,\nu)$.

We remark that there is another approach to entropy for actions of nonamenable groups called Rokhlin entropy, first investigated by Seward in [21]. Rokhlin entropy is easy to define and is defined for actions of arbitrary groups, but it is extremely hard to compute. There are no known instances where one can show that an action has positive Rokhlin entropy without knowing it has positive sofic entropy. In every case where the Rokhlin entropy has been computed and it is positive the computation has been done by first computing the sofic entropy, then using the general fact that sofic entropy is a lower bound for Rokhlin entropy, and finally showing (by methods that vary from case to case) that the sofic entropy is an upper bound for the Rokhlin entropy. Thus, in our opinion, there has yet to be a satisfactory, explicit computation of Rokhlin entropy which does not go through computing sofic entropy. In an analogous manner one can define the Rokhlin Pinsker factor and the outer Rokhlin Pinsker factor for a probability measure-preserving action of an arbitrary group. Alpeev in [1] showed that any probability measure-preserving action is weakly mixing over its Rokhlin Pinsker factor.

2. Proof of The Main Theorem

We start with the definition of a sofic group. For $d \in \mathbb{N}$, we use u_d for the uniform probability measure on $\{1,\ldots,d\}$.

Definition 2.1. Let Γ be a countable, discrete group. A *sofic approximation* of Γ is a sequence of functions $\sigma_i \colon \Gamma \to S_{d_i}$ (not assumed to be homomorphisms) so that

$$u_{d_i}(\{1 \le j \le d_i : \sigma_i(gh)(j) = \sigma_i(g)\sigma_i(h)(j)\}) \to 1 \text{ for all } g, h \in \Gamma,$$
$$u_{d_i}(\{1 \le j \le d_i : \sigma_i(g)(j) \ne j\}) \to 1 \text{ for all } g \in \Gamma \setminus \{e\}.$$

We say that Γ is *sofic* if it has a sofic approximation.

Intuitively, the first condition of a sofic approximation says that we have an "almost actions" of Γ on the finite set $\{1, \ldots, d_i\}$ and the second condition of a sofic approximation says that this action is "almost free." Since finite groups can be characterized as those groups which act freely on finite sets we may view soficity as the analogue of finiteness one obtains by replacing the exact algebra with approximate algebra. We now turn to some preliminaries needed for the definition of entropy in the presence. It will be important in this paper that we can reduce the computation of entropy (and extension entropy) to generating observables. If A, B are sets, we use A^B for the set of all functions $f \colon B \to A$.

Definition 2.2. Let $(X, \mathcal{M}, \mu)$ be a standard probability space. Let $\mathcal{S}$ be a subalgebra of $\mathcal{M}$ (here $\mathcal{S}$ is not necessarily a σ-algebra). A *finite $\mathcal{S}$-measurable observable* is a measurable map $\alpha \colon X \to A$ where A is a finite set and $\alpha^{-1}(\{a\}) \in \mathcal{S}$ for all $a \in A$. If $\mathcal{S} = \mathcal{M}$, we simply call α *a finite observable*. Another finite $\mathcal{S}$-measurable observable $\beta \colon X \to B$ is said to *refine* α, written $\alpha \le \beta$, if there is a $\omega \colon B \to A$ so that $\omega(\beta(x)) = \alpha(x)$ for almost every $x \in X$. If Γ is a countable discrete group and $\Gamma \curvearrowright (X, \mathcal{M}, \mu)$ by measure-preserving transformations. We say that $\mathcal{S}$ is generating if $\mathcal{M}$ is the σ-algebra generated by $\{gA : A \in \mathcal{S}\}$ (up to sets of measure zero).

Suppose we are given a standard probability space (X, μ), and a countable discrete group Γ with $\Gamma \curvearrowright (X, \mu)$ by measure-preserving transformations. Given a finite observable $\alpha \colon X \to A$ and a finite $F \subseteq \Gamma$, we let $\alpha^F \colon X \to A^F$ be defined by

$$\alpha^F(x)(g) = \alpha(g^{-1}x).$$

Definition 2.3. Let Γ be a countable discrete group and $\sigma \colon \Gamma \to S_d$ for some $d \in \mathbb{N}$ (note: σ is just a function, not a homomorphism). Let $(X, \mathcal{M}, \mu)$ be a standard probability space and let $\mathcal{S} \subseteq \mathcal{M}$ be a subalgebra. Let $\alpha \colon X \to A$ be a finite $\mathcal{S}$ measurable-observable. Given $F \subseteq \Gamma$ finite and $\delta > 0$, we let $\mathrm{AP}(\alpha, F, \delta, \sigma)$ be all $\phi \colon \{1, \ldots, d\} \to A^F$ so that

$$\sum_{a \in A^F} \left| u_{d_i}(\phi^{-1}(\{a\})) - \mu((\alpha^F)^{-1}(\{a\})) \right| < \delta,$$
$$u_{d_i}(\{1 \le j \le d_i : \phi(j)(g) \ne \phi(\sigma(g)^{-1}(j))(e)\}) < \delta \text{ for all } g \in F.$$

We can now define entropy in the presence. The following definition was given by Kerr in [12] and is a natural generalization of Bowen's original definition of measure entropy in [2]. For notation, if $f \colon B \to A$ and $C \subseteq B^X$ for some set X, we let $f \circ C = \{f \circ \phi : \phi \in C\}$.

Definition 2.4. Let Γ be a countable discrete sofic group with sofic approximation $\Sigma = (\sigma_i \colon \Gamma \to S_{d_i})$. Let $(X, \mathcal{M}, \mu)$ be a standard probability space and $\Gamma \curvearrowright (X, \mathcal{M}, \mu)$ by measure-preserving transformations. Let $\mathcal{S}, \mathcal{T}$ be subalgebras of $\mathcal{M}$. Assume that $\mathcal{S} \subseteq \mathcal{T}$. Let $\alpha \colon X \to A$ be a finite $\mathcal{S}$-measurable observable and let $\beta \colon X \to B$ be a $\mathcal{T}$-measurable observable refining α and $\omega \colon B \to A$ as in the definition of $\alpha \le \beta$. For a finite $F \subseteq \Gamma$, we define

$$\widetilde{\omega} \colon B^F \to A$$

by

$$\widetilde{\omega}(b) = \omega(b(e)).$$

We set

$$h_{\Sigma,\mu}(\alpha : \beta, F, \delta) = \limsup_{i \to \infty} \frac{1}{d_i} \log |\widetilde{\omega} \circ (\mathrm{AP}(\alpha, F, \delta, \sigma_i))|$$

$$h_{\Sigma,\mu}(\alpha : \beta, \Gamma) = \inf_{\substack{F \subseteq \Gamma \text{ finite,} \\ \delta > 0}} h_{\Sigma,\mu}(\alpha : \beta, F, \delta).$$

We then set

$$h_{\Sigma,\mu}(\alpha : \mathcal{T}, \Gamma) = \inf_{\alpha \leq \beta} h_{\Sigma,\mu}(\alpha; \beta, \Gamma)$$

$$h_{\Sigma,\mu}(\mathcal{S} : \mathcal{T}, \Gamma) = \sup_{\alpha} h_{\Sigma,\mu}(\alpha : \mathcal{S})$$

where the last infimum is over all $\mathcal{T}$-measurable observables, and the supremum is over all $\mathcal{S}$-measurable observables.

It is known that

$$h_{\Sigma,\mu}(\mathcal{S} : \mathcal{T}, \Gamma)$$

only depends upon the Γ-invariant sigma-algebra of sets generated by $\mathcal{S}, \mathcal{T}$. It is known that if $\mathcal{S}$ is a complete Γ-invariant sigma-subalgebra of $\mathcal{M}$, then there is factor map $\pi \colon (X, \mu) \to (Y, \nu)$ so that

$$\mathcal{S} = \{\pi^{-1}(A) : A \subseteq Y \text{ is } \nu\text{-measurable}\}.$$

Conversely, if we are given a factor map $\pi \colon (X, \mu) \to (Y, \nu)$ then

$$\{\pi^{-1}(A) : A \subseteq Y \text{ is } \nu\text{-measurable}\}$$

is a Γ-invariant sigma-subalgebra. Because of this, we will frequently blur the lines between observables, Γ-invariant sigma-subalgebras and factors. Thus if $\mathcal{A}$ is Γ-invariant sigma-algebra of measurable set in X and Y is the factor generated by this algebra, we shall use

$$h_{(\sigma_i)_i,\mu}(Y : X, \Gamma)$$

for

$$h_{(\sigma_i)_i,\mu}(\mathcal{A} : \mathcal{M}, \Gamma)$$

where $\mathcal{M}$ is the measurable subsets of X. We will call $h_{(\sigma_i)_i,\mu}(Y : X, \Gamma)$ the *entropy of Y in the presence of X* (with respect to $(\sigma_i)_i$). By [12] we have

$$h_{(\sigma_i)_i,\mu}(X : X, \Gamma) = h_{(\sigma_i),\mu}(X, \Gamma).$$

As in [10], we need to use a way to compute the entropy of Y in the presence of X using topological models for

$$X \to Y.$$

For this, we recall some terminology from [10]. Let X be a Polish space and $\Gamma \curvearrowright X$ an action of a countable discrete group Γ by homeomorphisms. We say that a continuous pseudometric Δ is *dynamically generating* if for every open subset U of X and every $x \in U$, there is a $\delta > 0$ and a finite $F \subseteq \Gamma$ so that

$$\bigcap_{g \in F} \{y \in X : \Delta(gx, gy) < \delta\} \subseteq U.$$

We note here that our definition of dynamically generating contains the assumption that Δ is continuous. Let (A, Δ) be a pseudometric space. For subsets C, B of A, and $\varepsilon > 0$ we say that C is ε-contained in B and write $C \subseteq_\varepsilon B$ if for all $c \in C$, there is a $b \in B$ so that $\Delta(c, b) < \varepsilon$. We say that $S \subseteq A$ is ε-dense if $A \subseteq_\varepsilon S$. We use $S_\varepsilon(A, \Delta)$ for the smallest cardinality of a ε-dense subset of A. If $C \subseteq_\delta B$ are subsets of A, then

$$S_{2(\varepsilon+\delta)}(C, \Delta) \leq S_\varepsilon(B, \Delta).$$

We say that $N \subseteq A$ is ε-separated if for every $n_1 \neq n_2$ in N we have $\Delta(n_1, n_2) > \varepsilon$. We use $N_\varepsilon(A, \Delta)$ for the smallest cardinality of a ε-separated subset of A. Note that

(1) $$N_{2\varepsilon}(A, \Delta) \leq S_\varepsilon(A, \Delta) \leq N_\varepsilon(A, \Delta),$$

and that if $A \subseteq B$, then

$$N_\varepsilon(A, \Delta) \leq N_\varepsilon(B, \Delta).$$

Definition 2.5. Let Γ be a countable discrete group and X a Polish space with $\Gamma \curvearrowright X$ by homeomorphisms. Let Δ be a bounded pseudometric on X. For a function $\sigma \colon \Gamma \to S_d$ for some $d \in \mathbb{N}$, a finite $F \subseteq \Gamma$ and a $\delta > 0$, we let $\mathrm{Map}(\Delta, F, \delta, \sigma)$ be all functions $\phi \colon \{1, \ldots, d\} \to X$ so that

$$\max_{g \in F} \Delta_2(\phi \circ \sigma(g), g\phi) < \delta.$$

Given a Polish space X, a finite $L \subseteq C_b(X)$, a $\delta > 0$ and $\mu \in \mathrm{Prob}(X)$, let

$$U_{L,\delta}(\mu) = \bigcap_{f \in L} \left\{ \nu \in \mathrm{Prob}(X) : \left| \int f \, d\nu - \int f \, d\mu \right| < \delta \right\}.$$

Then $U_{L,\delta}(\mu)$ form a basis of neighborhoods of μ for the weak topology. Here $C_b(X)$ is the space of bounded continuous functions on X.

Definition 2.6. Suppose that μ is a Γ-invariant Borel probability measure on X. For $F \subseteq \Gamma$ finite, a $\delta > 0$, a $L \subseteq C_b(X)$ finite and $\sigma \colon \Gamma \to S_d$ for some $d \in \mathbb{N}$, we let $\mathrm{Map}_\mu(\Delta, F, \delta, L, \sigma)$ be the set of all $\phi \in \mathrm{Map}(\Delta, F, \delta, \sigma)$ so that

$$\phi_*(u_d) \in U_{L,\delta}(\mu).$$

for all $f \in L$.

Recall that if X, Y are Polish spaces a continuous surjective map $\pi \colon X \to Y$ is a quotient map if $\{E \subseteq X : \pi^{-1}(E) \text{ is open}\}$ equals the set of open subsets of X.

Definition 2.7. Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i \colon \Gamma \to S_{d_i}$. Let X and Y be Polish spaces with $\Gamma \curvearrowright X, \Gamma \curvearrowright Y$ by homeomorphisms. Suppose that there exists a Γ-equivariant quotient map $\pi \colon X \to Y$. Let μ, ν be Γ-invariant Borel probability measures on X, Y with $\pi_*\mu = \nu$. Let Δ_X, Δ_Y be bounded dynamically generating pseudometrics for X, Y. Inductively define

$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \varepsilon, F, \delta, L) = \limsup_{i \to \infty} \frac{1}{d_i} \log N_\varepsilon(\pi \circ (\mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)), \Delta_Y)$$

$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \varepsilon) = \inf_{\substack{\text{finite } F \subseteq \Gamma, \\ \delta > 0, \\ \text{finite } L \subseteq C_b(X)}} h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \varepsilon, F, \delta, L)$$

$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X) = \sup_{\varepsilon > 0} h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \varepsilon).$$

We wish to prove that

$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X) = h_{(\sigma)_i, \mu}(Y : X, \Gamma)$$

for any choice of dynamically generating pseudometric Δ_Y, Δ_X. Much of the proof follows that of Theorem 3.12 in [10]. The following lemma follows exactly as in Lemma 3.9 of [10]. For notation, if X is a Polish space and $\Gamma \curvearrowright X$ we let $\mathrm{Prob}_\Gamma(X)$ be the set of Γ-invariant Borel probability measures on X.

Lemma 2.8. *Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i \colon \Gamma \to S_{d_i}$. Let X, Y be Polish spaces with $\Gamma \curvearrowright X, \Gamma \curvearrowright Y$ by homeomorphisms. Suppose there exists a topological factor map $\pi \colon X \to Y$ and $\mu \in \mathrm{Prob}_\Gamma(X), \nu \in \mathrm{Prob}_\Gamma(Y)$ with $\pi_*\mu = \nu$. For any pair of dynamically generating pseudometrics Δ_Y, Δ_X on Y, X, we can find compatible metrics Δ'_Y, Δ'_X so that*

$$h_{(\sigma_i)_i, \mu}(\Delta'_Y : \Delta'_X) = h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X).$$

We need the following lemma, which gives us a canonical way of producing microstates for a factor.

Lemma 2.9. *Let Γ be a countable, discrete, sofic group with sofic approximation $\sigma_i \colon \Gamma \to S_{d_i}$. Let X, Y be Polish spaces with $\Gamma \curvearrowright X, \Gamma \curvearrowright Y$ by homeomorphisms. Suppose there exists a topological factor map $\pi \colon X \to Y$ and $\mu \in \mathrm{Prob}_\Gamma(X), \nu \in \mathrm{Prob}_\Gamma(Y)$ with $\pi_*\mu = \nu$. Fix dynamically generating pseudometrics Δ_Y, Δ_X on Y, X. Then, for any finite $F \subseteq \Gamma, L \subseteq C_b(Y)$ and $\delta > 0$, there exists finite $F' \subseteq \Gamma, L' \subseteq C_b(X)$ and $\delta' > 0$ so that for all sufficiently large i,*

$$\pi \circ \mathrm{Map}_\mu(\Delta_X, F,', \delta', L', \sigma_i) \subseteq \mathrm{Map}_\nu(\Delta_Y, F, \delta, L, \sigma_i).$$

Proof. Fix finite $F \subseteq \Gamma, L \subseteq C_b(X)$ and $\delta > 0$. Let M be the diameter of Δ_Y. By Prokhorov's theorem, we may find a compact $K \subseteq X$ so that $\mu(X \setminus K) < \delta$. Choose $\eta > 0$ and a finite $E \subseteq \Gamma$ so that if $x, y \in K$ and

$$\max_{h \in E} \Delta_X(hx, hy) < \eta,$$

then $\Delta_Y(\pi(x), \pi(y)) < \delta$. By Lemma 3.10 in [10], we may find a $F'_0 \subseteq \Gamma, L'_0 \subseteq \Gamma$ finite and $\delta'_0 > 0$ so that

$$\phi_*(u_{d_i})(X \setminus K) \leq 2\delta, \text{ for all } \phi \in \mathrm{Map}_\mu(\Delta_X, F'_0, \delta'_0, L'_0, \sigma_i).$$

Let $\delta' > 0$ depend upon δ, η in manner to be determined later, we will at least assume that $\delta' < \min(\delta, \delta'_0)$. Set

$$L' = L'_0 \cup \{f \circ \pi : f \in L\},$$
$$F' = F'_0 \cup EF.$$

Now suppose that $\phi \in \mathrm{Map}_\mu(\Delta_X, F', \delta', L', \sigma_i)$, as $\phi_*(u_{d_i}) \in U_{L', \delta'}(\mu)$, we have $(\pi \circ \phi)_*(u_{d_i}) \in U_{L, \delta}(\nu)$. Fix $g \in F$ and let

$$C = \bigcap_{h \in E} \{1 \leq j \leq d_i : \Delta_X(hg\phi(j), h\phi(\sigma_i(g)(j))) < \eta\},$$

$$D = \phi^{-1}(X \setminus K).$$

For all $h \in E$, we have

$$\Delta_{X,2}(hg\phi, h\phi \circ \sigma_i(g)) \leq \Delta_{X,2}(hg\phi, \phi \circ \sigma_i(hg)) + \Delta_{X,2}(\phi \circ \sigma_i(hg), \phi \circ \sigma_i(h)\sigma_i(g))$$
$$+ \Delta_{X,2}(h\phi \circ \sigma_i(g), \phi \circ \sigma_i(h)\sigma_i(g))$$
$$= \Delta_{X,2}(hg\phi, \phi \circ \sigma_i(hg)) + \Delta_{X,2}(\phi \circ \sigma_i(hg), \phi \circ \sigma_i(h)\sigma_i(g)) + \Delta_{X,2}(h\phi, \phi \circ \sigma_i(h))$$
$$\leq 2\delta' + M u_{d_i}(\{j : \sigma_i(hg)(j) \neq \sigma_i(h)\sigma_i(g)(j)\}).$$

By soficity we have

$$\Delta_{X,2}(hg\phi, h\phi \circ \sigma_i(g)) \leq 3\delta',$$

if i is sufficiently large. Thus for all sufficiently large i, we have

$$u_{d_i}(D^c \cup C^c) \leq 2\delta + 9\left(\frac{\delta'}{\eta}\right)^2 |E|.$$

We may choose δ' sufficiently small so that

$$u_{d_i}(D^c \cup C^c) \leq 3\delta.$$

We then have for all sufficiently large i,

$$\Delta_{Y,2}(g\pi \circ \phi, \pi \circ \phi \circ \sigma_i(g))^2 \leq 3\delta M + \delta^2.$$

So $\pi \circ \phi \in \mathrm{Map}_\nu(\Delta_Y, F, L, (3\delta M + \delta^2)^{1/2}, \sigma_i)$ for all sufficiently large i.

$\square$

Before we prove that entropy in the presence can be expressed via dynamically generating pseudo-metrics, we need some more notation. If we are given $\sigma\colon \Gamma \to S_d$ and $\phi \in A^d$ we define, for a finite $F \subseteq \Gamma$,

$$\phi_\sigma^F\colon \{1, \cdots, d\} \to A^F$$

by

$$(\phi_\sigma^F)(j)(g) = \phi(\sigma(g)^{-1}j).$$

Given a Polish space X and $\mu \in \mathrm{Prob}_\Gamma(X)$, we let CO_μ the set of all Borel subsets of X so that

$$\mu(\mathrm{int}\, E) = \mu(\overline{E}).$$

Theorem 2.10. *Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i\colon \Gamma \to S_{d_i}$. Let X and Y be Polish spaces with $\Gamma \curvearrowright X, \Gamma \curvearrowright Y$ by homeomorphisms. Suppose that there exists a Γ-equivariant quotient map $\pi\colon X \to Y$. Let μ, ν be Γ-invariant Borel probability measures on X, Y with $\pi_*\mu = \nu$. Let Δ_X, Δ_Y be bounded, dynamically generating pseudometrics for X, Y. Then*

$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \Gamma) = h_{(\sigma_i)_i, \mu}(Y : X, \Gamma).$$

Proof. By Lemma 2.8, we may assume that Δ_Y, Δ_X are compatible. Let M_X, M_Y be the diameters of Δ_X, Δ_Y. We first prove that

$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \Gamma) \leq h_{(\sigma_i)_i, \mu}(Y : X, \Gamma).$$

Let $\varepsilon > 0$, since Y is Polish we can apply Prokhorov's Theorem to find a compact $K \subseteq Y$ so that

$$\nu(Y \setminus K) < \varepsilon.$$

By compactness of K, we may find $y_1, \ldots, y_n \in K$ and $\varepsilon > \delta_1, \ldots, \delta_n > 0$ so that

$$K \subseteq \bigcup_{j=1}^n B_{\Delta_Y}(y_j, \delta_j),$$

$$B_{\Delta_Y}(y_j, \delta_j) \in \mathcal{CO}_\nu.$$

Set

$$E = Y \setminus \bigcup_{j=1}^n B_{\Delta_Y}(y_j, \delta_j),$$

and define

$$\alpha\colon Y \to \{0, 1\}^{n+1}$$

by

$$\alpha(y)(k) = \begin{cases} \chi_{B_{\Delta_Y}(y_k, \delta_k)}(y), & \text{if } 1 \leq k \leq n \\ \chi_E(y), & \text{if } k = n + 1 \end{cases}.$$

Let $\beta\colon X \to B$ be any $\mathcal{CO}_\mu(X)$-measurable observable which refines $\alpha \circ \pi$. Since β refines $\alpha \circ \pi$, we can find a $\omega\colon B \to \{0, 1\}^{n+1}$ such that $\omega \circ \beta = \alpha \circ \pi$. Suppose we are given a finite $F \subseteq \Gamma$ and a $\delta > 0$. By Lemma 3.11 in [10] we may find finite $F' \subseteq \Gamma, L' \subseteq C_b(X)$ and a $\delta' > 0$ so that

$$\beta_{\sigma_i}^F(\mathrm{Map}_\mu(\Delta_X, F', \delta', L', \sigma_i)) \subseteq \mathrm{AP}(\beta, F, \delta, \sigma_i).$$

By Lemma 3.10 in [10] and Lemma 2.9 we may assume that L' is sufficiently large so that
$$(\pi \circ \phi)_*(u_{d_i})(Y \setminus E) \leq 2\varepsilon,$$
for all $\phi \in \mathrm{Map}_\mu(\Delta_X, F', \delta', L', \sigma_i)$ and all sufficiently large i. Choose an index set S and elements $\{\phi_s\}_{s \in S}$ so that
$$\phi_s \in \mathrm{Map}_\mu(\Delta_X, F', L', \delta', \sigma_i) \text{ for all } s \in S,$$
$$\{\alpha \circ \pi \circ \phi_s : s \in S\} = \alpha \circ \pi \circ \mathrm{Map}_\mu(\Delta_X, F', L', \delta', \sigma_i),$$
$$\alpha \circ \pi \circ \phi_s \neq \alpha \circ \pi \circ \phi_{s'}, \text{ for } s \neq s' \text{ in S.}$$
As $\omega \circ \beta \circ \phi_s = \alpha \circ \pi \circ \phi_s$, we have
$$|S| \leq |\widetilde{\omega} \circ (\mathrm{AP}(\beta, F', \delta', \sigma_i))|.$$
Now let $\phi \in \mathrm{Map}_\mu(\Delta_X, F', \delta', \sigma_i)$. Choose an $s \in S$ so that
$$\alpha \circ \pi \circ \phi_s = \alpha \circ \pi \circ \phi.$$
Then
$$\Delta_{Y,2}(\pi \circ \phi, \pi \circ \phi_s)^2 \leq 4M_Y^2 \varepsilon + \frac{1}{d_i} \sum_{j : \pi(\phi(j)), \pi(\phi_s(j)) \notin E} \Delta_Y(\pi(\phi(j)), \pi(\phi_s(j)))^2.$$
If $\phi(j), \phi_s(j)$ are not in E, then the fact that $\alpha^F(\pi(\phi(j)))(e) = \alpha^F(\pi(\phi_s(j)))(e)$ implies that
$$\Delta_Y(\pi(\phi(j)), \pi(\phi_s(j))) < \varepsilon,$$
so
$$\Delta_{Y,2}(\pi \circ \phi, \pi \circ \phi_s)^2 < 4M_Y^2 \varepsilon + \varepsilon^2.$$
Thus
$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, 2(4M_Y^2 \varepsilon + \varepsilon^2)^{1/2}) \leq h_{(\sigma_i)_i, \mu}(\alpha : \beta, F, \delta).$$
Taking the infimum over β, F, δ we find
$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, 2(4M_Y^2 \varepsilon + \varepsilon^2)^{1/2}) \leq h_{(\sigma_i)_i, \mu}(\alpha : \mathrm{CO}_\mu) \leq h_{(\sigma_i), \mu}(Y : X, \Gamma).$$
And letting $\varepsilon \to 0$ shows that
$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \Gamma) \leq h_{(\sigma_i), \mu}(Y : X, \Gamma).$$

We now turn to the reverse inequality. Let $\alpha \colon Y \to A$ be a CO_ν-measurable observable. Fix $\kappa > 0$ and let $\kappa' > 0$ depend upon κ in a manner to be determined later. By Lemma 3.10 in [10] we may choose an $\eta > 0$ and $L_0 \subseteq C_b(Y)$ finite so that if $\zeta \in \mathrm{Prob}(Y)$ and
$$\left| \int_Y f \, d\mu - \int_Y f \, d\zeta \right| < \eta$$
for all $f \in L_0$, then
$$|\zeta(\alpha^{-1}(\{a\})) - \mu(\alpha^{-1}(\{a\}))| < \kappa', \text{ for all } a \in A,$$
$$\zeta(\mathcal{O}_\eta(\alpha^{-1}(\{a\})) \setminus \alpha^{-1}(\{a\})_\eta) < \kappa' \text{ for all } a \in A.$$
Let $F' \subseteq \Gamma, L' \subseteq C_b(X)$ be given finite sets and $\delta' > 0$ be given. We assume that
$$L' \supseteq \{f \circ \pi : f \in L_0\}.$$
By Lemma 3.11 in [10], we may choose a refinement $\beta \colon X \to B$ of α, a finite $F \subseteq \Gamma$ and a $\delta > 0$ so that if $\beta^F \circ \phi \in \mathrm{AP}(\beta, F, \delta, \sigma_i)$, then $\phi \in \mathrm{Map}_\mu(\Delta_X, F', L', \delta', \sigma_i)$. Choose $\omega \colon B \to A$ so that $\beta \circ \omega = \alpha \circ \pi$ and choose a map $s \colon B^F \to X$ so that $\mathrm{Id} = \beta^F \circ s$. By construction, if $\phi \in \mathrm{AP}(\beta, F', \delta', \sigma_i)$, then we have
$$s \circ \phi \in \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i).$$

Let $\varepsilon > 0$ be sufficiently small depending upon η in a manner to be determined later. Let T be an index set and $\{\phi_t\}_{t \in T}$ be such that

$$\phi_t \in \mathrm{AP}(\beta, F', \delta', \sigma_i) \text{ for all } t \in T,$$

$$\{\pi \circ s \circ \phi_t : t \in T\} \text{ is } \varepsilon\text{-dense in } \{\pi \circ s \circ \phi : \phi \in \mathrm{AP}(\beta, F', \delta', \sigma_i)\}$$

$$\pi \circ s \circ \phi_t \neq \pi \circ s \circ \phi_{t'} \text{ if } t \neq t'.$$

We may choose such a T with

$$|T| \leq S_{\varepsilon/2}(\pi \circ \mathrm{Map}_\mu(\Delta_X, F', L', \delta', \sigma_i), \Delta_{Y,2}).$$

Then

$$\widetilde{\omega} \circ (\mathrm{AP}(\beta, F, \delta, \sigma_i)) \subseteq \bigcup_{t \in T} \alpha \circ (B_{\Delta_{Y,2}}(\pi \circ s \circ \phi_t, \varepsilon) \cap \pi \circ \mathrm{Map}_\mu(\Delta_X, F', L', \delta', \sigma_i)).$$

We thus have to bound $|\alpha \circ (B_{\Delta_{Y,2}}(\pi \circ s \circ \phi_t, \varepsilon) \cap \pi \circ \mathrm{Map}_\mu(\Delta_X, F', L', \delta', \sigma_i))|$ from above. Fix $t \in T$, suppose that $\phi \in \mathrm{Map}_\mu(\Delta_X, F', \delta', L', \sigma_i)$ and that $\Delta_{2,Y}(\pi \circ \phi, \pi \circ s \circ \phi_t) < \varepsilon$. Let

$$C_1 = \bigcup_{a \in A} \{1 \leq j \leq d_i : \pi(\phi(j)) \in \mathcal{O}_\eta(\alpha^{-1}(\{a\})) \setminus \alpha^{-1}(\{a\})_\eta\},$$

$$C_2 = \{1 \leq j \leq d_i : \pi(s(\phi_t(j))) \in \mathcal{O}_\eta(\alpha^{-1}(\{a\})) \setminus \alpha^{-1}(\{a\})_\eta\},$$

$$C = C_1 \cup C_2.$$

If we choose κ' sufficiently small, we then have that $u_{d_i}(C) \leq \kappa$. Let

$$D = \{1 \leq j \leq d_i : \Delta(\pi(\phi(j)), \pi(s(\phi_t(j)))) \geq \sqrt{\varepsilon}\},$$

so $u_{d_i}(D) \leq \sqrt{\varepsilon}$. For $j \in \{1, \ldots d_i\} \setminus (C \cup D)$ and $a = \alpha(\pi(\phi(j)))$, we have that $\pi(s(\phi_t(j))) \in \mathcal{O}_{\sqrt{\varepsilon}}(\alpha^{-1}(\{a\}))$. Hence if we choose $\sqrt{\varepsilon} < \eta$, then $\alpha(\pi(s(\phi_t(j)))) = a$ for all $j \in \{1, \ldots, d_i\} \setminus (C \cup D)$. So we can find a $\mathcal{V} \subseteq \{1, \ldots, d_i\}$ with $u_{d_i}(\mathcal{V}) \geq 1 - \kappa - \sqrt{\varepsilon}$ and $\alpha(\pi(s(\phi_t(j)))) = \alpha(\pi(\phi(j)))$ for all $j \in \mathcal{V}$. Thus

$$|\alpha \circ (B_{\Delta_{Y,2}}(\pi \circ s \circ \phi_t, \varepsilon) \cap \pi \circ \mathrm{Map}_\mu(\Delta_X, F', L', \delta', \sigma_i))| \leq \sum_{\substack{\mathcal{V} \subseteq \{1, \ldots, d_i\}, \\ |\mathcal{V}| \leq (\kappa + \sqrt{\varepsilon})d_i}} |A|^{|\mathcal{V}|}$$

$$\leq \sum_{l=1}^{\lfloor (\kappa + \sqrt{\varepsilon})d_i \rfloor} \binom{d_i}{l} |A|^l.$$

If $\kappa + \sqrt{\varepsilon} < 1/2$ then for all large i we have

$$\binom{d_i}{l} \leq \binom{d_i}{\lfloor \kappa + \sqrt{\varepsilon} \rfloor d_i}.$$

So by Stirling's Formula the above sum is at most

$$R(\kappa + \sqrt{\varepsilon})d_i \exp(d_i H(\kappa + \sqrt{\varepsilon})) |A|^{\kappa d_i}$$

for some constant $R > 0$, where

$$H(t) = -t \log t - (1 - t) \log(1 - t) \text{ for } 0 \leq t \leq 1.$$

Thus

$$h_{(\sigma_i)_i, \mu}(\alpha; \mathrm{CO}_\mu) \leq h_{\Sigma, \mu}(\alpha; \beta, F, \delta) \leq H(\kappa + \sqrt{\varepsilon}) + \kappa \log |A| + h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X; \varepsilon/2, F', \delta', L').$$

Taking the infimum over F', δ', L' and then letting $\kappa \to 0$ shows that

$$h_{(\sigma_i)_i, \mu}(\alpha; CO_\mu) \le h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \varepsilon/2).$$

Letting $\varepsilon \to 0$ and then taking the supremum over α shows that

$$h_{(\sigma_i)_i, \mu}(Y : X, \Gamma) = h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \Gamma).$$

$\square$

If Y is compact we use

$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \infty)$$

for the quantity defined in the same manner as

$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X)$$

replacing $\Delta_{Y,2}$ with $\Delta_{Y,\infty}$. We apply similar remarks for

$$h_{(\sigma_i)_i, \mu}(\Delta_Y, \varepsilon, F, \delta, L, \infty)$$
$$h_{(\sigma_i)_i, \mu}(\Delta_Y, \varepsilon, \infty).$$

Proposition 2.11. *Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i \colon \Gamma \to S_{d_i}$. Let X and Y be Polish spaces with $\Gamma \curvearrowright X, \Gamma \curvearrowright Y$ by homeomorphisms. Suppose that Y is compact and that there exists a Γ-equivariant quotient map $\pi \colon X \to Y$. Let μ, ν be Γ-invariant Borel probability measures on X, Y with $\pi_* \mu = \nu$. Let Δ_X, Δ_Y be bounded dynamically generating pseudometrics for X, Y. Then*

$$h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \infty, \Gamma) = h_{(\sigma_i)_i, \mu}(Y : X, \Gamma).$$

Proof. Fix $\varepsilon > 0$. Let $0 < \varepsilon' < \varepsilon$ be given. Let $E \subseteq Y$ be a finite ε-dense set with respect to Δ_Y, this may be done as Y is compact. Fix $F \subseteq \Gamma, L \subseteq C_b(X)$ finite and $\delta > 0$. Let S' be a ε'-dense subset of $\pi \circ \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)$ with respect to $\Delta_{Y,2}$ of minimal cardinality. Choose a $T \subseteq \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)$ so that $\pi \circ T = S$ and $|T| = |S|$. Given $\phi \in \mathrm{Map}_\mu(\Delta_X, F, L, \delta, \sigma_i)$, choose a $\psi \in T$ with

$$\Delta_2(\pi \circ \phi, \pi \circ \psi) < \varepsilon'.$$

Let

$$C = \{1 \le j \le d_i : \Delta_Y(\pi(\phi(j)), \pi(\psi(j))) < \varepsilon\}.$$

Then

$$u_{d_i}(C) \ge 1 - \left(\frac{\varepsilon'}{\varepsilon}\right)^2.$$

We thus see that

$$\pi \circ \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i) \subseteq_{\varepsilon, \Delta_{Y,\infty}} \bigcup_{\substack{\mathcal{V} \subseteq \{1,\ldots,d_i\}, \\ u_{d_i}(\mathcal{V}) \ge 1 - \left(\frac{\varepsilon'}{\varepsilon}\right)^2, \\ \psi \in S}} \{\phi \in Y^{d_i} : \phi|_\mathcal{V} = \psi, \phi|_{\mathcal{V}^c} \in E^{\mathcal{V}^c}\}.$$

As in the proof of the preceding theorem, we may find a $R > 0$ so that

$$S_{2\varepsilon}(\pi \circ \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i), \Delta_Y, \infty) \le R \left(\frac{\varepsilon'}{\varepsilon}\right)^2 \exp\left(H\left(\left(\frac{\varepsilon'}{\varepsilon}\right)^2\right) d_i\right) |E|^{\left(\frac{\varepsilon'}{\varepsilon}\right)^2 d_i} |S|.$$

Thus

$$h_{(\sigma_i), \mu}(\Delta_Y : \Delta_X, 4\varepsilon, F, \delta, L, \infty) \le H\left(\left(\frac{\varepsilon'}{\varepsilon}\right)^2\right) + \left(\frac{\varepsilon'}{\varepsilon}\right)^2 \log |E| + h_{(\sigma_i)_i, \mu}(\Delta_Y : \Delta_X, \varepsilon', F, \delta, L).$$

Taking the infimum over all F, δ, L we see that

$$h_{(\sigma_i),\mu}(\Delta_Y : \Delta_X, 4\varepsilon, \infty) \leq H\left(\left(\frac{\varepsilon'}{\varepsilon}\right)^2\right) + \left(\frac{\varepsilon'}{\varepsilon}\right)^2 \log|E| + h_{(\sigma_i)_i,\mu}(\Delta_Y : \Delta_X, \varepsilon')$$

Letting $\varepsilon' \to 0$ and then $\varepsilon \to 0$ shows that

$$h_{(\sigma_i)_i,\mu}(\Delta_Y : \Delta_X, \infty, \Gamma) \leq h_{(\sigma_i)_i,\mu}(Y : X, \Gamma).$$

Since the reverse inequality is trivial, the proof is complete.

$\square$

We use the definition of *singularity* of representations of a $*$-algebra as in [10] Definition 4.1. We first make a preliminary observation. Let A be a $*$-algebra and $\rho_j \colon A \to B(\mathcal{H}_j), j = 1, 2$ be $*$-representations of A on Hilbert spaces $\mathcal{H}_j, j = 1, 2$. Note that for $\xi \in \mathcal{H}_1$, we have that $\overline{A\xi}$ is singular with respect to $\mathcal{H}_2$ as a representation of A if and only if $T(\xi) = 0$ for all $T \in \mathrm{Hom}_A(\rho_1, \rho_2)$.

If $\mathcal{F}$ is a family of functions on X, we define the factor generated by $\mathcal{F}$ to be the factor associated to the sigma-algebra

$$\{gf^{-1}(E) : f \in \mathcal{F}, g \in \Gamma, E \subseteq \mathbb{C} \text{ is Borel}\}.$$

Given a factor $\Gamma \curvearrowright (Y, \nu)$ of $\Gamma \curvearrowright (X, \mu)$, we define the factor generated by $\mathcal{F}$ over Y to be the factor generated $\mathcal{F} \cup L^\infty(Y)$. If this factor is just X itself, we say that $\mathcal{F}$ generates X over Y.

Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i \colon \Gamma \to S_{d_i}$. Let X be a compact, metrizable space with $\Gamma \curvearrowright X$ by homeomorphisms. If $\phi \colon \{1, \ldots, d_i\} \to X$, and $f = \sum_{g \in \Gamma} f_g u_g \in C(X) \rtimes_{\mathrm{alg}} \Gamma$, we define $\phi \rtimes \sigma_i(f) \in M_{d_i}(\mathbb{C})$ by

$$\phi \rtimes \sigma_i(f) = \sum_{g \in \Gamma} m_{f_g \circ \phi} \sigma_i(g).$$

We let τ_ν be the trace on $L^\infty(Y, \nu) \rtimes_{\mathrm{alg}} \Gamma$ given by

$$\tau_\nu\left(\sum_{g \in \Gamma} f_g u_g\right) = \int_Y f_e \, d\nu.$$

Let Γ be a countable discrete group and X a compact metrizable space with $\Gamma \curvearrowright Y$ by homeomorphisms. Let μ be a Γ-invariant probability measure on X. We shall use the trace τ_μ on $C(X) \rtimes_{\mathrm{alg}} \Gamma$ defined by

$$\tau_\mu\left(\sum_{g \in \Gamma} f_g u_g\right) = \int_X f_e \, d\mu.$$

As in the case of $\mathbb{C}(\Gamma)$, we wish to produce an approximation of $C(Y) \rtimes_{\mathrm{alg}} \Gamma$ from a microstate ϕ, and a sofic approximation of Γ. Suppose now that Γ is sofic with sofic approximation $\sigma_i \colon \Gamma \to S_{d_i}$. If $\phi \colon \{1, \ldots, d_i\} \to X$, and $f = \sum_{g \in \Gamma} f_g u_g \in C(X) \rtimes_{\mathrm{alg}} \Gamma$, we define $\phi \rtimes \sigma_i(f) \in M_{d_i}(\mathbb{C})$ by

$$\phi \rtimes \sigma_i(f) = \sum_{g \in \Gamma} m_{f_g \circ \phi} \sigma_i(g),$$

regarding $\sigma_i(g)$ as a permutation matrix. Here if $f \in \mathbb{C}^{d_i}$ we use m_f for the diagonal matrix with entries $(f(1), \ldots, f(d_i))$. For $A \in M_k(\mathbb{C})$, we use $\mathrm{tr}(A) = \frac{1}{k}\sum_{j=1}^k A_{jj}$ and we use $\|A\|_2 = \mathrm{tr}(A^*A)^{1/2}$. If X is a compact space and $f \in C(X)$, we use $\|f\|$ for the supremum norm of f.

Lemma 2.12. *Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i \colon \Gamma \to S_{d_i}$. Let X be a compact metrizable space with $\Gamma \curvearrowright X$ by homeomorphisms fixing a Borel probability measure μ on X. Let Δ be a dynamically generating pseudometric on X. Let $f_1, \ldots, f_n \in C(X) \rtimes_{\mathrm{alg}} \Gamma$, and $\varepsilon > 0$. Then there exists a $\delta > 0$ and finite $F \subseteq \Gamma, L \subseteq C(X)$ so that for all large i and for all $\phi \in \mathrm{Map}_\mu(\Delta, F, \delta, L, \sigma_i)$ we have*

$$|\operatorname{tr}(\phi \rtimes \sigma_i(f_j)) - \tau_\mu(f_j)| < \varepsilon, \ \textit{for } j = 1, \ldots, n,$$
$$|\|\phi \rtimes \sigma_i(f_j)\|_{L^2(\mathrm{tr})} - \|f_j\|_{L^2(\tau_\mu)}| < \varepsilon, \ \textit{for } j = 1, \ldots, n.$$

Proof. We do the proof for $j = 1$, and use f instead of f_1. It is easy to see that this is enough. Write $f = \sum_{g \in \Gamma} f_g u_g$. We shall only prove the first statement. The second may be obtained from the first (albeit with different ε's). Note that if $g \in \Gamma \setminus \{e\}$, we have

$$|\operatorname{tr}(m_{f \circ \phi} \sigma_i(g))| = \left| \frac{1}{d_i} \sum_{j=1}^{d_i} f_g(\phi(j)) \delta_{j = \sigma_i(g)(j)} \right| \leq \|f_g\| u_{d_i}(\{j : \sigma_i(g)(j) = j\}) \to 0,$$

by soficity. We use $o(1)$ for any expression which tends to zero as $i \to \infty$. We thus see that for any $\phi \in \mathrm{Map}(\Delta, F, \delta, L, \sigma_i)$ we have

$$\operatorname{tr}(\phi \rtimes \sigma_i(f)) = o(1) + \operatorname{tr}(m_{f_e \circ \phi} \sigma_i(e)).$$

Suppose we assume $L \supseteq \{f_g : g \in \Gamma\}$ (as only finitely many of the f_g are nonzero this is possible). Then

$$\left| \operatorname{tr}(m_{f_e \circ \phi} \sigma_i(e)) - \int f_e \, d\mu \right| \leq \|f_e\| u_{d_i}(\{j : \sigma_i(e)(j) \neq j\}) + \left| \operatorname{tr}(m_{f_e}) - \int f_e \, d\mu \right|$$

$$= \|f_e\| u_{d_i}(\{j : \sigma_i(e)(j) \neq j\}) + \left| \frac{1}{d_i} \sum_{j=1}^{d_i} f_e(\phi(j)) - \int f_e \, d\mu \right|$$

$$\leq \|f_e\| u_{d_i}(\{j : \sigma_i(e)(j) \neq j\}) + \delta.$$

Since

$$u_{d_i}(\{j : \sigma_i(e)(j) \neq j\}) \to 0$$

as $i \to \infty$ and

$$\tau_\mu(f) = \int f_e \, d\mu$$

the proof is complete.

$\square$

Theorem 2.13. *Let Γ be a countable discrete group and $\sigma_i \colon \Gamma \to S_{d_i}$ a sofic approximation. Let (X, μ) be a standard probability space and $\Gamma \curvearrowright (X, \mu)$ a measure-preserving action. Let $\mathcal{H}$ be a $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$-subrepresentation of $L^2(X)$ and let (Z, ζ) be the intermediate factor between X and Y generated by $\mathcal{H} \cup L^\infty(Y)$. If $\mathcal{H}$ is singular with respect to $L^2(Y, \ell^2(\Gamma))$ as a representation of $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$, then*

$$h_{(\sigma_i)_i, \mu}(Z : X, \Gamma) = h_{(\sigma_i)_i, \mu}(Y : X, \Gamma).$$

In particular,

$$h_{(\sigma_i)_i, \zeta}(Z, \Gamma) \leq h_{(\sigma_i)_i, \nu}(Y, \Gamma).$$

Proof. We let

$$\rho \colon L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma \to B(L^2(X))$$

be defined as before the theorem. It is clear that

$$h_{(\sigma_i)_i, \mu}(Z : X, \Gamma) \geq h_{(\sigma_i)_i, \mu}(Y : X, \Gamma)$$

so it suffices to show that

$$h_{(\sigma_i)_i,\mu}(Z : X, \Gamma) \leq h_{(\sigma_i)_i,\mu}(Y : X, \Gamma).$$

Without loss of generality, suppose that Y is a compact metrizable space, that $\Gamma \curvearrowright Y$ by homeomorphisms, and that ν is a Borel measure on Y. Let $(\xi_n)_{n=1}^\infty$ be a dense sequence in $\mathcal{H}$.

We first reduce to the case that $\mathcal{H}$ is cyclic as a representation of $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$. So suppose that we can prove the theorem in the case that $\mathcal{H}$ is cyclic. For $n \geq 1$, let (Z_n, ζ_n) be the factor of X generated by $L^\infty(Y)$ and the functions

$$\{\xi_1, \ldots, \xi_n\}.$$

We use (Z_0, ζ_0) for (Y, ν). For $n \geq 1$, let $\mathcal{K}_n$ be the smallest closed $L^\infty(Z_{n-1}) \rtimes_{\mathrm{alg}} \Gamma$-invariant subspace of $L^2(X)$ containing ξ_n. We claim that $\mathcal{K}_n$ is singular with respect to $L^2(Z_{n-1}, \ell^2(\Gamma))$ as a representation of $L^\infty(Z_{n-1}) \rtimes_{\mathrm{alg}} \Gamma$. To see this, suppose that

$$T \colon \mathcal{K}_n \to L^2(Z_{n-1}, \ell^2(\Gamma))$$

is a $L^\infty(Z_{n-1}) \rtimes_{\mathrm{alg}} \Gamma$-equivariant bounded linear map. Then T is $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$-equivariant. Since $L^2(Z_{n-1}, \ell^2(\Gamma))$ embeds into

$$L^2(Y, \ell^2(\Gamma))^{\oplus\infty}$$

as a representation of $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$, and $\mathcal{H}$ is singular with respect to $L^2(Y, \ell^2(\Gamma))$, our observation before the theorem shows that $T(\xi_n) = 0$. Since T is $L^\infty(Z_{n-1}) \rtimes_{\mathrm{alg}} \Gamma$-equivariant and $\mathcal{K}_n$ is generated by ξ_n as a representation of $L^\infty(Z_{n-1}) \rtimes_{\mathrm{alg}} \Gamma$, we see that $T = 0$. Thus $\mathcal{K}_n$ is singular with respect to $L^2(Z_{n-1}, \ell^2(\Gamma))$ as a representation of $L^\infty(Z_{n-1}) \rtimes_{\mathrm{alg}} \Gamma$.

Since we are assuming we can prove the Theorem in the cyclic case we see inductively that

$$h_{(\sigma_i)_i,\mu}(Z_n : X) = h_{(\sigma_i)_i,\mu}(Y : X)$$

for all $n \geq 1$. As

$$h_{(\sigma_i)_i,\mu}(Z : X, \Gamma) \leq \liminf_{n\to\infty} h_{(\sigma_i)_i,\mu_n}(Z_n : X, \Gamma),$$

we have

$$h_{(\sigma_i)_i,\mu}(Z : X, \Gamma) \leq h_{(\sigma_i)_i,\mu}(Y : X, \Gamma).$$

Thus we may assume that $\mathcal{H}$ can be generated over Y by a single $\xi \in L^2(X) \ominus L^2(Y)$. Without loss of generality, we may assume that

$$\|\xi\|_2 \leq 1.$$

Define

$$\Phi_1 \colon X \to \mathbb{C}^\Gamma$$

by $\Phi_1(x)(g) = \xi(g^{-1}x)$ and define $\Phi \colon X \to \mathbb{C}^\Gamma \times Y$ by $\Phi(x) = (\Phi_1(x), y)$ and set $\eta = \Phi_*\mu$. Let $\Gamma \curvearrowright \mathbb{C}^\Gamma$ by left shifts and let $\Gamma \curvearrowright \mathbb{C}^\Gamma \times Y$ be the product action. Since ξ generates Z over Y, we know that $\Gamma \curvearrowright (\mathbb{C}^\Gamma \times Y, \eta) \cong \Gamma \curvearrowright (Z, \zeta)$. Hence, we may assume that $Z = \mathbb{C}^\Gamma \times Y$, that the factor map $\pi_Y \colon Z \to Y$ is projection onto the second factor, that $\Gamma \curvearrowright Z$ is the product action where $\Gamma \curvearrowright \mathbb{C}^\Gamma$ by left shifts, and that ξ is given by $\xi(z, y) = z(e)$. We may also assume that X is a Polish space and that the factor map $\pi_Z \colon X \to Z$ is continuous. Let Δ_Y be a compatible metric on Y and let Δ_Z be the dynamical generated metric on Z defined by

$$\Delta_Z((z_1, y_1), (z_2, y_2)) = \min(|z_1(e) - z_2(e)|, 1) + \Delta_Y(y_1, y_2).$$

Fix a dynamically generating pseudometric Δ_X on X. Using Proposition 4.2 in [10] and the density of $C(Y)$ inside $L^\infty(Y, \nu)$ in the weak operator topology, it is not hard to argue that $L^2(Z) \ominus L^2(Y)$ is singular with respect to $L^2(Z, \zeta, \ell^2(\Gamma))$ as a representation of $C(Y) \rtimes_{\mathrm{alg}} \Gamma$.

Let $1 > \varepsilon > 0$, and let $0 < \eta < \varepsilon$ be arbitrary. Since $\mathcal{H}$ is singular with respect to $L^2(Z, \zeta, \ell^2(\Gamma))$, by Proposition 4.2 of [10] we can find a $f \in C(Y) \rtimes_{\mathrm{alg}} \Gamma$ with

$$\|\rho(f)\| \leq 1,$$
$$\tau_\nu(f^*f) < \eta^2,$$
$$\|\rho(f^*f)\xi - \xi\|_2 < \eta.$$

Write

$$f = \sum_{g \in E} f_g u_g.$$

with E a finite subset of Γ and set $R = \max_{g \in E} \|f_g\|_\infty$. Let $M > 0$. Choose a $G \in C_c(\mathbb{C})$ so that $G(z) = z$ if $|z| \leq M$ and $\|G\|_{C_b(\mathbb{C})} \leq M$. We may suppose that M is sufficiently large so that

$$\|G \circ \xi - \xi\|_2 < \varepsilon$$
$$\mu(\{(z,y) : |z(e)| \geq M\}) \leq \varepsilon.$$

Since $\|\rho(f)\| \leq 1$ we have

$$\|\rho(f^*f)(G \circ \xi) - G \circ \xi\|_2 < 3\varepsilon.$$

For $\phi \in X^{d_i}$, we set $\phi_Y = \pi_Y \circ \pi_Z \circ \phi$, $\phi_Z = \pi_Z \circ \phi$. Fix finite $F \subseteq \Gamma, L \subseteq C_b(X)$ and $\delta > 0$. By Lemma 2.12, we may assume that F, δ, L are chosen appropriately so that for all large i and all $\phi \in \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)$ we have

$$\|(\phi_Y \rtimes \sigma_i)(f)^*(\phi_Y \rtimes \sigma_i)(f)(G \circ \xi \circ \phi_Z) - G \circ \xi \circ \phi_Z\|_2 < 4\varepsilon,$$
$$\|(\phi_Y \rtimes \sigma_i)(f)\|_2 < 2\eta,$$
$$\|G \circ \xi \circ \phi_Z\|_2 \leq 2,$$
$$u_{d_i}(\{j : |\xi(\phi(j))| \geq M\}) \leq 2\varepsilon.$$

For $\phi \in \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)$, set

$$p_\phi = \chi_{[0,\sqrt{\varepsilon}]}(|1 - (\phi_Y \rtimes \sigma_i)(f)^*(\phi_Y \rtimes \sigma_i)(f)|) = \chi_{[1-\sqrt{\varepsilon},1+\sqrt{\varepsilon}]}((\phi_Y \rtimes \sigma_i)(f)^*(\phi_Y \rtimes \sigma_i)(f)).$$

Thus for all large i,

$$\mathrm{tr}(p_\phi) \leq \frac{1}{(1 - \sqrt{\varepsilon})} \|(\phi_Y \rtimes \sigma_i)(f)\|_2^2 \leq \frac{4\eta^2}{(1 - \sqrt{\varepsilon})}.$$

By Lemma 2.7 of [10] we find, for all large i,

$$S_\phi \subseteq M p_\phi \mathrm{Ball}(\ell^2(d_i, u_{d_i}))$$

an ε-dense set with respect to $\|\cdot\|_2$ and so that

$$|S_\phi| \leq M^{8 \frac{\eta^2}{(1-\sqrt{\varepsilon})} d_i} \left(\frac{3 + 3\varepsilon}{\varepsilon}\right)^{\frac{8\eta^2}{(1-\sqrt{\varepsilon})} d_i}.$$

Let $\kappa > 0$ depend upon ε, E in a manner to be determined later. Let $\varepsilon > \varepsilon' > 0$ be such that if $a, b \in Y^{d_i}$ and $\Delta_{Y,\infty}(a, b) < \varepsilon'$, then

$$\|f_g \circ a - f_g \circ b\|_\infty < \kappa \text{ for all } g \in E.$$

Choose a $D \subseteq \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)$ so that $\{\psi_Y : \psi \in D\}$ is ε'-dense in

$$\{\phi_Y : \phi \in \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)\}$$

with respect to $\Delta_{Y,\infty}$ and

$$|D| = S_{\varepsilon'}(\{\phi_Y : \phi \in \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)\}, \Delta_{Y,\infty}).$$

Suppose that $\phi, \psi \in \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)$ and that

$$\Delta_{Y,\infty}(\phi_Y, \psi_Y) < \varepsilon'.$$

Then

$$\|(\phi_Y \rtimes \sigma_i)(f)^*(\phi_Y \rtimes \sigma_i)(f)(G \circ \xi \circ \phi_Z) - (\psi_Y \rtimes \sigma_i)(f)^*(\psi_Y \rtimes \sigma_i)(f)(G \circ \xi \circ \phi_Z)\|_2 \le$$

$$\sum_{g,h \in E} \|(f_g \circ \phi_Y) \cdot (f_h \circ \phi_Y \circ \sigma_i(g)) \cdot (\sigma_i(g)(G \circ \xi \circ \phi_Z)) - (f_g \circ \psi_Y) \cdot (f_h \circ \psi_Y \circ \sigma_i(g)) \cdot (\sigma_i(g)(G \circ \xi \circ \phi_Z))\|_2 \le$$

$$4R|E|^2\kappa,$$

as

$$\|G \circ \xi \circ \phi_Z\|_2 \le 2, \max_{g \in E} \|f_g\|_\infty \le R, \max_{g \in E} \|f_g \circ \phi_Y - f_g \circ \psi_Y\|_\infty < \kappa.$$

Now choose κ so that

$$4R|E|^2\kappa < \varepsilon.$$

Suppose that $\phi \in \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i)$ and that $\psi \in D$ has

$$\Delta_{Y,\infty}(\phi, \psi) < \varepsilon'.$$

By the above, we have

$$\|(\psi \rtimes \sigma_i)(f)^*(\psi \rtimes \sigma_i)(f)(G \circ \xi \circ \phi_Z) - G \circ \xi \circ \phi_Z\|_2 \le \varepsilon + \|(\phi \rtimes \sigma_i(f))^*(\phi \rtimes \sigma_i(f))(G \circ \xi \circ \phi_Z) - (G \circ \xi \circ \phi_Z)\|_2 < 5\varepsilon.$$

If i sufficiently large, we may choose a $w \in S_\psi$ so that

$$\|w - p_\psi(G \circ \xi \circ \phi_Z)\|_2 < \varepsilon.$$

Then

$$\|w - G \circ \xi \circ \pi_Z \circ \phi\|_2 \le \varepsilon + \|(p_\psi - 1)(G \circ \xi \circ \pi_Z \circ \phi)\|_2$$

$$= \varepsilon + \langle(1 - p_\psi)(G \circ \xi \circ \phi_Z), G \circ \xi \circ \phi_Z\rangle^{1/2}$$

$$\le \varepsilon + \frac{1}{\sqrt{\varepsilon}}\langle|1 - (\psi \rtimes \sigma_i)(f)^*(\psi \rtimes \sigma_i)(f)|^2(G \circ \xi \circ \pi_Z \circ \phi), G \circ \xi \circ \pi_Z \circ \phi\rangle^{1/2}$$

$$= \varepsilon + \frac{1}{\sqrt{\varepsilon}}\|(1 - (\psi \rtimes \sigma_i)(f)^*(\psi \rtimes \sigma_i)(f))(G \circ \xi \circ \phi_Z)\|_2$$

$$\le \varepsilon + \frac{5\varepsilon}{\sqrt{\varepsilon}}$$

$$< 6\sqrt{\varepsilon}.$$

Given $v \in \mathbb{C}^{d_i}$, define

$$\Theta_v \colon \{1, \ldots, d_i\} \to \mathbb{C}^\Gamma$$

by

$$\Theta_v(j)(g) = v(\sigma_i(g)^{-1}(j)).$$

Given $v \in \mathbb{C}^{d_i}, \beta \in Y^{d_i}$ define $\Gamma_{v,\beta} \in (\mathbb{C}^\Gamma \times Y)^{d_i}$ by

$$\Gamma_{v,\beta}(j) = (\Theta_v(j), \beta(y)).$$

We now estimate

$$\Delta_{Z,2}(\phi, \Gamma_{w,\psi_Y}).$$

We have

$$\Delta_{Z,2}(\phi, \Gamma_{w,\psi_Y}) \le \varepsilon' + \left(\frac{1}{d_i} \sum_{j=1}^{d_i} \min(1, |w(\sigma_i(e)^{-1}(j)) - \xi(\phi(j)|)^2 \right)^{1/2}$$

$$\le \varepsilon' + \left(u_{d_i}(\{j : \sigma_i(e)(j) \ne j\}) + 2\varepsilon + \sum_{j:\sigma_i(e)(j)=j, |\xi(\phi(j))| \le M} |w(j) - \xi(\phi(j))|^2 \right)^{1/2}$$

$$\le \varepsilon' + \sqrt{2\varepsilon} + u_{d_i}(\{j : \sigma_i(e)(j) \ne j\})^{1/2} + \|w - G \circ \xi \circ \phi_Z\|_2$$

$$\le \varepsilon + (6 + \sqrt{2})\sqrt{\varepsilon} + u_{d_i}(\{j : \sigma_i(e)(j) \ne j\})^{1/2}.$$

Since $\varepsilon < 1$, we see that for all large i we have

$$\Delta_{Z,2}(\phi, \Gamma_{w,\psi_Y}) < 9\sqrt{\varepsilon}.$$

Thus

$$\pi_Z \circ \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i) \subseteq_{9\sqrt{\varepsilon}} \{\Gamma_{w,\psi_Y} : \psi \in D, \xi \in S_\psi\}.$$

Thus

$$N_{18\sqrt{\varepsilon}}(\pi_Z \circ \mathrm{Map}(\Delta_X, F, \delta, L, \sigma_i), \Delta_{Z,2}) \le N_{\varepsilon'/2}(\pi^{d_i}(\mathrm{Map}(\Delta_Z : \Delta_X, F, \delta, L, \sigma_i)), \Delta_{Y,\infty})$$

$$\times M^{8\frac{\eta^2}{(1-\sqrt{\varepsilon})}d_i} \left(\frac{3 + 3\varepsilon}{\varepsilon} \right)^{\frac{8\eta^2}{(1-\sqrt{\varepsilon})}d_i},$$

so

$$h_{(\sigma_i),\mu}(\Delta_Z : \Delta_X, 18\sqrt{\varepsilon}, F, \delta, L) \le h_{(\sigma_i)_i,\mu}(\Delta_Y : \Delta_X; \varepsilon'/2, F, \delta, L, \infty) + \frac{8\eta^2}{(1 - \sqrt{\varepsilon})^2} \log\left(\frac{3 + 3\varepsilon}{\varepsilon} \right)$$

$$+ \frac{8\eta^2}{(1 - \sqrt{\varepsilon})^2} \log M.$$

Since this holds for all sufficiently large F, δ, L we can take the infimum over all F, δ, L to see that

$$h_{(\sigma_i),\mu}(\Delta_Z : \Delta_X, 18\sqrt{\varepsilon}, \sigma_i) \le h_{(\sigma_i)_i,\mu}(\Delta_Y : \Delta_X; \varepsilon'/2, \infty) + \frac{8\eta^2}{(1 - \sqrt{\varepsilon})^2} \log\left(\frac{3 + 3\varepsilon}{\varepsilon} \right) + \frac{8\eta^2}{(1 - \sqrt{\varepsilon})^2} \log M.$$

A fortiori,

$$h_{(\sigma_i),\mu}(\Delta_Z : \Delta_X, 18\sqrt{\varepsilon}, \sigma_i) \le h_{(\sigma_i)_i,\mu}(Y : X, \Gamma) + \frac{8\eta^2}{(1 - \sqrt{\varepsilon})^2} \log\left(\frac{3 + 3\varepsilon}{\varepsilon} \right) + \frac{8\eta^2}{(1 - \sqrt{\varepsilon})^2} \log M.$$

Since M only depends upon ε, and η can be any number less than ε, we can let $\eta \to 0$ to see that

$$h_{(\sigma_i),\mu}(\Delta_Z : \Delta_X, 18\sqrt{\varepsilon}, \sigma_i) \le h_{(\sigma_i)_i,\mu}(Y : X, \Gamma).$$

Taking the supremum over ε completes the proof.

The "in particular" part follows since we may take $X = Z$ to see that

$$h_{(\sigma_i)_i,\zeta}(Z, \Gamma) = h_{(\sigma_i)_i,\zeta}(Z : Z, \Gamma) = h_{(\sigma_i)_i,\zeta}(Y : Z, \Gamma) \le h_{(\sigma_i)_i,\nu}(Y, \Gamma).$$

$\square$

3. Mixing and Strong Ergodicity over the Outer Pinsker Factor

We give the definition of the *Pinsker Factor* and the *Outer Pinsker Factor*.

Definition 3.1. Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i \colon \Gamma \to S_{d_i}$. Let $(X, \mathcal{M}, \mu)$ be a standard probability space and $\Gamma \curvearrowright (X, \mathcal{M}, \mu)$ a measure-preserving action. We define $\Pi_{(\sigma_i)_i, \Gamma, X}$ to be the sigma-algebra generated by all finite observables α so that

$$h_{(\sigma_i)_i, \mu}(\alpha : \mathcal{M}, \Gamma) \leq 0.$$

Let Y be the factor corresponding to $\Pi_{(\sigma_i)_i, \Gamma, X}$, it is easy to see that

$$h_{(\sigma_i)_i, \mu}(Y : X, \Gamma) \leq 0.$$

And that Y is the maximal factor of X with $h_{(\sigma_i)_i, \mu}(Y : X, \Gamma) \leq 0$. We will call Y the *Outer Pinsker Factor* for $\Gamma \curvearrowright (X, \mu)$. Similarly, we can show that there is a largest factor $\Gamma \curvearrowright (Y_0, \nu_0)$ which has entropy zero with respect to $(\sigma_i)_i$. We will call this the *Pinsker factor of* $\Gamma \curvearrowright (X, \mu)$.

We have the following corollary of our first theorem.

Corollary 3.2. *Let Γ be a countable discrete sofic group, and $\sigma_i \colon \Gamma \to S_{d_i}$ a sofic approximation. Let (Y, ν) be the Pinsker factor for $\Gamma \curvearrowright (X, \mu)$. Then $L^2(X) \ominus L^2(Y)$ embeds into $L^2(Y, \nu, \ell^2(\Gamma))^{\oplus \infty}$ as a representation of the $*$-algebra $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$. Similarly, if (Y_0, ν_0) is the Outer Pinsker factor, then $L^2(X) \ominus L^2(Y_0)$ embeds into $L^2(Y_0, \ell^2(\Gamma))$ as a representation of $L^\infty(Y_0) \rtimes_{\mathrm{alg}} \Gamma$.*

Proof. We do the proof only for the Pinsker factor, as the proof for the Outer Pinsker factor is the same. As in Proposition 4.3 of [10] we may write

$$L^2(X) \ominus L^2(Y) = \mathcal{H}_a \oplus \mathcal{H}_s$$

where $\mathcal{H}_a, \mathcal{H}_s$ are subrepresentation of the $*$-algebra $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$, with $\mathcal{H}_a$ embedding into $L^2(Y, \nu, \ell^2(\Gamma))^{\oplus \infty}$ and $\mathcal{H}_s$ is singular with respect to $L^2(Y, \nu, \ell^2(\Gamma))$ as representations of $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$. Let $\xi \in \mathcal{H}_s$ and let $\mathcal{A}$ be the smallest σ-algebra of measurable subsets of X generated by Y and ξ. Let (Z, η) be the factor of (X, μ) corresponding to $\mathcal{A}$. Thus (Z, η) is an intermediate factor of $(X, \mu) \to (Y, \nu)$. We know $\mathcal{K} = \overline{\mathrm{Span}\{g\xi : g \in \Gamma\}}^{\|\cdot\|_2}$ generates Z over Y. Tautologically, $\mathcal{K}$ is singular with respect to $L^2(Y, \nu, \ell^2(\Gamma))$ as a representation of $L^\infty(Y, \nu) \rtimes_{\mathrm{alg}} \Gamma$. Thus by the preceding Theorem,

$$h_{(\sigma_i)_i, \nu}(Z : X) = h_{(\sigma_i)_i, \nu}(Y : X) \leq 0.$$

By maximality of the Outer Pinsker factor, we see that $Z = Y$. This implies that $\xi \in L^2(Y)$ and thus that $\mathcal{H}_s = 0$.

$\square$

We use the preceding Corollary to show mixing and ergodicity properties of the factor $\pi \colon (X, \mu) \to (Y, \nu)$. We first recall some definitions. We use $\mathbb{E}_Y(f)$ for the conditional expectation onto Y of $f \in L^1(X, \mu)$. We will typically view $L^2(Y)$ inside of $L^2(X)$ by $f \mapsto f \circ \pi$.

Definition 3.3. Let Γ be a countable discrete group. Let (X, μ) be a standard probability space and $\Gamma \curvearrowright (X, \mu)$ a measure-preserving action. Let $\Gamma \curvearrowright (Y, \nu)$ be a factor of (X, μ) and $\alpha \colon \Gamma \to \mathcal{U}(L^2(X, \mu))$ be the Koopman representation. We say that the extension $\Gamma \curvearrowright (X, \mu)$ is *mixing relative to* $\Gamma \curvearrowright (Y, \nu)$ if for all $f, h \in L^\infty(X, \mu)$ with $\mathbb{E}_Y(f) = 0 = \mathbb{E}_Y(h)$ we have

$$\lim_{g \to \infty} \|\mathbb{E}_Y(\alpha(g)(f)h)\|_{L^2(Y)} = 0.$$

We also that the extension $(X, \mu) \to (Y, \nu)$ is mixing. We say that the extension (X, μ) *has spectral gap over* (Y, ν) if for every sequence $\xi_n \in L^2(X)$ such that

$$\lim_{n \to \infty} \|\alpha(g)(\xi_n) - \xi_n\|_{L^2(X)} = 0 \text{ for all } g \in \Gamma,$$

we have

$$\lim_{n \to \infty} \|\xi_n - \mathbb{E}_Y(\xi_n)\|_{L^2(X)} = 0.$$

We remark that is easy to see that (X, μ) has spectral gap over (Y, ν) if and only if $L^2(X) \ominus L^2(Y)$ has spectral gap as a representation of Γ.

For the proof we introduce some notation. Let (Y, ν) be as in the proceeding definition. For $f, h \in L^\infty(Y, \nu, \ell^2(\Gamma))$ we let $\langle f, h \rangle_Y \in L^\infty(Y, \nu)$ be defined by

$$\langle f, h \rangle_Y(y) = \sum_{g \in \Gamma} f(y)(g)\overline{h(y)(g)}.$$

For $f \in L^\infty(Y, \nu), \xi \in \ell^2(\Gamma)$ we let $f \otimes \xi \in L^\infty(X, \mu, \ell^2(\Gamma))$ be defined by

$$(f \otimes \xi)(y)(g) = f(y)\xi(g).$$

We let $\lambda_Y \colon \Gamma \to B(L^\infty(Y, \nu, \ell^2(\Gamma)))$ be defined by

$$(\lambda_Y(g)f)(y)(h) = f(g^{-1}y)(g^{-1}h).$$

For $\xi \in L^\infty(Y, \nu, \ell^2(\Gamma))$ we use $\|\xi\|_\infty$ for the essential supremum in y of

$$\left(\sum_{g \in \Gamma} |\xi(y)(g)|^2 \right)^{1/2}$$

and we use $\|\xi\|_2$ for

$$\left(\int_Y \sum_{g \in \Gamma} |\xi(y)(g)|^2 \, d\nu(y) \right)^{1/2}.$$

Theorem 3.4. *Let Γ be a countable discrete sofic group with sofic approximation $\sigma_i \colon \Gamma \to S_{d_i}$, and let $\Gamma \curvearrowright (X, \mu)$ be a probability measure-preserving action. Let $(Y, \nu), (Y_0, \nu_0)$ be the Pinsker and Outer Pinsker factors, respectively, of $\Gamma \curvearrowright (X, \mu)$.*
(i): The extensions

$$\Gamma \curvearrowright (X, \mu) \to \Gamma \curvearrowright (Y, \nu), \Gamma \curvearrowright (X, \mu) \to \Gamma \curvearrowright (Y_0, \nu_0)$$

are mixing.
(ii): If Λ is any nonamenable subgroup of Γ, then

$$\Lambda \curvearrowright (X, \mu) \to \Lambda \curvearrowright (Y, \nu)$$

has spectral gap. In particular, the extension

$$\Lambda \curvearrowright (X, \mu) \to \Lambda \curvearrowright (Y_0, \nu_0)$$

has spectral gap.

Proof. Throughout, we shall let $\alpha_X \colon \Gamma \to \mathcal{U}(L^2(X))$, $\alpha_Y \colon \Gamma \to \mathcal{U}(L^2(Y))$ be the Koopman representations.

(i): We shall use direct integral theory. See [22] IV.8 for the appropriate background. By Corollary 3.2, there is a $L^\infty(Y, \nu) \rtimes_{\text{alg}} \Gamma$-equivariant isometric linear map

$$U \colon L^2(X) \ominus L^2(Y) \to L^2(Y, \mu, \ell^2(\Gamma))^{\oplus \infty}.$$

Let $\pi \colon (X, \mu) \to (Y, \nu)$ be the factor map. We may assume that X, Y are standard Borel spaces and that π is Borel. Let

$$(X, \mu) = \int_Y (X_y, \mu_y)\, d\nu(y)$$

be the disintegration. This means that $X_y = \pi^{-1}(\{y\})$, that μ_y is a Borel probability measure supported on X_y, that $y \mapsto \mu_y(E)$ is measurable for all $E \subseteq X$ measurable and that

$$\int_Y \mu_y(E)\, d\nu(y) = \mu(E)$$

for all $E \subseteq X$ measurable. This allows us to regard

$$L^2(X) \ominus L^2(Y) = \int_Y^\oplus L^2(X_y, \mu_y) \ominus \mathbb{C}1\, d\nu(y),$$

$$L^2(X, \ell^2(\Gamma)) = \int_Y^\oplus \ell^2(\Gamma)\, d\nu(y).$$

We may regard $\alpha_X(g)$ as a map $L^2(X_y, \mu_y) \to L^2(X_{gy}, \mu_{gy})$. Since U is $L^\infty(Y)$-equivariant, we may argue as in [22] Theorem IX.7.10 to see that there is a measurable field $U_y \in B(L^2(X_y) \ominus \mathbb{C}1, \ell^2(\Gamma))$ of isometric linear maps so that

$$U = \int_Y^\oplus U_y\, d\nu(y).$$

By Γ-equivariance we have that $U_{gy}(\alpha_X(g)(\xi)) = \lambda(g)U_y(\xi)$, for almost every $y \in Y$ and all $\xi \in L^2(X_y) \ominus \mathbb{C}1$. It is easy to see that for $f, h \in L^2(X) \ominus L^2(Y)$

$$\langle \lambda_Y(g)U(f), U(h) \rangle_Y = \mathbb{E}_Y(\alpha_X(g)(f)h^*).$$

Note that for every $k \in L^\infty(X, \mu)$ and almost every $y \in Y$ we have

$$\|k\|_{L^2(\mu_y)} \le \|k\|_{L^\infty(\mu_y)} \le \|k\|_{L^\infty}$$

(for the last inequality see e.g. [22] page 259 formula (6)). Thus $\|U_y(k)\|_2 \le \|k\|_\infty$ for almost every $y \in Y$. Now fix $f, h \in L^\infty(X, \mu)$ and $\mathbb{E}_Y(f) = 0 = \mathbb{E}_Y(h)$. Set $\xi = U(f), \eta = U(h)$. The above discussion shows that $\xi, \eta \in L^\infty(Y, \nu, \ell^2(\Gamma))$ with $\|\xi\|_\infty \le \|f\|_\infty, \|\eta\|_\infty \le \|h\|_\infty$. It thus suffices to show that

$$\lim_{g \to \infty} \|\langle \lambda_Y(g)\xi, \eta \rangle_Y\|_2 = 0$$

for $\xi, \eta \in L^\infty(X, \mu, \ell^2(\Gamma))$. We first do this when

$$\xi = \sum_{s \in \Gamma} \xi_s \otimes \delta_s$$

$$\eta = \sum_{s \in \Gamma} \eta_s \otimes \delta_s$$

with all but finitely many terms in each sum equal to 0. It is then easy to see that

$$\langle \lambda_Y(g)\xi, \eta \rangle = 0$$

for all g outside a finite subset of Γ.

Now fix $\xi, \eta \in L^\infty(X, \mu, \ell^2(\Gamma))$. For $s \in \Gamma$, define $\eta_s, \xi_s \in L^\infty(X, \mu)$ by $\xi_s(y) = \xi(y)(s)$, $\eta_s(y) = \eta(y)(s)$. For a finite $F \subseteq \Gamma$, set

$$\xi_F = \sum_{s \in F} \xi_s \otimes \delta_s, \eta_F = \sum_{s \in F} \eta_s \otimes \delta_s.$$

Then $\|\xi_F\|_\infty \leq \|\xi\|_\infty, \|\eta_F\| \leq \|\eta\|_\infty$ and by the dominated convergence theorem

$$\lim_F \|\xi - \xi_F\|_2 = 0, \lim_F \|\eta - \eta_F\|_2 = 0,$$

where the finite subsets of Γ are ordered by containment. Now fix $\varepsilon > 0$ and choose a finite $F \subseteq \Gamma$ so that

$$\|\xi - \xi_F\|_2 < \varepsilon, \|\eta - \eta_F\|_2 < \varepsilon.$$

By the first case, we may choose a finite $K \subseteq \Gamma$ so that $\langle \lambda_Y(g)\xi_F, \eta_F \rangle_Y = 0$ for all $g \in \Gamma \setminus K$. Then for $g \in \Gamma \setminus K$ we have

$$\begin{aligned}
\|\langle \lambda_Y(g)\xi, \eta \rangle\|_2 &\leq \varepsilon \|\eta\|_\infty + \|\langle \lambda_Y(g)\xi_F, \eta \rangle_Y\|_2 \\
&\leq \varepsilon(\|\eta\|_\infty + \|\xi\|_\infty) + \|\langle \lambda_Y(g)\xi_F, \eta_F \rangle_Y\|_2 \\
&= \varepsilon(\|\eta\|_\infty + \|\xi\|_\infty).
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, this proves that

$$\lim_{g \to \infty} \|\langle \lambda_Y(g)\xi, \eta \rangle_Y\|_2 = 0$$

as desired. This completes the proof that the extension

$$\Gamma \curvearrowright (X, \mu) \to \Gamma \curvearrowright (Y, \nu)$$

is mixing. The proof that $\Gamma \curvearrowright (X, \mu) \to \Gamma \curvearrowright (Y_0, \nu_0)$ is mixing is similar.

(ii): We consider the restriction of the action of $L^\infty(Y) \rtimes_{\mathrm{alg}} \Gamma$ to Γ. By Corollary 3.2, this unitary representation of Γ embeds into $L^2(Y, \ell^2(\Gamma))^{\oplus \infty}$. The unitary representation $L^2(Y, \ell^2(\Gamma))^{\oplus \infty}$ of Γ is canonically isomorphic to $(\alpha_Y \otimes \lambda_\Gamma)^{\oplus \infty}$. By Fell's absorption principle, we know that $\alpha_Y \otimes \lambda_\Gamma \leq \lambda_\Gamma^{\oplus \infty}$. So the unitary representation $L^2(X) \ominus L^2(Y)$ of Γ embeds into $\lambda_\Gamma^{\oplus \infty}$. If we regard $L^2(X) \ominus L^2(Y)$ as a representation of Λ then, by restriction, this representation embeds into $\lambda_\Lambda^{\oplus \infty}$. By nonamenability of Λ, we have that $\lambda_\Lambda^{\oplus \infty}$ has spectral gap. Thus the extension of Λ-actions

$$\Lambda \curvearrowright (X, \mu) \to \Lambda \curvearrowright (Y, \nu)$$

has spectral gap. The "in particular" part follows because

$$\|\xi - \mathbb{E}_{Y_0}(\xi)\|_2 \leq \|\xi - \mathbb{E}_Y(\xi)\|_2$$

for all $\xi \in L^2(X, \mu)$.

$\square$

Now, suppose we are given a standard probability space $(X, \mathcal{M}, \mu)$ and a countable discrete group Γ with $\Gamma \curvearrowright (X, \mathcal{M}, \mu)$ by measure-preserving transformations. Recall that the *Furstenberg Tower* is a tower of complete Γ-invariant sigma-subalgebras $\mathcal{M}_\alpha$ of $\mathcal{M}$ indexed by ordinals α less than or equal to some countable ordinal λ defined by:

(a): $\mathcal{M}_0$ is the sigma-algebra of sets which are either null or conull,

(b): if α is a successor ordinal and

$$(X_\alpha, \mu_\alpha) \to (X_{\alpha-1}, \mu_{\alpha-1})$$

is the factor map corresponding to $\mathcal{M}_{\alpha-1} \subseteq \mathcal{M}_\alpha$, then $\mathcal{M}_\alpha$ is the largest sigma-subalgebra of $\mathcal{M}$ with the property that the extension

$$(X_\alpha, \mu_\alpha) \to (X_{\alpha-1}, \mu_{\alpha-1})$$

is compact,

(c): if α is a limit-ordinal, then $\mathcal{M}_\alpha$ is the sigma-algebra generated by

$$\bigcup_{\alpha' < \alpha} M_{\alpha'},$$

(d): if (X_λ, μ_λ) is the factor of (X, μ) corresponding to the sigma-subalgebra $\mathcal{M}_\lambda$, then

$$(X, \mu) \to (X_\lambda, \mu_\lambda)$$

is a weakly mixing extension.

Motivated by Theorem 2.13, we make the following definition.

Definition 3.5. Let $(X, \mathcal{M}, \mu)$ be a standard probability space and Γ a countable discrete group with $\Gamma \curvearrowright (X, \mu)$ by automorphisms. Define, for every countable ordinal α, a family of complete Γ-invariant sigma-subalgebras of $\mathcal{M}$ as follows:

(i): $\mathcal{M}_0$ is the algebra of all null or conull sets

(ii): if α is successor ordinal, and $(X_{\alpha-1}, \mu_{\alpha-1})$ is the factor of (X, μ) corresponding to $\mathcal{M}_{\alpha-1}$, then we define $\mathcal{M}_\alpha$ to be the sigma-algebra generated by $\xi^{-1}(A)$, where A is a Borel subset of $\mathbb{C}$, and $\xi \in L^2(X, \mu)$ has the property that

$$\overline{\{\rho(f)\xi : f \in L^\infty(X_{\alpha-1}, \mu_{\alpha-1}) \rtimes_{\text{alg}} \Gamma\}}^{\|\cdot\|_2}$$

is singular with respect to $\lambda_{X_{\alpha-1}}$ as a representation of $L^\infty(X_{\alpha-1}, \mu_{\alpha-1}) \rtimes_{\text{alg}} \Gamma$.

(iii): if α is a limit ordinal, we let $\mathcal{M}_\alpha$ be the sigma-algebra generated by

$$\bigcup_{\alpha' < \alpha} \mathcal{M}_{\alpha'}.$$

We call $(\mathcal{M}_\alpha)_\alpha$ the *spectral tower*.

Note that if $(\mathcal{N}_\alpha), (\mathcal{M}_\alpha)$ are the Furstenberg-Zimmer and spectral towers, respectively, then $\mathcal{N}_\alpha \subseteq \mathcal{M}_\alpha$. Additionally, by iterated applications of Theorem 2.13, we find that if (X_α, μ_α) is the factor of (X, μ) corresponding to $\mathcal{M}_\alpha$, then

$$h_{(\sigma_i)_i, \mu_\alpha}(X_\alpha, \Gamma) \le 0.$$

This is another way to see that distal measure-theoretic actions have nonpositive sofic entropy. Additionally, if β is the first ordinal for which $\mathcal{M}_\beta = \mathcal{M}_{\beta+1}$, and (X_β, μ_β) is the factor corresponding to $\mathcal{M}_\beta$, then

$$L^2(X) \ominus L^2(X_\beta)$$

embeds into $L^2(X_\beta, \ell^2(\Gamma))^{\oplus\infty}$ as a representation of $L^\infty(X_\beta) \rtimes_{\text{alg}} \Gamma$. This gives another perspective of the proof of Corollary 3.2.

It appears that the analogues of our results are not known for Rokhlin entropy. Thus we ask the following question.

Question 1. The spectral tower makes sense for actions of arbitrary countable discrete groups. Is it true that if $(\mathcal{M}_\alpha)_\alpha$ is the spectral tower of a probability measure-preserving action $\Gamma \curvearrowright (X, \mathcal{M}, \mu)$ and if $\mathcal{M}_\alpha = \mathcal{M}$ for some α, then the Rokhlin entropy of $\Gamma \curvearrowright (X, \mu)$ is zero? Less ambitiously, is the Rokhlin entropy finite?

APPENDIX A. ENTROPY IN THE PRESENCE FOR ACTIONS OF AMENABLE GROUPS

The goal of this section is to prove that when Γ is amenable group, then for any extension

$$\Gamma \curvearrowright (X, \mu) \to \Gamma \curvearrowright (Y, \nu)$$

the entropy of Y in the presence of X is just the entropy of $\Gamma \curvearrowright (Y, \nu)$. Though technical, our argument is fairly intuitive: we wish to show that every microstate for $\Gamma \curvearrowright (Y, \nu)$ "lifts" to one for $\Gamma \curvearrowright (X, \mu)$.

The following lemma is a formal version of this intuitive statement. It is essentially a direct translation of a result of Popa in [20] (when the action is free the result is due to Paunescu in [17], but we will need the general version of Popa). We will use ultrafilters in our proof. If ω is a free ultrafilter on $\mathbb{N}$, then we say that $A \subseteq \mathbb{N}$ is ω-large if $A \in \omega$.

We also need the notion of a tracial ultraproduct of von Neumann algebras. Fix a free ultrafilter ω on $\mathbb{N}$. Suppose that $(M_n, \tau_n)_{n=1}^\infty$ is a sequence of tracial von Neumann algebras (i.e. M_n are von Neumann algebras and τ_n is a faithful, normal, tracial state on M_n). Let

$$M = \prod_{n \to \omega} (M_n, \tau_n) = \frac{\{(x_n)_{n=1}^\infty : x_n \in M_n, \sup_n \|x_n\| < \infty\}}{\{(x_n)_{n=1}^\infty : \sup_n \|x_n\| < \infty, \lim_{n \to \omega} \tau_n(x_n^* x_n) = 0\}}.$$

If $(x_n)_{n=1}^\infty$ is a sequence with $x_n \in M_n$ and $\sup_n \|x_n\| < \infty$, we use $(x_n)_{n \to \omega}$ for the image of $(x_n)_{n=1}^\infty$ in M under the quotient map. There is a well-defined trace

$$\tau_\omega \colon \prod_{n \to \omega} (M_n, \tau_n) \to \mathbb{C}$$

given by

$$\tau_\omega((x_n)_{n \to \omega}) = \lim_{n \to \omega} \tau_n(x_n).$$

We write

$$(M, \tau_\omega) = \prod_{n \to \omega} (M_n, \tau)$$

and say that (M, τ_ω) is the tracial ultraproduct of $(M_n, \tau_n)_{n=1}^\infty$. Note that M is a $*$-algebra with the operations defined coordinate wise. Let $L^2 \left(\prod_{n \to \omega} (M_n, \tau_n) \right)$ be the GNS completion of M with respect to τ_ω and let

$$\lambda \colon M \to B \left(L^2 \left(\prod_{n \to \omega} (M_n, \tau_n) \right) \right)$$

be given by the GNS representation. It turns out that $\lambda(M)$ is a von Neumann algebra acting on $B \left(L^2 \left(\prod_{n \to \omega} (M_n, \tau_n) \right) \right)$ (see [4] Lemma A.9). Thus we may regard $\prod_{n \to \omega} (M_n, \tau_n)$ as a tracial von Neumann algebra.

Lemma A.1. *Let Γ be a countable discrete amenable group and $\sigma_i \colon \Gamma \to S_{d_i}$ a sofic approximation. Let X, Y be compact metrizable spaces with $\Gamma \curvearrowright X, \Gamma \curvearrowright Y$ by homeomorphisms and suppose that there is a factor map $\pi \colon X \to Y$. Let Δ_X, Δ_Y be dynamically generating pseudometrics on X, Y respectively. Let $\mu \in \mathrm{Prob}_\Gamma(X), \nu \in \mathrm{Prob}_\Gamma(Y)$ be such that $\pi_* \mu = \nu$. Then for every finite $F \subseteq \Gamma, L \subseteq C(X)$ and $\delta, \varepsilon > 0$, there exists finite $F' \subseteq \Gamma, L' \subseteq C(Y)$ and $\delta' > 0$ so that for all large i*

$$\mathrm{Map}_\nu(\Delta_Y, F', \delta', L', \sigma_i) \subseteq_{\varepsilon, \Delta_{Y,2}} \pi \circ \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_i).$$

Proof. Write

$$\Gamma = \bigcup_{n=1}^{\infty} F'_n,$$

$$C(X) = \overline{\bigcup_{n=1}^{\infty} L'_n}$$

where $F'_n \subseteq \Gamma, L'_n \subseteq C(X)$ are increasing sequences of finite sets. Let $\delta'_n > 0$ be a decreasing sequence converging to zero. Suppose that the lemma is false, then we may find finite $F \subseteq \Gamma, L \subseteq C(X)$ and $\delta, \varepsilon > 0$ and a strictly increasing sequence of integers i_n so that for all n there is a $\phi_n \in \mathrm{Map}_\nu(\Delta_X, F'_n, \delta'_n, L'_n, \sigma_{i_n})$ so that

$$\Delta_{Y,2}(\phi_n, \pi \circ \psi) \geq \varepsilon \text{ for all } \psi \in \mathrm{Map}_\mu(\Delta_X, F, \delta, L, \sigma_{i_n}).$$

Since Γ is amenable, we know by [14] that $h_{(\sigma_i)_i, \mu}(X, \Gamma) \geq 0$ and thus we can find $\psi_n \in X^{d_{i_n}}$ so that

$$\Delta_{X,2}(g\psi_n, \psi_n \circ \sigma_{i_n}(g)) \to_{n \to \infty} 0,$$

$$(\psi_n)_*(u_{d_{i_n}}) \to \mu \text{ weak}^*.$$

Fix a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Let

$$(M, \mathrm{tr}_\omega) = \prod_{n \to \omega} (M_{d_{i_n}}(\mathbb{C}), \mathrm{tr}).$$

Define

$$\Phi \colon C(Y) \rtimes_{\mathrm{alg}} \Gamma \to M,$$
$$\Psi \colon C(X) \rtimes_{\mathrm{alg}} \Gamma \to M$$

by

$$\Phi(f u_g) = (m_{f \circ \phi_n} \sigma_{i_n}(g))_{n \to \omega}, \text{ for } f \in C(Y), g \in \Gamma,$$
$$\Psi(f u_g) = (m_{f \circ \psi_n} \sigma_{i_n}(g))_{n \to \omega}, \text{ for } f \in C(X), g \in \Gamma,$$

and extended by linearity. It is easy to see that Φ, Ψ are $*$-homomorphisms and that $\mathrm{tr}_\omega \circ \Phi = \tau_\nu, \mathrm{tr}_\omega \circ \Psi = \tau_\mu$. By uniqueness of GNS representations, we see that Φ, Ψ extend to normal $*$-homomorphisms

$$\overline{\Phi} \colon L^\infty(Y, \nu) \rtimes \Gamma \to M$$

$$\overline{\Psi} \colon L^\infty(X, \mu) \rtimes \Gamma \to M$$

so that $\mathrm{tr}_\omega \circ \overline{\Phi} = \tau_\nu, \mathrm{tr}_\omega \circ \overline{\Psi} = \tau_\mu$. We have an injective, trace-preserving, normal inclusion

$$\iota \colon L^\infty(Y, \nu) \rtimes \Gamma \to L^\infty(X, \mu) \rtimes \Gamma$$

defined densely by

$$\iota(f u_g) = f \circ \pi u_g.$$

By Corollary 5.2 of [20], we may find a $p \in \prod_{n \to \omega} S_{d_{i_n}}$ so that

$$\overline{\Phi}(x) = p\overline{\Psi}(\iota(x))p^{-1} \text{ for all } x \in L^\infty(Y, \nu) \rtimes \Gamma$$

(alternatively one can extend Ψ to an embedding of $L^\infty(\{0,1\}^\Gamma \times X) \rtimes \Gamma$ using the arguments of Theorem 8.1 of [2] and then use Proposition 1.20 of [17]). Write $p = (p_n)_{n \to \omega}$ with $p_n \in S_{d_{i_n}}$. The above equality then translates into

$$(m_{f \circ \phi_n})_{n \to \omega} = (m_{f \circ \pi \circ \psi_n \circ p_n})_{n \to \omega},$$
$$(p_n \sigma_{i_n}(g))_{n \to \omega} = (\sigma_{i_n}(g) p_n)_{n \to \omega}.$$

From the above it is easy to see that

(2) $$\lim_{n \to \omega} u_{d_{i_n}}(\{j : p_n(\sigma_{i_n}(g)(j)) = \sigma_{i_n}(p_n(j))\}) = 1.$$

We proceed to get a contradiction by establishing a few preliminary claims. We first claim that

$$\lim_{n \to \omega} \Delta_{Y,2}(\pi \circ \psi_n \circ p_n, \phi_n) = 0.$$

To see this, let $\eta > 0$ and let M be the diameter of Δ_Y. Since Y may be identified with the Gelfand spectrum of $C(Y)$, we see that we may find a $\kappa > 0$ and $f_1, \ldots, f_n \in C(Y)$ so that if $y_1, y_2 \in Y$ have

$$\max_{1 \le l \le n} |f_l(y_1) - f_l(y_2)| < \kappa,$$

then

$$\Delta_Y(y_1, y_2) < \eta.$$

Since for all $1 \le l \le n$

$$\lim_{n \to \omega} \|f_l \circ \phi_n - f_l \circ \pi \circ \psi_n \circ p_n\|_2 = 0,$$

we see that for all $1 \le l \le n$

$$\lim_{n \to \omega} u_{d_{i_n}}(\{j : |f_l(\phi_n(j)) - f_l(\pi \circ \psi_n \circ p_n(j))| < \kappa\}) = 1.$$

Thus, there is an ω-large set of n so that

$$u_{d_{i_n}}\left(\bigcap_{l=1}^{n} \{j : |f_l(\phi_n(j)) - f_l(\pi \circ \psi_n \circ p_n(j))| < \kappa\} \right) \ge 1 - \eta.$$

By our choice of $f_1, \ldots, f_n, \kappa$, we thus see that

$$\Delta_{Y,2}(\pi \circ \psi_n \circ p_n, \phi_n)^2 < \eta^2 + M^2 \eta$$

for an ω-large set of n. Thus

$$\lim_{n \to \omega} \Delta_{Y,2}(\pi \circ \psi_n \circ p_n, \phi_n) \le \eta^2 + M^2 \eta$$

and letting $\eta \to 0$ proves the first claim.

The second claim is that

$$\psi_n \circ p_n \in \mathrm{Map}_\mu(\Delta_{X,2}, F, \delta, L, \sigma_i).$$

for an ω-large set of n. Suppose we grant this claim. The combination of the first and second claims imply that there is an ω-large set of n so that

$$\psi_n \circ p_n \in \mathrm{Map}_\mu(\Delta_{X,2}, F, \delta, L, \sigma_i)$$

and

$$\Delta_{Y,2}(\phi_n, \pi \circ \psi_n \circ p_n) < \varepsilon,$$

which contradicts our choice of ϕ_n. It thus remains to prove the second claim. By standard properties of ultrafilters, to prove the second claim it is enough to show that

(3) $$\lim_{n \to \omega} \Delta_{X,2}(\psi_n \circ p_n \circ \sigma_{i_n}(g), g\psi_n \circ p_n) = 0 \text{ for all } g \in \Gamma,$$

(4) $$\lim_{n \to \omega} (\psi_n \circ p_n)_*(u_{d_{i_n}}) = \mu \text{ weak}^*.$$

Equation (4) is a trivial consequence of the fact that $(\psi_n)_*(u_{d_{i_n}}) \to \mu$ in the weak* topology. Equation (3) is straightforward to argue from (2) and the fact that

$$\Delta_{X,2}(\psi_n \circ \sigma_{i_n}(g), g\psi_n) \to_{n \to \infty} 0 \text{ for all } g \in \Gamma.$$

This completes the proof.

$\square$

We can now prove that entropy in the presence agrees with entropy of the factor in the amenable case.

Theorem A.2. *Let Γ be a countable, discrete, amenable group and $\sigma_i\colon \Gamma \to S_{d_i}$ a sofic approximation. Let $(X,\mu),(Y,\nu)$ be standard probability spaces with $\Gamma \curvearrowright (X,\mu), \Gamma \curvearrowright (Y,\nu)$ by measure-preserving transformations and suppose that $\pi\colon X \to Y$ is a factor map. Then*

$$h_{(\sigma_i)_i,\mu}(Y : X, \Gamma) = h_{\nu}(Y, \Gamma).$$

Proof. By [3],[14] it is enough to show that

$$h_{(\sigma_i)_i,\mu}(Y : X, \Gamma) = h_{(\sigma_i)_i,\nu}(Y, \Gamma).$$

It is clear that

$$h_{(\sigma_i)_i,\mu}(Y : X, \Gamma) \leq h_{(\sigma_i)_i,\nu}(Y, \Gamma),$$

so we focus on proving the opposite inequality.

We may assume that X, Y are compact, metrizable spaces, that the action of Γ is by homeomorphisms, that $\mu \in \mathrm{Prob}_\Gamma(X), \nu \in \mathrm{Prob}_\Gamma(Y)$ and that π is continuous. Fix dynamically generating pseudometrics Δ_Y, Δ_X on Y, X. Let $\delta, \varepsilon > 0$ and finite $F \subseteq \Gamma, L \subseteq C(X)$ be given. Let δ', F', L' be as in the preceding lemma. We then have that

$$h_{(\sigma_i)_i,\nu}(\Delta_Y, F', \delta', L', 2\varepsilon) \leq h_{(\sigma_i)_i,\mu}(\Delta_Y : \Delta_X, F, \delta, L, \varepsilon).$$

A fortiori,

$$h_{(\sigma_i)_i,\nu}(\Delta_Y, 2\varepsilon) \leq h_{(\sigma_i)_i,\mu}(\Delta_Y : \Delta_X, F, \delta, L, \varepsilon)$$

and taking the infimum over all F, δ, L we have

$$h_{(\sigma_i)_i,\nu}(\Delta_Y, 2\varepsilon) \leq h_{(\sigma_i)_i,\mu}(\Delta_Y : \Delta_X, \varepsilon).$$

Letting $\varepsilon \to 0$ completes the proof.

$\square$

References

[1] A. Alpeev. On Pinsker factors for Rokhlin entropy. In *Representation theory, dynamical systems, combinatorial methods. Part XXIV*, volume 432 of *Zap. Nauchn. Sem. POMI*, pages 30–35, St. Petersburg, 2015. POMI.

[2] L. Bowen. Measure conjugacy invariants for actions of countable sofic groups. *J. Amer. Math. Soc*, 23:217–245, 2010.

[3] L. Bowen. Sofic entropy and amenable groups. *Ergodic Theory Dynam. Systems*, 32, 2011.

[4] N. Brown and N. Ozawa. C^*-Algebras and Finite-Dimensional Approximations. American Mathematical Society, 1994.

[5] P. Burton. Completely positive entropy actions of sofic groups with $\mathbb{Z}$ in their center. *arXiv:1502.01861*.

[6] A. Connes. Classification of injective factors cases. *Ann. of Math.*, 104(1):73–115, 1976.

[7] A. Connes, J. Feldman, and B. Weiss. An amenable equivalence relations is generated by a single transformation. *Ergodic Th. Dyan. Sys*, 10(4):431–450, 1983.

[8] K. Dykema, D. Kerr, and M. Pichot. Sofic dimension for discrete measurable groupoids. *Trans. Amer. Math. Soc*, 366(2):707–748, 2013.

[9] G. Elek and E. Szabo. Sofic representations of amenable groups. *Proceedings of the AMS*, 139(2011):4285–4291.

[10] B. Hayes. Polish models and sofic entropy. *J. Inst. Math. Jussieu to appear*.

[11] B. Hayes. Relative entropy and the Pinsker product formula for sofic groups. *arXiv:1605.01747*.

[12] D. Kerr. Sofic measure entropy via finite partitions. *Groups Geom. Dyn*, 7(617-632), 2013.

[13] D. Kerr. Bernoulli actions of sofic groups have completely positive entropy. *Israel J. Math*, 202(1):461–474, 2014.

[14] D. Kerr and H.Li. Soficity, amenability, and dynamical entropy. *Amer. J. Math*, 135(3):721–761, 2013.

[15] J. Kieffer. A generalized Shannon-McMillan theorem for the action of an amenable group on a probability space. *Ann.Prob*, 3(6):1031–1037, 1975.

[16] D. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse. Math.*, 48:1–141, 1987.

[17] L. Paunescu. On sofic actions and equivalence relations. *Journal of Functional Analysis*, 261(9):2461–2485, November 2011.

[18] S. Popa. Strong rigidity of II$_1$ factors arising from malleable actions of w-rigid groups. II. *Invent. Math.*, 101:593–629, 1990.

[19] S. Popa. On the superrigidity of malleable actions with spectral gap. *J. Amer. Math. Soc.*, 21(4):981–1000, 2008.

[20] S. Popa. Independence properties in sublagebras of ultraproduct II$_1$ factors. *Journal of Functional Analysis*, 266(9):5818–5846, 2014.

[21] B. Seward. Krieger's finite generator theorem for ergodic actions of countable groups I. *arXiv:1405.3604*.

[22] M. Takesaki. *The Theory of Operator Algebras I*. Springer-Verlag, Berlin-Heidelberg-New York, 2002.

[23] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

Stevenson Center, Nashville, TN 37240

E-mail address: benjamin.r.hayes@vanderbilt.edu