CHAPTER 7

AREA MINIMIZING CURRENTS

This chapter provides an introduction to the theory of area minimizing currents. In the first section (§33) of the chapter we derive some basic preliminary properties, and in particular we discuss the fact that the integer multiplicity varifold corresponding to a minimizing current is stable (and indeed minimizing in a certain sense). In §34 there are some existence and compactness results, including the important theorem that if $\{T_j\}$ is a sequence of minimizing currents in U with $\sup_{j\geq 1}(\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)) < \infty$ $\forall\, W \subset\subset U$, and if $T_j \rightharpoonup T \in \mathcal{D}_n(U)$, then T is also minimizing in U and the corresponding varifolds converge in the measure theoretic sense of §15. This enables us to discuss tangent cones and densities in §35, and in particular make some regularity statements for minimizing currents in §36. Finally, in §37 we develop the standard codimension 1 regularity theory, due originally to De Giorgi [DG], Fleming [FW], Almgren [A4], J. Simons [SJ] and Federer [FH2].

§33. BASIC CONCEPTS

Suppose A is any subset of $\mathbb{R}^{n+k}$, $A \subset U$, U open in $\mathbb{R}^{n+k}$, and $T \in \mathcal{D}_n(U)$ an integer multiplicity current.

33.1 DEFINITION We say that T is minimizing in A if

$$\underline{\underline{M}}_W(T) \leq \underline{\underline{M}}_W(S)$$

whenever $W \subset\subset U$, $\partial S = \partial T$ (in U) and $\mathrm{spt}(S-T)$ is a compact subset of $A \cap W$.

There are two especially important cases of this definition:

(1) when $A = U$

(2) when $A = N \cap U$, N an $(n+k_1)$-dimensional embedded submanifold of $\mathbb{R}^{n+k}$ (in the sense of §7).

As a matter of fact, these are the only cases we are interested in here.

Corresponding to the current $T = \underline{\underline{\tau}}(M,\theta,\xi) \in \mathcal{D}_n(U)$ we have the integer multiplicity varifold $V = \underline{\underline{v}}(M,\theta)$. As one would expect, V is stationary in U if T is minimizing in U and $\partial T = 0$; indeed we show more:

33.2 LEMMA *Suppose* T *is minimizing in* $N \cap U$, *where* N *is an* $(n+k_1)$*-dimensional* C^2 *submanifold of* $\mathbb{R}^{n+k}$ $(k_1 \le k)$ *and suppose* $\partial T = 0$ *in* U. *Then* V *is stationary in* $N \cap U$ *in the sense of* 16.4, *so that in particular* V *has locally bounded generalized mean curvature in* U *(in the sense of* 16.5*).*

In fact V *is minimizing in* $N \cap U$ *in the sense that*

$$(*) \qquad \underline{\underline{M}}_W(V) \le \underline{\underline{M}}_W(\phi_\# V),$$

whenever $W \subset\subset U$ *and* ϕ *is a diffeomorphism of* U *such that* $\phi(N\cap U) \subset N\cap U$ *and* $\phi | U \sim K = \underline{\underline{1}}_{U\sim K}$ *for some compact* $K \subset W \cap N$.

<u>Note</u>: Of course $N = U$ (when $k_1 = k$) is an important special case; then V is stationary and in fact stable in U.

33.3 REMARK In view of 33.2 (together with the fact that $\theta \ge 1$) we can apply the theory of chapters 4 and 5 to V; in particular we can represent $T = \underline{\underline{\tau}}(M_*,\theta_*,\xi)$ where M_* is a relatively *closed* countably n-rectifiable subset of U, and θ_* is an upper semi-continuous function on M_* with $\theta_* \ge 1$ everywhere on M_* (and θ_* integer-valued $\mathcal{H}^n$-a.e. on M_*).

Proof of 33.2 Evidently (in view of the discussion of §16) the first claim in 33.2 follows from (*) (by taking $\phi = \phi_t$ in (*) , ϕ_t is in 16.1 with $U \cap N$ in place of U) .

To prove (*) we first note that, for any W, ϕ as in the statement of the theorem,

$$(1) \qquad \underline{\underline{M}}_W(\phi_{\#}V) = \underline{\underline{M}}_W(\phi_{\#}T)$$

by Remark 27.2(3). Also, since $\partial T = 0$ (in U) , we have

$$(2) \qquad \partial\phi_{\#}T = \phi_{\#}\partial T = 0 .$$

Finally,

$$(3) \qquad \operatorname{spt}(T-\phi_{\#}T) \subset K \subset W .$$

By virtue of (2), (3) we are able to use the inequality of 33.1 with $S = \phi_{\#}T$. This gives (*) as required by virtue of (1).

We conclude this section with the following useful *decomposition lemma:*

33.4 LEMMA *Suppose* $T_1, T_2 \in \mathcal{D}_n(U)$ *are integer multiplicity and suppose* $T_1 + T_2$ *is minimizing in* A , $A \subset U$, *and*

$$\underline{\underline{M}}_W(T_1+T_2) = \underline{\underline{M}}_W(T_1) + \underline{\underline{M}}_W(T_2)$$

for each $W \subset\subset U$. *Then* T_1 , T_2 *are both minimizing in* A .

Proof Let $X \in \mathcal{D}_n(U)$ be integer multiplicity with $\operatorname{spt} X \subset K$, K a compact subset of $A \cap W$, and with $\partial X = 0$. Because $T_1 + T_2$ is minimizing in A we have (by Definition 33.1)

$$\underline{\underline{M}}_W(T_1+T_2+X) \geq \underline{\underline{M}}_W(T_1+T_2) .$$

However since $\underline{\underline{M}}_W(T_1+T_2) = \underline{\underline{M}}_W(T_1) + \underline{\underline{M}}_W(T_2)$, and $\underline{\underline{M}}_W(T_1+T_2+X) \le \underline{\underline{M}}_W(T_1+X) + \underline{\underline{M}}_W(T_2)$, this gives

$$\underline{\underline{M}}_W(T_1) \le \underline{\underline{M}}_W(T_1+X) .$$

In view of the arbitrariness of X , this establishes that T_1 is minimizing in $A \cap W$ (in accordance with Definition 33.1). Interchanging T_1, T_2 in the above argument, we likewise deduce that T_2 is minimizing in $A \cap W$.

§34. EXISTENCE AND COMPACTNESS RESULTS

We begin with a result which establishes the rich abundance of area minimizing currents in Euclidean space.

34.1 LEMMA *Let* $S \in \mathcal{D}_{n-1}(\mathbb{R}^{n+k})$ *be integer multiplicity with* spt S *compact and* $\partial S = 0$. *Then there is an integer multiplicity current* $T \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *such that* spt T *is compact and* $\underline{\underline{M}}(T) \le \underline{\underline{M}}(R)$ *for each integer multiplicity* $R \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *with* spt R *compact and* $\partial R = S$.

34.2 REMARKS

(1) Of course T is minimizing in $\mathbb{R}^{n+k}$ in the sense of Definition 33.1.

(2) By virtue of 33.2 and the convex hull property 19.2 we have automatically that spt $T \subset$ convex hull of spt S .

(3) $$\underline{\underline{M}}(T)^{\frac{n-1}{n}} \le c\underline{\underline{M}}(S)$$

by virtue of the isoperimetric theorem 30.1.

Proof of 34.1 Let

$$\mathcal{I}_S = \{R \in \mathcal{D}_n(\mathbb{R}^{n+k}) : R \text{ is integer multiplicity, spt } R \text{ compact, } \partial R = S\} .$$

Evidently $\mathcal{I}_S \neq \emptyset$. (e.g. $0 \times\!\!\!\times S \in \mathcal{I}_S$.) Take any sequence $\{R_q\} \subset \mathcal{I}_S$ with

(1) $$\lim_{q\to\infty} \underline{\underline{M}}(R_q) = \inf_{R\in\mathcal{I}_S} \underline{\underline{M}}(R) ,$$

let $B_R(0)$ be any ball in $\mathbb{R}^{n+k}$ such that $\operatorname{spt} S \subset B_R(0)$, and let $f : \mathbb{R}^{n+k} \to \bar{B}_R(0)$ be the nearest point (radial) retract of $\mathbb{R}^{n+k}$ onto $\bar{B}_R(0)$. Then $\operatorname{Lip} f = 1$ and hence

(2) $$\underline{\underline{M}}(f_{\#}R_q) \le \underline{\underline{M}}(R_q) .$$

On the other hand $\partial f_{\#}R_q = f_{\#}\partial R_q = f_{\#}S = S$, because $f|B_R(0) = \underline{\underline{1}}_{B_R(0)}$ and $\operatorname{spt} S \subset B_R(0)$. Thus $f_{\#}R_q \subset \mathcal{I}_S$ and by (1), (2) we have

(3) $$\lim_{q\to\infty} \underline{\underline{M}}(f_{\#}R_q) = \inf_{R\in\mathcal{I}_S} \underline{\underline{M}}(R) .$$

Now by the compactness theorem 27.3 there is a subsequence $\{q'\} \subset \{q\}$ and an integer multiplicity current $T \in \mathcal{D}_n(\mathbb{R}^{n+k})$ such that $f_{\#}R_{q'} \rightharpoonup T$ and (by (3) and lower semi-continuity of mass with respect to weak convergence)

(4) $$\underline{\underline{M}}(T) \le \inf_{R\in\mathcal{I}_S} \underline{\underline{M}}(R) .$$

However $\operatorname{spt} T \subset \overline{B_R}(0)$ and $\partial T = \lim \partial f_{\#}R_{q'} = \lim f_{\#}\partial R_{q'} = S$, so that $T \in \mathcal{I}_S$, and the lemma is established (by (4)) .

The proof of the following lemma is similar to that of 34.1 (and again based on 27.3), and its proof is left to the reader.

34.3 LEMMA *Suppose* N *is an* $(n+k_1)$*-dimensional compact* C^1 *submanifold embedded in* $\mathbb{R}^{n+k}$ *and suppose* $R_1 \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *is given such that* $\partial R_1 = 0$, $\operatorname{spt} R_1 \subset N$ *and*

$$\mathcal{I}_{R_1} = \{R \in \mathcal{D}_n(\mathbb{R}^{n+k}) : R - R_1 = \partial S$$

for some integer multiplicity $S \in \mathcal{D}_{n+1}(\mathbb{R}^{n+k})$ *with* $\operatorname{spt} S \subset N\} \neq \emptyset$.

Then there is $T \in \mathcal{I}_{R_1}$ *such that*

$$\underline{\underline{M}}(T) = \inf_{R \in \mathcal{I}_{R_1}} \underline{\underline{M}}(R)$$

34.4 REMARKS

(1) $R - R_1 = \partial S$ with S integer multiplicity and $\operatorname{spt} S \subset N$ means that R, R_1 represent homologous cycles in the n-th singular homology class (with integer coefficients) of N . (See [FH1] or [FF] for discussion.)

(2) It is quite easy to see that T is *locally* minimizing in N ; thus for each $\xi \in \operatorname{spt} T$ there is a neighbourhood U of ξ such that T is minimizing in $N \cap U$.

We conclude this section with the following important compactness theorem for minimizing currents:

34.5 THEOREM *Suppose* $\{T_j\}$ *is a sequence of minimizing currents in* U *with* $\sup_{j\geq 1}(\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)) < \infty$ *for each* $W \subset\subset U$, *and suppose* $T_j \rightharpoonup T \in \mathcal{D}_n(U)$. *Then* T *is minimizing in* U *and* $\mu_{T_j} \to \mu_T$ *(in the usual sense of Radon measures in* U*)* .

34.6 REMARKS

(1) Note that $\mu_{T_j} \to \mu_T$ means the corresponding sequence of varifolds converge in the measure theoretic sense of §15 to the varifold associated with T . (T is automatically integer multiplicity by 27.3.)

(2) If the hypotheses are as in the theorem, except that $\operatorname{spt} T_j \subset N_j \subset U$ and T_j is minimizing in N_j , $\{N_j\}$ a sequence of C^1 embedded $(n+k_1)$-dimensional submanifolds of $\mathbb{R}^{n+k}$ converging in the C^1 sense to

N, $N \subset U$ an embedded $(n+k_1)$-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$,(*) then T minimizes in N (and we still have $\mu_{T_j} \to \mu_T$ in the sense of Radon measures in U). We leave this modification of 34.5 to the reader. (It is easily checked by using suitable local representations for the N_j and by obvious modifications of the proof of 34.5 given below.)

Proof of 34.5 Let $K \subset U$ be an arbitrary compact set and choose a smooth $\phi : U \to [0,1]$ such that $\phi \equiv 1$ in some neighbourhood of K, and $\operatorname{spt} \phi \subset \{x \in U : \operatorname{dist}(x,K) < \varepsilon\}$, where $0 < \varepsilon < \operatorname{dist}(K,\partial U)$ is arbitrary. For $0 < \lambda < 1$, let

$$W_\lambda = \{x \in U : \phi(x) > \lambda\} .$$

Then

(1) $$K \subset W_\lambda \subset\subset U$$

for each λ, $0 \le \lambda < 1$.

By virtue of 31.2 we know that $d_W(T_j,T) \to 0$ for each $W \subset\subset U$, hence in particular we have

(2) $$T - T_j = \partial R_j + S_j , \quad \underline{\underline{M}}_{W_0}(R_j) + \underline{\underline{M}}_{W_0}(S_j) \to 0$$

($W_0 = \{x \in U : \phi(x) > 0\}$).

By the slicing theory (and in particular by 28.5) we can choose $0 < \alpha < 1$ and a subsequence $\{j'\} \subset \{j\}$ (subsequently denoted simply by $\{j\}$) such that

(3) $$\partial(R_j \llcorner W_\alpha) = (\partial R_j) \llcorner W_\alpha + P_j$$

where $\operatorname{spt} P_j \subset \partial W_\alpha$, P_j is integer multiplicity, and

(*) Thus $\exists\, \psi_j : U \to U$, $\psi_j|N_j$ in a diffeomorphism onto N, and $\psi_j \to \underline{\underline{1}}_U$ locally in U with respect to the C^1 metric.

(4) $$\underline{\underline{M}}(P_j) \to 0 .$$

We can also of course choose α to be such that

(5) $$\underline{\underline{M}}(T_j \llcorner \partial W_\alpha) = 0 \qquad \forall j \ \textit{and} \ \underline{\underline{M}}(T \llcorner \partial W_\alpha) = 0 .$$

Thus, combining (2), (3), (4) we have

(6) $$T \llcorner W_\alpha = T_j \llcorner W_\alpha + \partial \tilde{R}_j + \tilde{S}_j$$

with $\tilde{R}_j$, $\tilde{S}_j$ integer multiplicity $(\tilde{R}_j = R_j \llcorner W_\alpha, \tilde{S}_j = S_j \llcorner W_\alpha + P_j)$ with

(7) $$\underline{\underline{M}}(\tilde{R}_j) + \underline{\underline{M}}(\tilde{S}_j) \to 0 .$$

Now let $X \in \mathcal{D}_n(U)$ be any integer multiplicity current with $\partial X = 0$ and $\operatorname{spt} X \subset K$. We want to prove

(8) $$\underline{\underline{M}}_{W_\alpha}(T) \le \underline{\underline{M}}_{W_\alpha}(T+X) .$$

(In view of the arbitrariness of K, X this will evidently establish the fact that T is minimizing in U.)

By (6), we have

(9) $$\underline{\underline{M}}_{W_\alpha}(T+X) = \underline{\underline{M}}_{W_\alpha}(T_j+X+\partial\tilde{R}_j+\tilde{S}_j)$$

$$\ge \underline{\underline{M}}_{W_\alpha}(T_j+X+\partial\tilde{R}_j) - \underline{\underline{M}}(\tilde{S}_j) .$$

Now since T_j is minimizing and $\partial(X+\partial\tilde{R}_j) = 0$ with $\operatorname{spt}(X+\partial\tilde{R}_j) \subset \bar{W}_\alpha$, we have

(10) $$\underline{\underline{M}}_{W_\lambda}(T_j+X+\partial\tilde{R}_j) \ge \underline{\underline{M}}_{W_\lambda}(T_j)$$

for $\lambda > \alpha$. But by (3) we have $\underline{\underline{M}}(\partial\tilde{R}_j \llcorner \partial W_\alpha) = \underline{\underline{M}}(P_j) \to 0$, and by (5) $\underline{\underline{M}}(T_j \llcorner \partial W_\alpha) = 0$, $\underline{\underline{M}}(T \llcorner \partial W_\alpha) = 0$. Hence letting $\lambda \downarrow \alpha$ in (10) we get

$$\underline{\underline{M}}_{W_\alpha}(T_j+X+\partial\tilde{R}_j) \geq \underline{\underline{M}}_{W_\alpha}(T_j) - \underline{\underline{M}}(P_j) ,$$

and therefore from (9) we obtain

(11) $$\underline{\underline{M}}_{W_\alpha}(T+X) \geq \underline{\underline{M}}_{W_\alpha}(T_j) - \varepsilon_j , \quad \varepsilon_j \downarrow 0 .$$

In particular, setting $X = 0$, we have

(12) $$\underline{\underline{M}}_{W_\alpha}(T) \geq \underline{\underline{M}}_{W_\alpha}(T_j) - \varepsilon_j , \quad \varepsilon_j \downarrow 0 .$$

Using the lower semi-continuity of mass with respect to weak convergence in (11), we then have (8) as required.

It thus remains only to prove that $\mu_{T_j} \to \mu_T$ in the sense of Radon measures in U. First note that by (12) we have

$$\limsup \underline{\underline{M}}_{W_\alpha}(T_j) \leq \underline{\underline{M}}_{W_\alpha}(T) ,$$

so that (since $K \subset W_\alpha \subset \{x : \text{dist}(x,K)<\varepsilon\}$ by construction)

$$\limsup \mu_{T_j}(K) \leq \underline{\underline{M}}_{\{x:\text{dist}(x,K)<\varepsilon\}}(T) .$$

Hence, letting $\varepsilon \downarrow 0$

(13) $$\limsup \mu_{T_j}(K) \leq \mu_T(K) .$$

(We actually only proved this for some subsequence, but we can repeat the argument for a subsequence of any given subsequence, hence it holds for the original sequence $\{T_j\}$.)

By the lower semi-continuity of mass with respect to weak convergence, we have

(14) $$\mu_T(W) \leq \liminf \mu_{T_j}(W) \qquad \forall \text{ open } W \subset\subset U .$$

Since (13), (14) hold for arbitrary compact K and open $W \subset U$, it now easily follows (by a standard approximation argument) that $\int f d\mu_{T_j} \to \int f d\mu_T$ for each continuous f with compact support in U, as required.

§35. TANGENT CONES AND DENSITIES

In this section we prove the basic results concerning tangent cones and densities of area minimizing currents. All results depend on the fact that (by virtue of 33.2) the varifold associated with a minimizing current is stationary. This enables us to bring into play the important monotonicity results of Chapter 4.

Subsequently we take N to be a smooth (at least C^2) embedded $(n+k_1)$-dimensional submanifold of $\mathbb{R}^{n+k}$ $(k_1 \le k)$, U open in $\mathbb{R}^{n+k}$ and $(\bar{N} \sim N) \cap U = \emptyset$. Notice that an important case is when $N = U$ (when $k_1 = k$).

35.1 THEOREM *Suppose* $T \in \mathcal{D}_n(U)$ *is minimizing in* $U \cap N$, $\operatorname{spt} T \subset U \cap N$, *and* $\partial T = 0$ *in* U. *Then*

(1) $\Theta^n(\mu_T, x)$ *exists everywhere in* U *and* $\Theta^n(\mu_T, \cdot)$ *is upper semi-continuous in* U;

(2) *For each* $x \in \operatorname{spt} T$ *and each sequence* $\{\lambda_j\} \downarrow 0$, *there is a subsequence* $\{\lambda_{j'}\}$ *such that* $\eta_{x,\lambda_{j'}\#} T \rightharpoonup C$ *in* $\mathbb{R}^{n+k}$, *where* $C \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *is integer multiplicity and minimizing in* $\mathbb{R}^{n+k}$, $\eta_{0,\lambda\#} C = C \quad \forall\, \lambda > 0$, *and* $\Theta^n(\mu_C, 0) = \Theta^n(\mu_T, x)$.

35.2 REMARKS

If C is as in (2) above, we say that C is a *tangent cone* for T

at x. If $\text{spt}\, C$ is an n-dimensional subspace P (notice that since C is integer multiplicity and $\partial C = 0$, it then follows from 26.27 that $C = m[\![P]\!]$ for some $m \in \mathbb{Z}$, assuming P has constant orientation) then we call C a *tangent plane* for T at x.

(2) Notice that is *not* clear whether or not there is a *unique* tangent cone for T at x; thus it is an open question whether or not C depends on the particular sequence $\{\lambda_j\}$ or subsequence $\{\lambda_{j'}\}$ used in its definition. Recently it has been shown ([SL3]) that if C is a tangent cone of T at x such that $\Theta^n(\mu_C, x) = 1$ for *all* $x \in \text{spt}\, C \sim \{0\}$, then C is the unique tangent cone for T at x, and hence $\eta_{x,\lambda\#}T \rightharpoonup C$ as $\lambda \downarrow 0$. Also B. White [WB] has shown in case $n = 2$ that C is always unique (with $\text{spt}\, C$ consisting of a union of 2-planes meeting transversely at 0).

Proof of 35.1 By virtue of Lemma 33.2 we can apply the monotonicity formula of 17.6 (with $\alpha = 1$) and Corollary 17.8 in order to deduce that $\Theta^n(\mu_T, x)$ exists for every $x \in U$ and is an upper semi-continuous function of x in U.

Similarly the existence of C as in part (2) of 35.1 follows directly from Theorem 19.3(*) and the compactness theorem 34.5 (more particularly from Remark 34.6 with $N_j = \eta_{x,\lambda_j\#}N$). Notice that Remark 34.6 establishes first that C is minimizing only in the $(n+k_1)$-dimensional subspace $T_xN \subset \mathbb{R}^{n+k}$. However since orthogonal projection of $\mathbb{R}^{n+k}$ onto T_xN does not increase area, and since $\text{spt}\, C \subset T_xN$, it then follows that C is area minimizing in $\mathbb{R}^{n+k}$.

(*) Actually 19.3 gives $\eta_{0,\lambda\#}V_C = V_C$ for the varifold V_C associated with C, but then $x \wedge \vec{C}(x) = 0$ and hence $\eta_{0,\lambda\#}C = C$ by 26.22 with $h(t,x) = t\lambda x + (1-t)x$.

35.3 THEOREM* *Suppose* $T \in \mathcal{D}_n(U)$ *is minimizing in* $U \cap N$, $\operatorname{spt} T \subset U \cap N$, *and* $\partial T = 0$ (*in* U). *Then*

(1) $\Theta^n(\mu_T, x) \in \mathbb{Z}$ *for all* $x \in U \sim E$, *where* $\mathcal{H}^{n-3+\alpha}(E) = 0 \quad \forall\, \alpha > 0$;

(2) *There is a set* $F \subset E$ (E *as in* (1)) *with* $\mathcal{H}^{n-2+\alpha}(F) = 0$ $\forall\, \alpha > 0$ *and such that for each* $x \in \operatorname{spt} T \sim F$ *there is a tangent plane (see* 35.2(1) *above for terminology) for* T *at* x.

<u>Note</u>: We do not claim E, F are *closed.*

The proof of both parts is based on the abstract dimension reducing argument of Appendix A. In order to apply this in the context of currents we need the observation of the following remark.

35.4 REMARK Given an integer multiplicity current $S \in \mathcal{D}_n(\mathbb{R}^{n+k})$, there is an associated function $\phi_S = (\phi_S^0, \phi_S^1, \ldots, \phi_S^N) : \mathbb{R}^{n+k} \to \mathbb{R}^{N+1}$, where $N = \binom{n+k}{n}$, such that (writing $\theta_S(x) = \Theta^{*n}(\mu_S, x)$)

$$\phi_S^0(x) = \theta_S(x) \ , \ \phi_S^j(x) = \theta_S(x)\xi_S^j(x) \ , \ j = 1, \ldots, N \ ,$$

where $\xi_S^j(x)$ is the j^{th} component of the orientation $\vec{S}(x)$ relative to the usual orthonormal basis $e_{i_1} \wedge \ldots \wedge e_{i_n}$, $1 \le i_1 < i_2 < \ldots < i_n \le n+k$ for $\Lambda_n(\mathbb{R}^{n+k})$ (ordered in any convenient manner). Evidently, for any $x \in \mathbb{R}^{n+k}$,

$$\phi_S(x+\lambda y) = \phi_{\eta_{x,\lambda\#}S}(y) \ , \ y \in \mathbb{R}^{n+k} \ ,$$

and, given a sequence $\{S_i\} \subset \mathcal{D}_n(I+R^{n+k})$ of such integer multiplicity currents, we trivially have

$$\phi_{S_i}^j \, d\mathcal{H}^n \to \phi_S^j \, d\mathcal{H}^n \ \ \forall\, j \in \{1, \ldots, N\} \ \Leftrightarrow \ S_i \rightharpoonup S$$

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* Cf. Almgren [A2]

and

$$\phi^0_{S_i}\, dH^n \to \phi^0_S\, dH^n \;\Leftrightarrow\; \mu_{S_i} \to \mu_S$$

(where $\psi_i dH^n \to \psi dH^n$ means $\int f\psi_i dH^n \to \int f\psi dH^n \;\forall\, f \in C_c(\mathbb{R}^{n+k})$) .

We shall also need the following simple lemma, the proof of which is left to the reader.

35.5 LEMMA *Suppose* S *is minimizing in* $\mathbb{R}^{n+k}$, $\partial S = 0$, *and*

$$\eta_{x,1\#}S = S \qquad \forall\, x \in \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{n+k}$$

for some positive integer $m < n$. *(Recall* $\eta_{x,1} : y \mapsto y-x$, $y \in \mathbb{R}^{n+k}$.) *Then*

$$S = [\![\mathbb{R}^m]\!] \times S_0 ,$$

where $\partial S_0 = 0$ *and* S_0 *is minimizing in* $\mathbb{R}^{n+k-m}$.

Furthermore if S *is a cone* (i.e. $\eta_{0,\lambda\#}S = S$ *for each* $\lambda > 0$) , *then so is* S_0 .

Proof of 35.3(1) For each positive integer m and $\beta \in (0,\frac{1}{2})$ let

$$U_{m,\beta} = \{x \in U : \Theta^n(\mu_T, x) < m-\beta\} .$$

Now T is minimizing in $U \cap N$, so by the monotonicity formula of 17.6 (which can be applied by virtue of 33.2) we have, firstly, that $U_{m,\beta}$ is open, and secondly that for each $x \in U_{m,\beta}$, there is some ball $B_{2\rho}(x) \subset U_{m,\beta}$ such that

$$(1) \qquad \frac{\mu_T(B_\sigma(y))}{\omega_n \sigma^n} \le m-\beta/2 \qquad \forall\, \sigma < \rho , \quad y \in B_\rho(x) .$$

We ultimately want to prove

$$\mathcal{H}^{n-3+\alpha}\left(\bigcup_{m=1}^{\infty} \{x \in U_{m,\beta} : m-1+\beta < \Theta^n(\mu_T, x) < m-\beta\}\right) = 0$$

for each sufficiently small $\alpha, \beta > 0$, and, in view of (1), by a rescaling and translation it will evidently suffice to assume

(2) $$B_2(0) = U\ , \quad \frac{\mu_T(B_\sigma(y))}{\omega_n \sigma^n} \le m-\beta \quad \forall\ \sigma < 1\ ,\ y \in B_1(0)\ ,$$

and then prove

(3) $$\mathcal{H}^{n-3+\alpha}\{x \in B_1(0) : \Theta^n(\mu_T, x) \ge m-1+\beta\} = 0\ .$$

We consider the set $\mathcal{T}$ of weak limit points of sequences $S_i = \eta_{x_i,\lambda_i\#}T$ where $|x_i| < 1-\lambda_i$, $0 < \lambda_i < 1$, with $\lim x_i \in \overline{B_1}(0)$ and $\lim \lambda_i = \lambda \ge 0$ both existing. For any such sequence S_i we have (by (2))

$$\limsup \underline{\underline{M}}_W(S_i) < \infty$$

for each $W \subset\subset \eta_{x,\lambda}(U)$ in case $\lambda > 0$, and for each $W \subset\subset \mathbb{R}^{n+k}$ in case $\lambda = 0$. Hence we can apply the compactness theorem 34.5 to conclude that each element S of $\mathcal{T}$ is integer multiplicity and

(4) S minimizes in $\eta_{x,\lambda}U \cap \eta_{x,\lambda}N$ in case $S = \lim \eta_{x_i,\lambda_i\#}T$

with $\lim x_i = x$ and $\lim \lambda_i = \lambda > 0$, and

(5) S minimizes in all of $\mathbb{R}^{n+k}$ in case $S = \lim \eta_{x_i,\lambda_i\#}T$

with $\lim x_i = x$ and $\lim \lambda_i = 0$. (Cf. the discussion in the proof of 35.1(2).)

For convenience we define

$$(6) \qquad U_S = \begin{cases} \eta_{x,\lambda}U & \text{in case } \lim \lambda_i > 0 \text{ (as in (4))} \\ \mathbb{R}^{n+k} & \text{in case } \lim \lambda_i = 0 \text{ (as in (5))} , \end{cases}$$

so that $S \in \mathcal{D}_n(U_S)$ for each $S \in \mathcal{T}$.

Now by definition one readily checks that

$$(7) \qquad \eta_{x,\lambda\#}\mathcal{T} = \mathcal{T} , \; 0 < \lambda < 1 , \; |x| < 1-\lambda ,$$

and, by (2),

$$(8) \qquad \Theta^n(\mu_S, y) \le m-\beta \quad \forall \, y \in U_S , \; S \in \mathcal{T} .$$

Furthermore by using 34.5 together with the monotonicity formula 17.6, one readily checks that if $S_i \rightharpoonup S$ $(S_i, S \in \mathcal{T})$ and if $y, y_i \in B_1(0)$ with $\lim y_i = y$, then

$$(9) \qquad \Theta^n(\mu_S, y) \ge \limsup \Theta^n(\mu_{S_i}, y_i) .$$

It now follows from (7), (8), (9) and 34.5 that all the hypotheses of Theorem A.4 (of Appendix A) are satisfied with (using notation of Remark 35.4)

$$F = \{\phi_S : S \in \mathcal{T}\}$$

and with sing defined by

$$\text{sing } \phi_S = \{x \in U_S : \Theta^n(\mu_S, \cdot) \ge m-1+\beta\}$$

for $S \in \mathcal{T}$. We claim that in this case the additional hypothesis is satisfied with $d = n-3$. Indeed suppose $d \ge n-2$; then there is $S \in \mathcal{T}$ and $\eta_{y,\lambda\#}S = S$ $\forall \, y \in L$, $\lambda > 0$ with L an $(n-2)$-dimensional subspace of $\mathbb{R}^{n+k}$, $L \subset \text{sing } \phi_S$. Since we can make a rotation of $\mathbb{R}^{n+k}$ to bring L into coincidence with $\mathbb{R}^{n-2} \times \{0\}$, we assume that $L = \mathbb{R}^{n-2} \times \{0\}$. Then by Lemma 35.4 we have

$$S = [\![\mathbb{R}^{n-2}]\!] \times S_0 ,$$

where $S_0 \in \mathcal{D}_2(\mathbb{R}^N)$, $N = 2+k$, with S_0 a 2-dimensional area minimizing cone in $\mathbb{R}^N$. Then spt S_0 is contained in a finite union $\bigcup_{i=1}^{q} P_i$ of 2-planes, with $P_i \cap P_j = \{0\}$ $\forall i \neq j$. (For a formal proof of this characterization of 2 dimensional area minimizing cones, see for example [WB].) In particular, since $\Theta^n(\mu_S, \cdot)$ is constant on $P_i \sim \{0\}$ (by the constancy theorem 26.27), we have that $\Theta^n(\mu_S, y) \in \mathbb{Z}$ for *every* $y \in \mathbb{R}^{n+k}$, and by (8) it follows that $\Theta^n(\mu_S, y) \leq m-1$ $\forall$ y $\mathbb{R}^{n+k}$. That is, sing $\phi_S = \emptyset$, a contradiction, hence we can take $d = n-3$ as claimed. We have thus established (3) as required.

Proof of 35.3(2) The proof goes similarly to 35.3(1). This time we assume (again without loss of generality) that

(1) $$U = B_2(0) ,$$

and we prove that T has a tangent plane at all points of spt $T \cap B_1(0)$ except for a set $F \subset$ spt $T \cap B_1(0)$ with

(2) $$\mathcal{H}^{n-2+\alpha}(F) = 0 \quad \forall \alpha > 0 .$$

$\mathcal{T}$ is as described in the proof of 35.3(1), and for any $S \in \mathcal{T}$ and $\beta > 0$ we let

$$R_\beta(S) = \{x \in \operatorname{spt} S : \overline{B_\rho}(x) \subset U_S \text{ and}$$
$$h(\operatorname{spt} S, L, \rho, x) < \beta\rho \text{ for some } \rho > 0$$
$$\text{and some } n\text{-dimensional subspace } L \text{ of } \mathbb{R}^{n+k}\} ,$$

where U_S is as in the proof of 35.3(1) (so that $S \in \mathcal{D}_n(U_S)$), and where we define

$$h(\mathrm{spt}S,L,\rho,x) = \sup_{y\in \mathrm{spt}S\cap B_\rho(x)} |q(y-x)| ,$$

with q the orthogonal projection of $\mathbb{R}^{n+k}$ onto $L^\perp$.

Now notice that (Cf. the proof of 35.3(1))

(2) $$\eta_{x,\lambda\#}\mathcal{T} = \mathcal{T} \qquad \forall\ 0<\lambda<1 ,\ |x|<1-\lambda ,$$

and

(3) $$\eta_{x,\lambda}R_\beta(S) = R_\beta(\eta_{x,\lambda\#}S) ,\ S\in\mathcal{T} .$$

Furthermore if $S_j \to S$, $S_j, S\in\mathcal{T}$, then by the monotonicity formula 17.6 it is quite easy to check that if $y\in R_\beta(S)$ and if $y_j\in \mathrm{spt}\,S_j$ with $y_j\to y$, then $y_j\in R_\beta(S_j)$ for all sufficiently large j. Because of this, and because of (2), (3) above, it is now straightforward to check that the hypotheses of Theorem A.4 hold with (again in notation of Remark 35.4)

$$\mathcal{F} = \{\phi_S : S\in\mathcal{T}\}$$

and

$$\mathrm{sing}\ \phi_S = \mathrm{spt}\ \Theta^n(\mu_S,\cdot)\cap U_S \sim R_\beta(S) .$$

(Notice that $R_\beta(S)$ is completely determined by $\Theta^n(\mu_S,\cdot)$, and hence this makes sense.) In this case we claim that $d\le n-2$. Indeed if $d>n-2$ (i.e. $d=n-1$) then $\exists\ S\in\mathcal{T}$ such that

$$\eta_{x,\lambda\#}S = S \qquad \forall\ x\in L ,\ \lambda>0 ,\ \text{and}\ \ L\subset \mathrm{sing}\ \phi_S$$

where L is an $(n-1)$-dimensional subspace. Then, supposing without loss of generality that $L=\mathbb{R}^{n-1}\times\{0\}$, we have by Lemma 35.5 that

(3) $$S = [\![\mathbb{R}^{n-1}]\!]\times S_0 ,$$

where S_0 is a 1-dimensional minimizing cone in $\mathbb{R}^{k+1}$. However it is easy to check that such a 1-dimensional minimizing cone necessarily has the form

$$S_0 = m[\![\ell]\!] ,$$

where $m \in \mathbb{Z}$ and ℓ is a 1-dimensional subspace of $\mathbb{R}^{k+1}$. Thus (3) gives that $S = m[\![L]\!]$ where L is an n-dimensional subspace and hence $\operatorname{sing} \phi_S = \emptyset$, a contradiction, so $d \le n-2$ as claimed.

We therefore conclude from Theorem A.4 that for each $S \in \mathcal{T}$

$$\mathcal{H}^{n-2+\alpha}(\operatorname{spt} S \sim R_\beta(S) \cap B_1(0)) = 0 \qquad \forall\, \alpha > 0 .$$

If $\beta_j \downarrow 0$ we thus conclude in particular that

$$(4) \qquad \mathcal{H}^{n-2+\alpha}(\operatorname{spt} T \sim \bigcap_{j=1}^{\infty} R_{\beta_j}(T) \cap B_1(0)) = 0 \qquad \forall\, \alpha > 0 .$$

However by (1) we see that

$$x \in \bigcap_{j=1}^{\infty} R_{\beta_j}(T) \Leftrightarrow T \text{ has a tangent plane at } x ,$$

and therefore (4) gives (2) as required.

§36. SOME REGULARITY RESULTS (Arbitrary Codimension)

In this section, for $T \in \mathcal{D}_n(U)$ any integer multiplicity current, we define a relatively closed subset $\operatorname{sing} T$ of U by

36.1 $$\operatorname{sing} T = \operatorname{spt} T \sim \operatorname{reg} T ,$$

where $\operatorname{reg} T$ denotes the set of points $\xi \in \operatorname{spt} T$ such that for some $\rho > 0$ there is an $m \in \mathbb{Z}$ and an embedded n-dimensional oriented C^1 submanifold M of $\mathbb{R}^{n+k}$ with $T = m[\![M]\!]$ in $B_\rho(\xi)$.

Recently F.J. Almgren [A2] has proved the very important theorem that $\mathcal{H}^{n-2+\alpha}(\text{sing } T) = 0 \;\; \forall \alpha > 0$ in case $\text{spt } T \subset N$, $\partial T = 0$ and T is minimizing in N, where N is a smooth embedded $(n+k_1)$-dimensional submanifold of $\mathbb{R}^{n+k}$. The proof is very non-trivial and requires development of a whole new range of results for minimizing currents. We here restrict ourselves to more elementary results.

Firstly, the following theorem is an immediate consequence of Theorem 24.4 and Lemma 33.2.

36.2 THEOREM *Suppose* $T \in \mathcal{D}_n(U)$ *is integer multiplicity and minimizing in* $U \cap N$ *for some embedded* C^2 $(n+k_1)$*-dimensional submanifold* N *of* $\mathbb{R}^{n+k}$, $(\bar{N} \sim N) \cap U = \emptyset$, *and suppose* $\text{spt } T \subset U \cap N$, $\partial T = 0$ *(in* U*). Then* $\text{reg } T$ *is dense in* $\text{spt } T$.

(Note that by definition $\text{reg } T$ is relatively open in $\text{spt } T$.)

The following is a useful fact; however its applicability is limited by the hypothesis that $\Theta^n(\mu_T, y) = 1$.

36.3 THEOREM *Suppose* $\{T_i\} \subset \mathcal{D}_n(U)$, $T \in \mathcal{D}_n(U)$ *are integer multiplicity currents with* T_i *minimizing in* $U \cap N_i$, T *minimizing in* $U \cap N$, N, N_i *embedded* $(n+k_1)$*-dimensional* C^2 *submanifolds, and* $\text{spt } T_i \subset N_i$, $\text{spt } T \subset N$, $\partial T_i = \partial T = 0$ *(in* U*). Suppose also that* N_i *converges to* N *in the* C^2 *sense in* U, $T_j \rightharpoonup T$ *in* $\mathcal{D}_n(U)$, *and suppose* $y \in N \cap U$ *with* $\Theta^n(\mu_T, y) = 1$, $y = \lim y_j$, *where* y_j *is a sequence such that* $y_j \in \text{spt } T_j$ $\forall j$. *Then* $y \in \text{reg } T$ *and* $y_j \in \text{reg } T_j$ *for all sufficiently large* j.

Proof By virtue of the monotonicity formula 17.6(1) (which is applicable by 33.2) it is easily checked that

$$\limsup \Theta^n(\mu_{T_j}, y_j) \le \Theta^n(\mu_T, y) = 1 ,$$

hence (since $\Theta^n(\mu_{T_j}, y_j) \ge 1$ by 17.8) we conclude $\Theta^n(\mu_{T_j}, y_j) \to \Theta^n(\mu_T, y) = 1$. Hence by Allard's theorem 24.2 we have $y \in \operatorname{reg} T$ and $y_j \in \operatorname{reg} T_j$ for all sufficiently large j. (33.2 justifies the use of 24.2.)

Next we have the following consequence of Theorem A.4 of Appendix A.

36.4 THEOREM *Suppose* T *is as in* 36.2, *and in addition suppose* $\xi \in \operatorname{spt} T$ *is such that* $\Theta^n(\mu_T, \xi) < 2$. *Then there is a* $\rho > 0$ *such that*

$$\mathcal{H}^{n-2+\alpha}(\operatorname{sing} T \cap B_\rho(\xi)) = 0 \qquad \forall\, \alpha > 0 .$$

Proof Let $\alpha = \frac{1}{2}(2-\Theta^n(\mu_T, \xi))$ and let $B_\rho(\xi)$ be such that $B_{2\rho}(\xi) \subset U$ and

$$(1) \qquad \frac{\mu_T(B_\sigma(\zeta))}{\omega_n \sigma^n} < 2(1-\alpha/2)$$

$\forall\, \zeta \in \operatorname{spt} T \cap B_\rho(\xi)$, $0 < \sigma < \rho$. (Notice that such ρ exists by virtue of the monotonicity formula 17.6(1), which can be applied by 33.2.) Assume without loss of generality that $\xi = 0$, $\rho = 1$ and $U = B_2(0)$, and define $\mathcal{T}$ to be the set of weak limits S of sequences $\{S_i\}$ of the form $S_i = \eta_{x_i, \lambda_i \#} T$, $|x_i| < (1-\lambda_i)$, $0 < \lambda_i < 1$, where $\lim x_i$ and $\lim \lambda_i \equiv \lambda$ are assumed to exist. Notice that

$$\limsup \underset{=}{M}_W(S_i) < \infty$$

for each $W \subset\subset \eta_{x,\lambda}(U)$ in case $\lambda > 0$ and for each $W \subset\subset \mathbb{R}^{n+k}$ in case $\lambda = 0$. Hence by the compactness theorem 34.5 any such S is integer multiplicity in U_S

$$(U_S = \eta_{x,\lambda} U \text{ in case } \lambda > 0,\ U_S = \mathbb{R}^{n+k} \text{ in case } \lambda = 0)$$

and (Cf. the proof of 35.1(2))

(2) S minimizes in $\eta_{x,\lambda}U \cap \eta_{x,\lambda}N$ in case $\lambda > 0$

(3) S minimizes in $\mathbb{R}^{n+k}$ in case $\lambda = 0$.

One readily checks that, by definition of $\mathcal{T}$,

(4) $$\eta_{y,\tau\#}\mathcal{T} = \mathcal{T} \ , \ 0 < \tau < 1 \ , \ |y| < 1-\tau$$

Furthermore we note that (by (1))

(5) $$\Theta^n(\mu_S, x) = 1 \ , \ \mu_S\text{-a.e. } x \in U_S \ ,$$

and by Allard's theorem 24.2 there is $\delta > 0$ such that

(6) $$\text{sing } S = \{x \in U_S : \Theta^n(\mu_S, x) \geq 1+\delta\} \ , \ S \in \mathcal{T} \ .$$

Now in view of (2), (3), (4), (5), (6) and the upper semi-continuity of Θ^n as in (9) of the proof of 35.3(1), all the hypotheses of Theorem A.4 of Appendix A are satisfied with $\mathcal{F} = \{\phi_S : S \in \mathcal{T}\}$ (notation as in Remark 35.4) and with $\text{sing } \phi_S = \{x \in U_S : \Theta^n(\mu_S, x) \geq 1+\delta\}$ ($\equiv$ sing S by (6)). In fact we claim that in this case we may take $d = n-2$, because if $d = n-1$ $\exists$ $S \in \mathcal{T}$ and $\eta_{x,\lambda\#}S = S$ $\forall x \in L$, $\lambda > 0$, where $L \subset \text{sing } S$ is an $(n-1)$dimensional subspace of $\mathbb{R}^{n+k}$, then (Cf. the last part of the proof of 35.3(2)) we have $S = m[\![Q]\!]$ for some n-dimensional subspace Q . Hence sing $S = \emptyset$, a contradiction.

The following lemma is often useful:

36.5 THEOREM *Suppose* $C \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *is minimizing in* $\mathbb{R}^{n+k}$, $\partial C = 0$, *and* C *is a cone:* $\eta_{0,\lambda\#}C = C$ $\forall \lambda > 0$. *Suppose further that* spt $C \subset \bar{H}$ *where* H *is an open* $\frac{1}{2}$*-space of* $\mathbb{R}^{n+k}$ *with* $0 \in \partial H$. *Then* spt $C \subset \partial H$.

36.6 REMARK The reader will see that the theorem here is actually valid with any stationary rectifiable varifold V in $\mathbb{R}^{n+k}$ satisfying $\eta_{0,\lambda\#}V = V$ in place of C .

Proof of 36.5 Since the varifold V associated with C is stationary (by 33.2) in $\mathbb{R}^{n+k}$ we have by 18.1 (since $(Dr)^{\perp} = 0$ by virtue of the fact that C is a cone),

$$(1) \qquad \frac{d}{d\rho}\left(\rho^{-n}\int_{\mathbb{R}^{n+k}} h\phi(r/\rho)\,d\mu_C\right) = \rho^{-n-1}\int_{\mathbb{R}^{n+k}} x\cdot(\nabla^C h)\,\phi(r/\rho)\,d\mu_C$$

for each $\rho > 0$, where $r = |x|$ and ϕ is a non-negative C^1 function on $\mathbb{R}$ with compact support, and h is an arbitrary $C^1(\mathbb{R}^{n+k})$ function. ($\nabla^C h(x)$ denotes the orthogonal projection of $\mathrm{grad}_{\mathbb{R}^{n+k}} h(x)$ onto the tangent space T_xV of V at x.)

Now suppose without loss of generality that $H = \{x = (x^1,\ldots,x^{n+k}) : x^1 > 0\}$ and select $h(x) \equiv x^1$. Then $x\cdot\nabla^C h = e_1^T\cdot x = e_1\cdot x^T = re_1\cdot\nabla^C r$, where v^T denotes orthogonal projection of v onto T_xV. Thus the term on the right side of (1) can be written $-\int_{\mathbb{R}^{n+k}}(e_1\cdot\nabla^C r)(r\phi(r/\rho))\,d\mu_C$, which in turn can be written $-\int_{\mathbb{R}^{n+k}} e_1\cdot\nabla^C\psi_\rho\,d\mu_C$, where $\psi_\rho(x) = \int_{|x|}^{\infty} r\phi(r/\rho)\,dr$. (Thus ψ_ρ has compact support in $\mathbb{R}^{n+k}$.) But $e_1\cdot\nabla^C\psi_\rho \equiv \mathrm{div}_V(\psi_\rho e_1)$, and hence the term on the right of (1) actually vanishes by virtue of the fact that V is stationary. Thus (1) gives

$$\rho^{-n}\int_{\mathbb{R}^{n+k}} x_1\phi(r/\rho)\,d\mu_C = \text{const.},\quad 0<\rho<\infty .$$

In view of the arbitrariness of ϕ, this implies

$$\rho^{-n}\int_{B_\rho(0)} x_1\,d\mu_C \equiv \text{const.}$$

However trivially we have $\lim_{\rho\downarrow 0}\rho^{-n}\int_{B_\rho(0)} x_1\,d\mu_C = 0$, and hence we deduce

$$\rho^{-n}\int_{B_\rho(0)} x_1\,d\mu_C = 0 \qquad \forall\ \rho > 0 .$$

Thus since $x_1 \geq 0$ on spt C $(\subset \bar{H})$, we conclude spt $C \subset \partial H$ $(= \{x : x^1 = 0\})$.

The following corollary of 36.5 follows directly by combining 36.5 and 35.1(2).

36.6 COROLLARY *If* T *is as in* 36.2, *if* $\xi \in$ spt T, *if* Q *is a* C^1 *hypersurface in* $\mathbb{R}^{n+k}$ *such that* $\xi \in Q$ *and if* spt T *is locally on one side of* Q *near* ξ, *then all tangent cones* C *of* T *at* ξ *satisfy* spt $C \subset T_\xi Q \cap T_\xi N$.

§37. CODIMENSION 1 MINIMIZING CURRENTS

We begin by looking at those integer multiplicity currents $T \in \mathcal{D}_n(U)$ with spt $T \subset N \cap U$, N an (n+1)-dimensional oriented embedded submanifold of $\mathbb{R}^{n+k}$ with $(\bar{N} \sim N) \cap U = \emptyset$ and such that

$$(*) \qquad \partial T = [\![E]\!]$$

(in U), where E is an $\mathcal{H}^{n+1}$-measurable subset of N. (We know by 27.8, 33.4 that all minimizing currents $T \in \mathcal{D}_n(U)$ with $\partial T = 0$ and spt T in N can be locally decomposed into minimizing currents of this special form.)

37.1 REMARK The fact that T has the form (*) and T is integer multiplicity evidently is equivalent to the requirement that if $V \subset U$ is open, and if ϕ is a C^2 diffeomorphism of V onto an open subset of $\mathbb{R}^{n+k}$ such that $\phi(V \cap N) = G$, G open in $\mathbb{R}^{n+1}$, then $\phi(E)$ has locally finite perimeter in G. This is an easy consequence of Remark 26.28, and in fact we see from this and Theorem 14.3 that any T of the form (*) with $\underline{\underline{M}}_W(T) < \infty$ $\forall\, W \subset\subset U$ is automatically integer multiplicity with

(**) $$\Theta^n(T,x) = 1 \ , \ \mu_T\text{-a.e. } x \in U \ .$$

We shall here develop the theory of minimizing currents of the form (*) ; indeed we show this is naturally done using only the more elementary facts about currents. In particular we shall not in this section have any need for the compactness theorem 27.3 (instead we use only the elementary compactness theorem 6.3 for BV functions), nor shall we need the deformation theorem and the subsequent material of Chapter 6.

The following theorem could be derived from the general compactness theorem 34.5, but here (as we mentioned above) we can give a more elementary treatment. In this theorem, and subsequently, we take $U \subset \mathbb{R}^{n+k}$ to be open, and $\mathcal{O}$ will denote the collection of $(n+1)$-dimensional oriented embedded C^2 submanifolds N of $\mathbb{R}^{n+k}$ with $(\bar{N} \sim N) \cap U = \emptyset$, $N \cap U \neq \emptyset$. A sequence $\{N_j\} \subset \mathcal{O}$ is said to converge to $N \in \mathcal{O}$ in the C^2 sense in U if there are orientation preserving C^2 embeddings $\psi_j : N \cap U \to N_j$ with $\psi_j \to \underline{\underline{1}}_{N \cap U}$ locally relative to the C^2 metric in $N \cap U$. In particular if $x \in N$ then $\eta_{x,\lambda} N$ converges to $T_x N$ in the C^2 sense in W as $\lambda \downarrow 0$, for each $W \subset\subset \mathbb{R}^{n+k}$.

In the following theorem p is a proper C^2 map $U \to N \cap U$ such that, in some neighbourhood $V \subset U$ of $N \cap U$, p coincides with the nearest point projection of V onto N . (Since the nearest point projection is C^2 in some neighbourhood of $N \cap U$ it is clear that such p exists.)

37.2 THEOREM (Compactness theorem for minimizing T as in (*))

Suppose $T_j \in \mathcal{D}_n(U)$, $T_j = \partial [\![E_j]\!]$ *(in* U*)* , E_j $\mathcal{H}^{n+1}$*-measurable subsets of* $N_j \cap U$, $N_j \in \mathcal{O}$, $N_j \to N \in \mathcal{O}$ *in the* C^2 *sense described above, and suppose* T_j *is integer multiplicity and minimizing in* $U \cap N_j$.

Then there is a subsequence $\{T_{j'}\}$ *with* $T_{j'} \rightharpoonup T$ *in* $\mathcal{D}_n(U)$, T *integer multiplicity,* $T = \partial[\![E]\!]$ *(in* U) , $\chi_{p(E_{j'})} \to \chi_E$ *in* $L^1_{loc}(\mathcal{H}^{n+1}, U)$, $\mu_{T_{j'}} \to \mu_T$ *(in the usual sense of Radon measures) in* U , *and* T *is minimizing in* $N \cap U$.

37.3 REMARKS

(1) Recall (from Remark 37.1) that the hypothesis that T_j is integer multiplicity is automatic if we assume merely that $\underline{\underline{M}}_W(T_j) < \infty \quad \forall\ W \subset\subset U$.

(2) We make no *a-priori* assumptions on local boundedness of the mass of the T_j (we see in the proof that this is automatic for minimizing currents as in (*)).

(3) Let $h(x,t) = x + t(p(x) - x)$, $x \in U$, $0 \le t \le 1$. Using the homotopy formula 26.22 (and in particular the inequality 26.23) together with the fact that $N_j \to N$ in the C^2 sense in U , it is straightforward to check that

$$T_j - T = \partial R_j \ , \ R_j = h_\#([\![(0,1)]\!] \times T_j) + p_\#[\![E_j]\!] - [\![E]\!]$$

with

$$\underline{\underline{M}}_W(R_{j'}) \to 0 \qquad \forall\ W \subset\subset U ,$$

provided that $\chi_{p(E_{j'})} \to \chi_E$ as claimed in the theorem. Thus once we establish $\chi_{p(E_{j'})} \to \chi_E$ for some E , then we can use the argument of 34.5 (with $S_j = 0$) in order to conclude

(1) T is minimizing in U

(2) $\mu_{T_{j'}} \to \mu_T$ in U .

(Notice we have not had to use the deformation theorem here.)

In the following proof we therefore concentrate on proving $\chi_{P(E_{j'})} \to \chi_E$ in $L^1_{loc}(\mathcal{H}^{n+1}, N \cap U)$ for some subsequence $\{j'\}$ and some E such that $\partial[\![E]\!]$ has locally finite mass in U. (T is then automatically integer multiplicity by Remark 37.1.)

Proof of 37.2 We first establish a local mass bound for the T_j in U: if $\xi \in N$ and $B_{\rho_0}(\xi) \subset U$, then

$$(1) \qquad \underline{\underline{M}}(T_j \llcorner B_\rho(\xi)) \le \frac{1}{2}\mathcal{H}^n(\partial B_\rho(\xi) \cap N) , \quad \mathcal{L}^1 \text{ a.e. } \rho \in (0,\rho_0) .$$

This is proved by simple area comparison as follows:

With $r(x) = |x-\xi|$, by the elementary slicing theory of 28.5(1), (2) we have that, for $\mathcal{L}^1$-a.e. $\rho \in (0,\rho_0)$, the slice $\langle [\![E_j]\!], r, \rho\rangle$ (i.e. the slice of $[\![E_j]\!]$ by $\partial B_\rho(\xi)$) is integer multiplicity, and (using $T_j = \partial[\![E_j]\!]$),

$$\partial[\![E_j \cap B_\rho(\xi)]\!] = T_j \llcorner B_\rho(\xi) + \langle [\![E_j]\!], r, \rho\rangle .$$

Hence (applying ∂ to this identity)

$$\partial(T_j \llcorner B_\rho(\xi)) = -\partial\langle [\![E_j]\!], r, \rho\rangle , \quad \mathcal{L}^1\text{-a.e. } \rho \in (0,\rho_0) .$$

But by definition 33.1 of minimizing we then have

$$\underline{\underline{M}}(T_j \llcorner B_\rho(\xi)) \le \underline{\underline{M}}\langle [\![E_j]\!], r, \rho\rangle , \quad \mathcal{L}^1\text{-a.e. } \rho \in (0,\rho_0) .$$

Similarly, since $-T_j$ is also minimizing in $N \cap U$,

$$\underline{\underline{M}}(T_j \llcorner B_\rho(\xi)) \le \underline{\underline{M}}\langle [\![\tilde{E}_j]\!], r, \rho\rangle , \quad \mathcal{L}^1\text{-a.e. } \rho \in (0,\rho_0) ,$$

where $\tilde{E}_j = N \cap U \sim E_j$. Thus

$$(2) \qquad \underline{\underline{M}}(T_j \llcorner B_\rho(\xi)) \le \min\{\underline{\underline{M}}\langle [\![E_j]\!], r, \rho\rangle , \underline{\underline{M}}\langle [\![\tilde{E}_j]\!], r, \rho\rangle\}$$

for $\mathcal{L}^1$-a.e. $\rho \in (0,\rho_0)$. Now of course $[\![\tilde{E}_j]\!] + [\![E_j]\!] = [\![N \cap U]\!]$, so that (for a.e. $\rho \in (0,\rho_0)$)

$$<[\![E_j]\!],r,\rho> + <[\![\tilde{E}_j]\!],r,\rho> = <N,r,\rho>$$

and hence (2) gives (1) as required (because $\underline{\underline{M}}(<N,r,\rho>) \le H^n(N \cap \partial B_\rho(\xi))$ by virtue of the fact that $|Dr| = 1$, hence $|\nabla^N r| \le 1$) .

Now by virtue of (1) and Remark 37.1 we deduce from the BV compactness theorem 6.3 that some subsequence $\{\chi_{p(E_{j'})}\}$ of $\{\chi_{p(E_j)}\}$ converges in $L^1_{loc}(H^{n+1}, N \cap U)$ to χ_E , where $E \subset N$ is H^{n+1}-measurable and such that $\partial[\![E]\!]$ is integer multiplicity (in U) . The remainder of the theorem now follows as described in Remark 37.3(3).

37.4 THEOREM (Existence of tangent cones)

Suppose $T = \partial[\![E]\!] \in \mathcal{D}_n(U)$ *is integer multiplicity, with* $E \subset N \cap U$, $N \in \mathcal{O}$, *and* T *is minimizing in* $U \cap N$. *Then for each* $x \in \text{spt}\, T$ *and each sequence* $\{\lambda_j\} \downarrow 0$ *there is a subsequence* $\{\lambda_{j'}\}$ *and an integer multiplicity* $C \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *with* C *minimizing in* $\mathbb{R}^{n+k}$, $0 \in \text{spt}\, C \subset T_x N$, $\Theta^n(\mu_C,0) = \Theta^n(\mu_T,x)$, $C = \partial[\![F]\!]$, F *an* H^{n+1}*-measurable subset of* $T_x N$,

(1) $\mu_{\eta_{x,\lambda_{j'}\#}T} \to \mu_C$ *in* $\mathbb{R}^{n+k}$, $\chi_{p(\eta_{x,\lambda_{j'}}(E))} \to \chi_F$ *in* $L^1_{loc}(H^{n+1}, T_x N)$,

where p *is the orthogonal projection of* $\mathbb{R}^{n+k}$ *onto* $T_x N$, *and*

(2) $$\eta_{0,\lambda\#}C = C \, , \ \eta_{0,\lambda}F = F \qquad \forall\, \lambda > 0 \, .$$

37.5 REMARK The proof given here is independent of the general tangent cone existence theorem 35.1.

Proof of Theorem 37.4 As we remarked prior to Theorem 37.2, $\eta_{x,\lambda_j}N$ converges to $T_x N$ in the C^2 sense in W for each $W \subset\subset \mathbb{R}^{n+k}$. By the compactness theorem 37.2 we then have a subsequence $\lambda_{j'}$ such that all the required conclusions, except possibly for 37.4(2) and the fact that $0 \in \operatorname{spt} C$, hold. To check that $0 \in \operatorname{spt} C$ and that 37.4(2) is valid, we first note by 33.2 that the varifold V associated with T is stationary in $N \cap U$ (and that V therefore has locally bounded generalized mean curvature $\underline{\underline{H}}$ in $N \cap U$). Therefore by the monotonicity formula 17.6(1), and by 17.8, we have

(1) $$\Theta^n(\mu_V, x) \text{ exists and is } \geq 1 .$$

Since $\mu_{\eta_{x,\lambda_j}\#T} \to \mu_C$, we then have $\Theta^n(\mu_C, 0) = \Theta^n(\mu_T, x) \geq 1$, so $0 \in \operatorname{spt} C$, and by 19.3 we deduce that the varifold V_C associated with C is a cone. Then in particular $x \wedge \vec{C}(x) = 0$ for μ_C-a.e. $x \in \mathbb{R}^{n+k}$ and hence, if we let h be the homotopy $h(t,x) = tx + (1-t)\lambda x$, we have $h_\#([\![(0,1)]\!] \times C) = 0$, and then by the homotopy formula 26.22 (since $\partial C = 0$) we have $\eta_{0,\lambda\#}C = C$ as required. Finally since $\operatorname{spt} C$ has locally finite $\mathcal{H}^n$-measure (indeed by 17.8 $\operatorname{spt} C$ is the closed set $\{y \in \mathbb{R}^{n+k} : \Theta^n(\mu_C, y) \geq 1\}$), we have

$$[\![F]\!] = [\![\tilde{F}]\!] ,$$

where $\tilde{F}$ is the (open) set $\{y \in T_xN \sim \operatorname{spt} C : \Theta^{n+1}(\mathcal{H}^{n+1}, T_xN, y) = 1\}$. Evidently $\eta_{0,\lambda}(\tilde{F}) = \tilde{F}$ (because $\eta_{0,\lambda}(\operatorname{spt} C) = \operatorname{spt} C$). Hence the required result is established with $\tilde{F}$ in place of F.

37.6 COROLLARY* *Suppose* T *is as in* 37.4 *and in addition suppose there is an* n*-dimensional submanifold* Σ *embedded in* $\mathbb{R}^{n+k}$ *with* $x \in \Sigma \subset N \cap U$ *for some* $x \in \operatorname{spt} T$, *and suppose* $\operatorname{spt} T \sim \Sigma$ *lies locally, near* x, *on one side of* Σ. *Then* $x \in \operatorname{reg} T$. (reg T is as in 36.1.)

Proof Let $C = \partial[\![F]\!]$ $(F \subset T_xN)$ be any tangent cone for T at x. By assumption, $\operatorname{spt}[\![F]\!] \subset \bar{H}$, where H is an open $\frac{1}{2}$-space in T_xN with $0 \in \partial H$. Then, by 36.5, $\operatorname{spt} C \subset \partial H$ and hence by the constancy theorem 26.27,

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* Cf. Miranda [MM1]

since C is integer multiplicity rectifiable, it follows that $C = \pm\partial[\![H]\!]$. However $\operatorname{spt}[\![F]\!] \subset \bar{H}$, hence $C = +\partial[\![H]\!]$. Then $\Theta^n(\mu_C, y) \equiv 1$ for $y \in \partial H$, and in particular $\Theta^n(\mu_C, 0)$ $(= \Theta^n(\mu_T, x)) = 1$, so that $x \in \operatorname{reg} T$ (by Allard's theorem 24.2) as required.

We next want to prove the main regularity theorem for codimension 1 currents. We continue to define $\operatorname{sing} T$, $\operatorname{reg} T$ as in 36.1.

37.7 THEOREM *Suppose* $T = \partial[\![E]\!] \in \mathcal{D}_n(U)$ *is integer multiplicity, with* $E \subset N \cap U$, $N \in \mathcal{O}$, *and* T *minimizing in* $N \cap U$. *Then* $\operatorname{sing} T = \emptyset$ *for* $n \le 6$, $\operatorname{sing} T$ *is locally finite in* U *for* $n = 7$, *and* $\mathcal{H}^{n-7+\alpha}(\operatorname{sing} T) = 0$ $\forall\, \alpha > 0$ *in case* $n > 7$.

Proof We are going to use the abstract dimension reducing argument of Appendix A (Cf. the proof of Theorem 36.4).

To begin we note that it is enough (by re-scaling, translation, and restriction) to assume that

(1) $$U = B_2(0)$$

and to prove that

(2) $$\begin{cases} \operatorname{sing} T \cap B_1(0) = \emptyset \text{ if } n \le 6, \ \operatorname{sing} T \cap B_1(0) \text{ discrete if } n = 7, \\ \mathcal{H}^{n-7+\alpha}(\operatorname{sing} T \cap B_1(0)) = 0 \quad \forall\, \alpha > 0 \text{ if } n > 7. \end{cases}$$

Let $\mathcal{T}$ be the set of currents as defined in the proof of 36.4,* and for each $S \in \mathcal{T}$ let ϕ_S be the function : $\mathbb{R}^{n+k} \to \mathbb{R}^{n+1}$ associated with S as in Remark 35.4. Also, let

$$F = \{\phi_S : S \in \mathcal{T}\}$$

and define

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* We still have $\Theta^n(\mu_S, x) = 1$ for μ_S-a.e. $x \in U_S$, this time by 37.2 and 37.1 (**).

$$\text{sing } \phi_S = \text{sing } S .$$

(sing S as defined in 36.1.)

By Theorem A.4 we then have either sing $S = \emptyset$ for all $S \in \mathcal{T}$ (and hence sing $T = \emptyset$) or

(3) $$\dim B_1(0) \cap \text{sing } S \leq d ,$$

where $d \in [0,n-1]$ is the integer such that

$$\dim B_1(0) \cap \text{sing } S \leq d \quad \text{for all} \quad S \in \mathcal{T}$$

and such that there is $S \in \mathcal{T}$ and a d-dimensional subspace L of $\mathbb{R}^{n+k}$ such that

$$\eta_{x,\lambda\#} S = S \qquad \forall\, x \in L ,\ \lambda > 0$$

and

(4) $$\text{sing } S = L .$$

Supposing without loss of generality that $L = \mathbb{R}^d \times \{0\}$, we then (by Lemma 35.5) have

(5) $$S = [\![R^d]\!] \times S_0$$

where $\partial S_0 = 0$, S_0 is minimizing in $\mathbb{R}^{n+k-1}$, and sing $S_0 = \{0\}$. (With S as in (5), sing $S_0 = \{0\} \Leftrightarrow$ (4).) Also, by definition of $\mathcal{T}$, spt $S \subset$ some (n+1)-dimensional subspace of $\mathbb{R}^{n+k}$, hence without loss of generality we have that S_0 is an (n-d)-dimensional minimizing cone in $\mathbb{R}^{n-d+1}$ with sing $S_0 = \{0\}$. Then by the result of J.Simons (see Appendix B) we have $n-d > 6$; i.e. $d \leq n-7$. Notice that this contradicts $d \geq 0$ in case $n < 7$.

Thus for $n < 7$ we must have $\text{sing}\, T = \emptyset$ as required. If $n = 7$, $\text{sing}\, T$ is discrete by the last part of Theorem A.4.

37.8 COROLLARY *If* T *is as in* 37.7, *and if* $T_1 \in \mathcal{D}_n(U)$ *is obtained by equipping a component of* $\text{reg}\, T$ *with multiplicity* 1 *and with the orientation of* T, *then* $\partial T_1 = 0$ *(in* U*) and* T_1 *is minimizing in* $U \cap N$.

37.9 REMARK Notice that this means we can write

(*) $$T = \sum_{j=1}^{\infty} T_j ,$$

where each T_j is obtained by equipping a component M_j of $\text{reg}\, T$ with multiplicity 1 and with the orientation of T ; then $M_i \cap M_j = \emptyset$ $\forall i \neq j$, $\partial T_j = 0$, and T_j is minimizing in U $\forall j$. Furthermore (since $\mu_{T_j}(B_\rho(x)) \geq c\rho^n$ for $B_\rho(x) \subset U$ and $x \in \text{spt}\, T_j$ by virtue of 33.2 and the monotonicity formula 17.6(1)) only finitely many T_j can have support intersecting a given compact subset of U .

Proof of 37.8 The main point is to prove

(1) $$\partial T_1 = 0 \text{ in } U .$$

The fact that T_1 is minimizing in U will then follow from 33.4 and the fact that $\underline{\underline{M}}_W(T_1) + \underline{\underline{M}}_W(T - T_1) = \underline{\underline{M}}_W(T)$ $\forall\, W \subset\subset U$.

To check (1) let $\omega \in \mathcal{D}^{n-1}(U)$ be arbitrary and note that if $\zeta \equiv 0$ in some neighbourhood of $\text{spt}\, T \sim M_1$

(2) $$T_1(d(\zeta\omega)) = T(d(\zeta\omega)) = \partial T(\zeta\omega) = 0 .$$

Now corresponding to any $\varepsilon > 0$ we construct ζ as follows: since $\mathcal{H}^{n-1}(\text{sing}\, T) = 0$ (by 37.7) and since $\text{sing}\, T \cap \text{spt}\, \omega$ is compact, we can find a finite collection $\{B_{\rho_j}(\xi_j)\}_{j=1,\ldots,P}$ of balls with $\xi_j \in \text{sing}\, T \cap \text{spt}\, \omega$

and $\sum_{j=1}^{P} \rho_j^{n-1} < \varepsilon$. For each $j = 1,\ldots,P$ let $\phi_j \in C_C^\infty(\mathbb{R}^{n+k})$ be such that $\phi_j \equiv 1$ on $\bar{B}_{\rho_j}(\xi_j)$, $\phi_j = 0$ on $\mathbb{R}^{n+k} \sim B_{2\rho_j}(\xi_j)$, and $0 \le \phi_j \le 1$ everywhere. Now choose $\zeta = \prod_{j=1}^{P} \phi_j$ in a neighbourhood of spt T_1 and so that $\zeta \equiv 0$ in a neighbourhood of spt $T \sim$ spt T_1 . Then $d\zeta = \sum_{i=1}^{P} \prod_{j\neq i}^{P} \phi_j d\phi_i$ on spt T_1 , and hence

$$|d(\zeta\omega) - \zeta d\omega| \le c|\omega| \sum_{j=1}^{P} \rho_j^{n-1} \le c\varepsilon|\omega| \quad \text{on spt } T_1 .$$

Then letting $\varepsilon \downarrow 0$ in (2), and noting that $\zeta d\omega \to d\omega$ $\mathcal{H}^n$-a.e. in spt $T_1 \cap N \cap$ spt ω (and using $|\zeta| \le 1$) , we conclude $T_1(d\omega) = 0$. That is $\partial T_1 = 0$ in U as required.

Finally we have the following lemma.

37.10 LEMMA *If* $T_1 = \partial[\![E_1]\!]$, $T_2 = \partial[\![E_2]\!] \in \mathcal{D}_n(U)$, U *bounded,* $E_1, E_2 \subset U \cap N$, N *of class* C^4 , $N \in \mathcal{O}$, T_1 , T_2 *minimizing in* $U \cap N$, reg T_1 , reg T_2 *are connected, and* $E_1 \cap V \subset E_2 \cap V$ *for some neighbourhood* V *of* ∂U , *then* spt $[\![E_1]\!] \subset$ spt$[\![E_2]\!]$ *and either* $[\![E_1]\!] = [\![E_2]\!]$ *or* spt $T_1 \cap$ spt $T_2 \subset$ sing $T_1 \cap$ sing T_2 .

Proof Since $\mathcal{H}^{n+1}(\text{spt } T_j) = 0$ (in fact spt T_j has locally finite $\mathcal{H}^n$-measure in U by virtue of the fact that $\Theta^n(\mu_{T_j}, x) \ge 1$ $\forall\, x \in$ spt T_j) , we may assume that E_1 and E_2 are open with $U \cap \partial E_j = U \cap \partial \bar{E}_j =$ spt T_j , $j = 1,2$.

Let S_1 , $S_2 \in \mathcal{D}_n(U)$ be the currents defined by

$$S_1 = \partial[\![E_1 \cap E_2]\!] , \quad S_2 = \partial[\![E_1 \cup E_2]\!] .$$

Using the hypothesis concerning V we have

$$\text{(1)} \qquad S_j \llcorner (V \cap U) = T_j \llcorner (V \cap U) , \quad j = 1,2 .$$

On the other hand we trivially have

$$[\![E_1 \cap E_2]\!] + [\![E_1 \cup E_2]\!] = [\![E_1]\!] + [\![E_2]\!] \ ,$$

so (applying ∂) we get

$$S_1 + S_2 = T_1 + T_2 \ . \tag{2}$$

Furthermore $E_1 \cap E_2 \subset E_1 \cup E_2$, so

$$\begin{aligned} \underline{\underline{M}}_W(S_1) + \underline{\underline{M}}_W(S_2) &= \underline{\underline{M}}_W(S_1+S_2) \\ &= \underline{\underline{M}}_W(T_1+T_2) \quad \text{(by (2))} \\ &\leq \underline{\underline{M}}_W(T_1) + \underline{\underline{M}}_W(T_2) \end{aligned} \tag{3}$$

$\forall\ W \subset\subset U$. On the other hand, choosing an open V_0 so that $\partial U \subset V_0 \subset\subset V$, and using (1) together with the fact that T_1 is minimizing, we have

$$\underline{\underline{M}}_W(S_1) \geq \underline{\underline{M}}_W(T_1) \ , \ W = U \sim \bar{V}_0 \ ,$$

and hence (combining this with (3))

$$\underline{\underline{M}}_W(S_2) \leq \underline{\underline{M}}_W(T_2)$$

for $W = U \sim \bar{V}_0$. Thus (using (1) with $j = 2$) S_2 is minimizing in U. Likewise S_1 is minimizing in U.

We next want to prove that either $T_1 = T_2$ or $\operatorname{reg} T_1 \cap \operatorname{reg} T_2 = \emptyset$. Suppose $\operatorname{reg} T_1 \cap \operatorname{reg} T_2 \neq \emptyset$. If the tangent spaces of $\operatorname{reg} T_1$ and $\operatorname{reg} T_2$ coincide at every point of their intersection, then using suitable local coordinates $(x,z) \in \mathbb{R}^n \times \mathbb{R}$ for N near a point $\xi \in \operatorname{reg} T_1 \cap \operatorname{reg} T_2$, we can write

$$\operatorname{reg} T_j = \operatorname{graph} u_j \ , \ j = 1,2 \ ,$$

where $Du_1 = Du_2$ at each point where $u_1 = u_2$, and where both u_1, u_2 are (weak) C^1 solutions of the equation

$$\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial p_i}(x,u,Du)\right) - \frac{\partial F}{\partial z}(x,u,Du) = 0 ,$$

where $F = F(x,z,p)$, $(x,z,p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, is the area functional for graphs $z = u(x)$ relative to the local coordinates x, z for N. Since N is C^4 we then deduce (from standard quasilinear elliptic theory - see e.g. [GT]) that u_1, u_2 are $C^{3,\alpha}$. Now the difference $u_1 - u_2$ of the solutions evidently satisfies an equation of the general form

$$D_j(a_{ij}D_iu) + b_iD_iu + cu = 0 ,$$

where a_{ij}, b_i, c are $C^{2,\alpha}$. By standard unique continuation results (see e.g. [PM]) we then see that $Du_1 = Du_2$ at each point where $u_1 = u_2$ is impossible if $u_1 - u_2$ changes sign. On the other hand the Harnack inequality ([GT]) tells us that either $u_1 \equiv u_2$ or $|u_1-u_2| > 0$ in case $u_1 - u_2$ does not change sign. Thus we deduce that either $T_1 = T_2$ or $\text{reg}\, T_1 \cap \text{reg}\, T_2 = \emptyset$ or there is a point $\xi \in \text{reg}\, T_1 \cap \text{reg}\, T_2$ such that $\text{reg}\, T_1$ and $\text{reg}\, T_2$ intersect *transversely* at ξ. But then we would have $\mathcal{H}^{n-1}(\text{sing}\, \partial[\![E_1 \cap E_2]\!]) > 0$, which by virtue of 37.7 contradicts the fact (established above) that $\partial[\![E_1 \cap E_2]\!]$ is minimizing in U.

Thus either $T_1 = T_2$ or $\text{reg}\, T_1 \cap \text{reg}\, T_2 = \emptyset$, and it follows in either case that $E_1 \subset E_2$. On the other hand we then have $\text{sing}\, T_1 \cap \text{reg}\, T_2 = \emptyset$ and $\text{sing}\, T_2 \cap \text{reg}\, T_1 = \emptyset$ by virtue of Corollary 37.6. Thus we conclude that $E_1 \subset E_2$ and $\text{spt}\, T_1 \cap \text{spt}\, T_2 \subset \text{sing}\, T_1 \cap \text{sing}\, T_2$ as required.