CHAPTER 4

THEORY OF RECTIFIABLE n-VARIFOLDS

Let M be a countably n-rectifiable, $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+k}$, and let θ be a positive locally $\mathcal{H}^n$-integrable function on M. Corresponding to such a pair (M, θ) we define the rectifiable n-varifold $\underline{\underline{v}}(M,\theta)$ to be simply the equivalence class of all pairs $(\tilde{M},\tilde{\theta})$, where $\tilde{M}$ is countably n-rectifiable with $\mathcal{H}^n((M\sim\tilde{M})\cup(\tilde{M}\sim M)) = 0$ and where $\tilde{\theta} = \theta$ $\mathcal{H}^n$-a.e. on $M\cap\tilde{M}$.* θ is called the *multiplicity function* of $\underline{\underline{v}}(M,\theta)$. $\underline{\underline{v}}(M,\theta)$ is called an integer multiplicity rectifiable n-varifold (more briefly, an *integer* n-*varifold*) if the multiplicity function is integer-valued $\mathcal{H}^n$-a.e.

In this chapter and in Chapter 5 we develop the theory of general n-rectifiable varifolds, particularly concentrating on *stationary* (see §16) rectifiable n-varifolds, which generalize the notion of classical minimal submanifolds of $\mathbb{R}^{n+k}$. The key section is §17, in which we obtain the monotonicity formulae; much of the subsequent theory is based on these and closely related formulae.

§15. BASIC DEFINITIONS AND PROPERTIES

Associated to a rectifiable n-varifold $V = \underline{\underline{v}}(M,\theta)$ (as described above) there is a Radon measure μ_V (called the *weight measure* of V) defined by

15.1 $$\mu_V = \mathcal{H}^n \llcorner \theta ,$$

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* We shall see later that this is essentially equivalent to Allard's ([AW1]) notion of n-dimensional rectifiable varifold. In case $M\subset U$, U open in $\mathbb{R}^{n+k}$ and θ is locally $\mathcal{H}^n$-integrable in U, we say $V = \underline{\underline{v}}(M,\theta)$ (as defined above) is *a rectifiable* n-*varifold in* U.

where we adopt the convention that $\theta \equiv 0$ on $\mathbb{R}^{n+k} \sim M$. Thus for $\mathcal{H}^n$-measurable A,

$$\mu_V(A) = \int_{A\cap M} \theta \, d\mathcal{H}^n ,$$

The *mass* (or weight) of V, $\underline{\underline{M}}(V)$, is defined by

15.2 $$\underline{\underline{M}}(V) = \mu_V(\mathbb{R}^{n+k}) .$$

Notice that, by virtue of Theorem 11.8, an abstract Radon measure μ is actually μ_V for some rectifiable varifold V if and only if μ has an approximate tangent space T_x with multiplicity $\theta(x) \in (0,\infty)$ for μ-a.e. $x \in \mathbb{R}^{n+k}$. (See the statement of Theorem 11.8 for the terminology.) In this case $V = \underline{\underline{v}}(M,\theta)$, where $M = \{x : \Theta^{*n}(\mu,x) > 0\}$.

Given a rectifiable n-varifold $V = \underline{\underline{v}}(M,\theta)$, we define the tangent space T_xV to be the approximate tangent space of μ_V (as defined in the statement of Theorem 11.8) whenever this exists. Thus

15.3 $$T_xV = T_xM \quad \mathcal{H}^n\text{-a.e. } x \in M$$

where T_xM is the approximate tangent space of M with respect to the multiplicity θ. (See 11.4, 11.5.)

We also define, for $V = \underline{\underline{v}}(M,\theta)$,

15.4 $$\operatorname{spt} V = \operatorname{spt} \mu_V ,$$

and for any $\mathcal{H}^n$-measurable subset $A \subset \mathbb{R}^{n+k}$, $V \llcorner A$ is the rectifiable n-varifold defined by

15.5 $$V \llcorner A = \underline{\underline{v}}(M\cap A, \theta|(M\cap A)) .$$

Given $V = \underline{\underline{v}}(M,\theta)$ and a sequence $V_k = \underline{\underline{v}}(M_k,\theta_k)$ of rectifiable n-varifolds, we say that $V_k \to V$ provided $\mu_{V_k} \to \mu_V$ in the usual sense of Radon measures. (Notice that this is *not* varifold convergence in the sense of Chapter 8.)

Next we want to discuss the notion of mapping a rectifiable n-varifold relative to a Lipschitz map. Suppose $V = \underline{\underline{v}}(M,\theta)$, $M \subset U$, U open in $\mathbb{R}^{n+k}$, W open in $\mathbb{R}^{n+k_1}$ and suppose $f : \operatorname{spt} V \cap U \to W$ is proper*, Lipschitz and 1:1. Then we define the *image* varifold $f_\# V$ by

15.6 $$f_\# V = \underline{\underline{v}}(f(M), \theta \circ f^{-1}) .$$

We leave it to the reader to check using 12.5 that $f(M)$ is countably n-rectifiable and that $\theta \circ f^{-1}$ is locally $\mathcal{H}^n$-integrable in W, and therefore that 15.6 does define a rectifiable n-varifold in W. More generally if f satisfies the conditions above, except that f is not necessarily 1:1, then we define $f_\# V$ by

$$f_\# V = \underline{\underline{v}}(f(M), \tilde{\theta}) ,$$

where $\tilde{\theta}$ is defined on $f(M)$ by $\sum_{x \in f^{-1}(y) \cap M} \theta(x) \left(\equiv \int_{f^{-1}(y) \cap M} \theta \, d\mathcal{H}^0 \right)$. Notice that $\tilde{\theta}$ is locally $\mathcal{H}^n$-integrable in W by virtue of the area formula (see §12), and in fact

15.7 $$\underline{\underline{M}}(f_\# V) = \int_{f(M)} \tilde{\theta} \, d\mathcal{H}^n$$

$$\equiv \int_M J_M f \, \theta \, d\mathcal{H}^n ,$$

where $J_M f$ is the Jacobian of f relative to M as defined in §12; that is

$$J_M f = \sqrt{\det (d^M f_x)^* \circ d^M f_x}$$

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* i.e. $f^{-1}(K) \cap \operatorname{spt} V$ is compact whenever K is a compact subset of W.

where $d^M f_x : T_x M \to \mathbb{R}^{n+k_1}$ is the linear map induced by f as described in §12.

§16. FIRST VARIATION

Suppose $\{\phi_t\}_{-\varepsilon<t<\varepsilon}$ $(\varepsilon>0)$ is a 1-parameter family of diffeomorphisms of an open set U of $\mathbb{R}^{n+k}$ satisfying

16.1

(i) $\phi_0 = \underline{\underline{1}}_U$, $\exists$ compact $K \subset U$ such that $\phi_t | U \sim K = \underline{\underline{1}}_{U\sim K}$ $\forall$ $t \in (-\varepsilon,\varepsilon)$

(ii) $(x,t) \to \phi_t(x)$ is a smooth map $U \times (-\varepsilon,\varepsilon) \to U$.

Then if $V = \underline{\underline{v}}(M,\theta)$ is a rectifiable n-varifold and if $K \subset U$ is compact as in (i) above, we have, according to 15.7 above,

$$\underline{\underline{M}}(\phi_{t\#}(V \llcorner K)) = \int_{M\cap K} J_M \, \phi_t \, \theta \, d\mathcal{H}^n ,$$

and we can compute the *first variation* $\frac{d}{dt} \underline{\underline{M}}(\phi_{t\#}(V \llcorner K))\Big|_{t=0}$ exactly as in §9. We thus deduce

16.2
$$\frac{d}{dt} \underline{\underline{M}}(\phi_{t\#}(V \llcorner K))\Big|_{t=0} = \int_M \operatorname{div}_M X \, d\mu_V ,$$

where $X|_x = \frac{\partial}{\partial t} \phi(t,x)\Big|_{t=0}$ is the initial velocity vector for the family $\{\phi_t\}$ and where $\operatorname{div}_M X$ is as in §7:

$$\operatorname{div}_M X = \nabla_j^M \, X^j (\equiv e_j \cdot (\nabla^M X^j)) .$$

($\nabla^M X^j$ as in §12) we can therefore make the following definition.

16.3 DEFINITION $V = \underline{\underline{v}}(M,\theta)$ is *stationary in* U if $\int \operatorname{div}_M X \, d\mu_V = 0$ for any C^1 vector field X on U having compact support in U .

More generally if N is an $(n+k_1)$-dimensional submanifold of $\mathbb{R}^{n+k}$ $(k_1 \le k)$, if U is an open subset of N, if $M \subset N$, and if the family $\{\phi_t\}$ is as in 16.1, then by Lemma 9.6 it is reasonable to make the following definition (in which $\bar{B}$ is the second fundamental form of N).

16.4 DEFINITION If $U \subset N$ is open and $M \subset N$ is as above, then we say $V = \underline{\underline{v}}(M,\theta)$ is *stationary in* U if

$$\int_U \operatorname{div}_M X \, d\mu_V = -\int_U X \cdot \bar{\underline{\underline{H}}}_M \, d\mu_V$$

whenever X is a C^1 vector field in U with compact support in U; here $\bar{\underline{\underline{H}}}_M = \sum_{i=1}^{n} \bar{B}_x(\tau_i,\tau_i)$, $\tau_1,\ldots,\tau_n$ any orthonormal basis for the approximate tangent space T_xM of M at x. (Notice that by 16.2, which remains valid when $U \subset N$, this is equivalent to $\frac{d}{dt}\underline{\underline{M}}(\phi_{t\#}(V \llcorner K))\Big|_{t=0} = 0$ whenever $\{\phi_t\}$ are as in 16.1 with $U \subset N$.)

It will be convenient to generalize these notions of stationarity in the following way:

16.5 DEFINITION Suppose $\underline{\underline{H}}$ is a locally μ_V-integrable function on $M \cap U$ with values in $\mathbb{R}^{n+k}$. We say that $V(= \underline{\underline{v}}(M,\theta))$ has *generalized mean curvature* $\underline{\underline{H}}$ in U (U open in $\mathbb{R}^{n+k}$) if

$$\int_U \operatorname{div}_M X \, d\mu_V = -\int_U X \cdot \underline{\underline{H}} \, d\mu_V$$

whenever X is a C^1 vector field on U with compact support in U.

16.6 REMARKS

(1) Notice that in case M is smooth with $(\bar{M} \sim M) \cap U = \emptyset$, and when $\theta \equiv 1$, the generalized mean curvature of V is exactly the ordinary mean

curvature of M as described in §7 (see in particular 7.6).

(2) V is stationary in U (U open in $\mathbb{R}^{n+k}$) in the sense of 16.3 precisely when it has zero generalized mean curvature in U, and V is stationary in U (U open in N) in the sense of 16.4 precisely when it has generalized mean curvature $\bar{\underline{\underline{H}}}_M$.

§17. MONOTONICITY FORMULAE AND BASIC CONSEQUENCES

In this section we assume that U is open in $\mathbb{R}^{n+k}$, $V = \underline{\underline{v}}(M,\theta)$ has generalized mean curvature $\underline{\underline{H}}$ in U (see 16.5), and we write μ for μ_V ($= \mathcal{H}^n \llcorner \theta$ as in 15.1).

Our aim is to obtain information about V by making appropriate choices of X in the formula (see 16.5)

17.1 $$\int \operatorname{div}_M X \, d\mu = -\int X \cdot \underline{\underline{H}} \, d\mu , \quad X \in C_c^1(U;\mathbb{R}^{n+k}) .$$

First we choose $X_x = \gamma(r)(x-\xi)$, where $\xi \in U$ is fixed, $r = |x-\xi|$, and γ is a $C^1(\mathbb{R})$ function with

$$\gamma'(t) \le 0 \quad \forall\, t , \quad \gamma(t) \equiv 1 \quad \text{for} \quad t \le \rho/2 , \quad \gamma(t) \equiv 0 \quad \text{for} \quad t > \rho ,$$

where $\rho > 0$ is such that $\bar{B}_\rho(\xi) \subset U$. (Here and subsequently $B_\rho(\xi)$ denotes the *open* ball in $\mathbb{R}^{n+k}$ with centre ξ and radius ρ.)

For any $f \in C^1(U)$ and any $x \in M$ such that T_xM exists (see 11.4 - 11.6) we have (by 12.1) $\nabla^M f(x) = \sum_{j,\ell=1}^{n+k} e^{j\ell} D_\ell f(x) e_j$, where $D_\ell f$ denotes the partial derivative $\frac{\partial f}{\partial x^\ell}$ of f taken in U and where $(e^{j\ell})$ is the matrix of the orthogonal projection of $\mathbb{R}^{n+k}$ onto T_xM . Thus, writing $\nabla_j^M = e_j \cdot \nabla^M$ (as in §16), with the above choice of X we deduce

$$\operatorname{div}_M X \equiv \sum_{j=1}^{n+k} \nabla_j^M X^j = \gamma(r) \sum_{j=1}^{n+k} e^{jj} + r\gamma'(r) \sum_{j,\ =1}^{n+k} \frac{(x^j-\xi^j)}{r} \frac{(x^\ell-\xi^\ell)}{r} e^{j\ell} .$$

Since $(e^{j\ell})$ represents orthogonal projection onto T_xM we have $\sum_{j=1}^{n+k} e^{jj} = n$ and $\sum_{j,\ell=1}^{n+k} \frac{(x^j-\xi^j)}{r} \frac{(x^\ell-\xi^\ell)}{r} e^{j\ell} = 1 - |D^\perp r|^2$, where $D^\perp r$ denotes the orthogonal projection of Dr (which is a vector of length = 1) onto $(T_xM)^\perp$. The formula 17.1 thus yields

$$(*) \qquad n\int \gamma(r)\,d\mu + \int r\gamma'(r)\,d\mu = -\int \underline{\underline{H}} \cdot (x-\xi)\ \gamma(r)\,d\mu + \int r\gamma'(r)\,|(Dr)^\perp|^2 d\mu$$

provided $\bar{B}_\rho(\xi) \subset U$. Now take ϕ such that $\phi(t) \equiv 1$ for $t \le 1/2$, $\phi(t) = 0$ for $t \ge 1$ and $\phi'(t) \le 0$ for all t . Then we can use (*) with $\gamma(r) = \phi(r/\rho)$. Since $r\gamma'(r) = r\rho^{-1}\phi'(r/\rho) = -\rho \frac{\partial}{\partial\rho} [\phi(r/\rho)]$ this gives

$$n\, I(\rho) - \rho\, I'(\rho) = J'(\rho) - L(\rho)$$

where

$$I(\rho) = \int \phi(r/\rho)\,d\mu , \quad L(\rho) = \int \phi(r/\rho)(x-\xi)\cdot \underline{\underline{H}}\ d\mu$$

$$J(\rho) = \int \phi(r/\rho)\,|(Dr)^\perp|^2 d\mu .$$

Thus, multiply by ρ^{-n-1} and rearranging we have

17.2 $$\frac{d}{d\rho} [\rho^{-n} I(\rho)] = \rho^{-n} J'(\rho) + \rho^{-n-1} L(\rho) .$$

Thus letting ϕ increase to the characteristic function of the interval $(-\infty,1)$, we obtain, in the distribution sense, 17.3 $$\frac{d}{d\rho}\left(\rho^{-n}\mu(B_\rho(\xi))\right) = \frac{d}{d\rho}\int_{B_\rho(\xi)} \frac{|D^\perp r|^2}{r^n}\,d\mu + \rho^{-n-1}\int_{B_\rho(\xi)} (x-\xi)\cdot\underline{\underline{H}}\,d\mu \;.$$

This is the fundamental *monotonicity identity*; since $\mu(B_\rho(\xi))$ and $\int_{B_\rho(\xi)} \frac{|D^\perp r|^2}{r^n}$ are *increasing* in ρ it also holds in the *classical sense* for a.e. $\rho > 0$ such that $\bar{B}_\rho(\xi) \subset U$. Notice that if $\underline{\underline{H}} \equiv 0$ then 17.3 tells us that the ratio $\rho^{-n}\mu(B_\rho(\xi))$ is non-decreasing in ρ. Generally, by integrating with respect to ρ in 17.3 we get the identity

17.4 $$\sigma^{-n}\,\mu(B_\sigma(\xi)) = \rho^{-n}\,\mu(B_\rho(\xi)) - \int_{B_\rho(\xi)\sim B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n}\,d\mu$$
$$- \frac{1}{n}\int_{B_\rho(\xi)} (x-\xi)\cdot\underline{\underline{H}}\left(\frac{1}{r_\sigma^n} - \frac{1}{\rho^n}\right) d\mu \;,$$

for all $0 < \sigma \le \rho$ with $\bar{B}_\rho(\xi) \subset U$, where $r_\sigma = \max\{r,\sigma\}$, so that if $\underline{\underline{H}} \equiv 0$ we have the particularly interesting identity

17.5 $$\sigma^{-n}\,\mu(B_\sigma(\xi)) = \rho^{-n}\,\mu(B_\rho(\xi)) - \int_{B_\rho(\xi)\sim B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n}\,d\mu \;.$$

We now want to examine the important question of what 17.3 tells us in case we assume boundedness and L^p conditions on $\underline{\underline{H}}$.

17.6 THEOREM *If* $\xi \in U$, $0 < \alpha \le 1$, $\Lambda \ge 0$, *and if*

(*) $$\alpha^{-1}\int_{B_\rho(\xi)} |\underline{\underline{H}}|\,d\mu \le \Lambda(\rho/R)^{\alpha-1}\,\mu(B_\rho(\xi)) \quad \text{for all } \rho \in (0,R) \;,$$

where $\bar{B}_R(\xi) \subset U$, *then* $e^{\Lambda R^{1-\alpha}\rho^\alpha}\,\rho^{-n}\mu(B_\rho(\xi))$ *is a non-decreasing function of* $\rho \in (0,R)$, *and in fact*

(1) $$e^{\Lambda R^{1-\alpha}\sigma^\alpha}\,\sigma^{-n}\mu(B_\sigma(\xi)) \le e^{\Lambda R^{1-\alpha}\rho^\alpha}\,\rho^{-n}\mu(B_\rho(\xi)) - \int_{B_\rho(\xi)\sim B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n}\,d\mu$$

whenever $0 < \sigma < \rho \leq R$. *Also,*

$$(2) \qquad e^{-\Lambda R^{1-\alpha}\sigma^{\alpha}}\sigma^{-n}\mu(B_\sigma(\xi)) \geq e^{-\Lambda R^{1-\alpha}\rho^{\alpha}}\rho^{-n}\mu(B_\rho(\xi)) - \int_{B_\rho(\xi)\sim B_\sigma(\xi)} \frac{|D^{\perp}r|^2}{r^n}\,d\mu$$

whenever $0 < \sigma < \rho \leq R$.

Proof To get (1) we simply multiply the identity 17.3 by the integrating factor $e^{\Lambda R^{1-\alpha}\rho^{\alpha}}$, whereupon, after using (*), we obtain

$$\frac{d}{d\rho}\left(e^{\Lambda R^{1-\alpha}\rho^{\alpha}}\rho^{-n}\mu(B_\rho(\xi))\right) \geq \frac{d}{d\rho}\int_{B_\rho(\xi)} \frac{|D^{\perp}r|^2}{r^n}\,d\mu ,$$ in the sense of distributions.

(2) is proved similarly except that this time we multiply through in 17.3 by the integrating factor $e^{-\Lambda R^{1-\alpha}\rho^{\alpha}}$.

17.7 THEOREM *If* $\xi \in U$, *and* $\left(\int_{B_R(\xi)} |\underline{\underline{H}}|^p\,d\mu\right)^{1/p} \leq \Gamma$, *where* $\overline{B_R}(\xi) \subset U$ *and* $p > n$, *then*

$$(\sigma^{-n}\mu(B_\sigma(\xi)))^{1/p} \leq (\rho^{-n}\mu(B_\rho(\xi)))^{1/p} + \frac{\Gamma}{p-n}(\rho^{1-n/p}-\sigma^{1-n/p})$$

whenever $0 < \sigma < \rho \leq R$.

Proof Using the Hölder inequality, we obtain from 17.2 that

$$\frac{d}{d\rho}(\rho^{-n}I(\rho)) \geq -\rho^{-n}\Gamma(I(\rho))^{1-1/p}$$

for L^1-a.e. $\rho \in (0,R)$. Hence

$$\frac{d}{d\rho}(\rho^{-n}I(\rho))^{1/p} \geq -p\,\rho^{-n/p}\,\Gamma .$$

Thus, integrating over (σ,ρ) and letting ϕ increase to the characteristic function of $(-\infty,1)$ as before, we deduce the required inequality.

17.8 COROLLARY *If* $\underline{\underline{H}} \in L^p_{loc}(\mu)$ *in* U *for some* $p > n$ *, then the density* $\Theta^n(\mu,x) = \lim_{\rho\downarrow 0} \frac{\mu(\bar{B}_\rho(x))}{\omega_n \rho^n}$ *exists at every point* $x \in U$ *, and* $\Theta^n(\mu,\cdot)$ *is an upper-semi-continuous function in* U *:*

$$\Theta^n(\mu,x) \geq \limsup_{y\to x} \Theta^n(\mu,y) \qquad \forall\, x \in U .$$

Proof The inequality 17.7 tells us that $(\rho^{-n}\mu(B_\rho(\xi))^{1/p} + \frac{1}{p-n}\Gamma\,\rho^{1-n/p}$ is a non-decreasing function of ρ ; hence $\lim_{\rho\downarrow 0}\rho^{-n}\mu(B_\rho(\xi))$ exists (and is the same as $\lim_{\rho\downarrow 0}\rho^{-n}\mu(\bar{B}_\rho(\xi)))$. We also deduce that

$$(\sigma^{-n}\mu(B_\sigma(y)))^{1/p} \leq (\rho^{-n}\mu(B_\rho(y)))^{1/p} + c\,\rho^{1-n/p}$$

$$\leq (\rho^{-n}\mu(B_{\rho+\varepsilon}(x)))^{1/p} + c\,\rho^{1-n/p}$$

whenever $\sigma < \rho$, $\varepsilon > 0$, $B_{\rho+\varepsilon}(x) \subset U$ and $|y-x| < \varepsilon$. Letting $\sigma \downarrow 0$ we thus have

$$(\Theta^n(\mu,y))^{1/p} \leq (\omega_n^{-1}(\rho+\varepsilon)^{-n}\mu(B_{\rho+\varepsilon}(x)))^{1/p}(1+\varepsilon/\rho)^{n/p} + c\,\rho^{1-n/p} .$$

Now let $\delta > 0$ be given and choose $\varepsilon \ll \rho < \delta$ so that

$$(\omega_n^{-1}(\rho+\varepsilon)^{-n}\mu(B_{\rho+\varepsilon}(x)))^{1/p}(1+\varepsilon/\rho)^{n/p} < (\Theta^n(\mu,x))^{1/p} + \delta .$$

Then the above inequality gives

$$(\Theta^n(\mu,y))^{1/p} \leq (\Theta^n(\mu,x))^{1/p} + c\,\rho^{1-n/p}$$

(c depends on x but is independent of δ, ε) provided $|y-x| < \varepsilon$. Thus the required upper-semi-continuity is proved.

17.9 REMARKS

(1) If $\theta \ge 1$ μ - a.e. in U, then $\Theta^n(\mu,x) \ge 1$ at *each* point of $\operatorname{spt}\mu \cap U$, and hence we can write $V \llcorner U = \underline{\underline{v}}(M_*,\theta_*)$ where $M_* = \operatorname{spt}\mu \cap U$, $\theta_*(x) = \Theta^n(\mu,x)$, $x \in U$. Thus $V \llcorner U$ is represented in terms of a relatively *closed* countably n-rectifiable set with *upper-semi-continuous* multiplicity function.

(2) If $\xi \in U$, $\Theta^n(\mu,\xi) \ge 1$, and $\left(\omega_n^{-1}\int_{B_R(\xi)} |\underline{\underline{H}}|^p d\mu\right)^{1/p} \le \Gamma(1-n/p)$, where $\bar{B}_R(\xi) \subset U$ and $p > n$, then both inequalities 17.6(1), (2) hold with $\Lambda = 2\Gamma R^{-n/p}$ and $\alpha = 1 - n/p$, provided $\Gamma\rho^{1-n/p} \le 1/2$. To see this, just use Hölder's inequality to give

$$(*) \qquad \int_{B_\rho(\xi)} |\underline{\underline{H}}|\, d\mu \le \Gamma(\mu(B_\rho(\xi)))^{1-1/p} = \Gamma\, \mu(B_\rho(\xi))(\mu(B_\rho(\xi)))^{-1/p} .$$

On the other hand, letting $\sigma \downarrow 0$ in 17.7 we have

$$\mu(B_\rho(\xi)) \ge \omega_n\rho^n(1-\Gamma\rho^{1-n/p})^p ,$$

so that $\mu(B_\rho(\xi)) \ge \frac{1}{2}p\,\omega_n\rho^n$ for $\Gamma\rho^{1-n/p} \le \frac{1}{2}$, and (*) gives $\int_{B_\rho(\xi)} |\underline{\underline{H}}|\, d\mu \le 2\,\Gamma\,\mu(B_\rho(\xi))\rho^{-n/p}$. Thus the hypotheses of 17.6 hold with $\Lambda = 2\,\Gamma\,R^{-n/p}$.

(3) Notice that either 17.6(1) or 17.7 give bounds of the form $\mu(B_\sigma(\xi)) \le \beta\sigma^n$, $0 < \sigma < R$ for suitable constant β. Such an inequality implies

$$\int_{B_\rho(\xi)} |x-\xi|^{\alpha-n} d\mu \le n\beta\alpha^{-1}\rho^\alpha$$

for any $\rho \in (0,R)$ and $0 < \alpha < n$. This is proved by using the following general fact with $f(t) = t^{-1}$, $t_0 = \rho^{-1}$, and with $n-\alpha$ in place of α.

17.10 LEMMA *If* X *is an abstract space,* μ *is a measure on* X , $\alpha > 0$, $f \in L^1(\mu)$, $f \geq 0$, *and if* $A_t = \{x \in X : f(x) > t\}$, *then*

$$\int_0^\infty t^{\alpha-1}\mu(A_t)\,dt = \alpha^{-1}\int_{A_0} f^\alpha\,d\mu .$$

More generally

$$\int_{t_0}^\infty t^{\alpha-1}\mu(A_t)\,dt = \alpha^{-1}\int_{A_{t_0}} (f^\alpha - t_0^\alpha)\,d\mu$$

for each $t_0 \geq 0$.

This is proved simply by applying Fubini's theorem on the product space $A_{t_0} \times [t_0,\infty)$ for $t_0 > 0$.

The observation of the following lemma is important.

17.11 LEMMA *Suppose* $\theta \geq 1$ μ-a.e. *in* U , $\underline{\underline{H}} \in L^p_{loc}(\mu)$ *in* U *for some* $p > n$. *If the approximate tangent space* T_xV (see §15) *exists at a given point* $x \in U$, *then* T_xV *is a "classical" tangent plane for* $\operatorname{spt}\mu$ *in the sense that*

$$\lim_{\rho\downarrow 0}\,(\sup\{\rho^{-1}\operatorname{dist}(y,T_xV) : y \in \operatorname{spt}\mu \cap B_\rho(x)\}) = 0 .$$

Proof For sufficiently small R (with $B_{2R}(x) \subset U$), 17.7, 17.8 (with $\sigma \downarrow 0$) evidently imply

(1) $$\omega_n^{-1}\rho^{-n}\mu(B_\rho(\xi)) \geq 1/2 , \quad 0 < \rho < R , \quad \xi \in \operatorname{spt}\mu \cap B_R(x) .$$

Using this we are going to prove that if $\alpha \in (0,1/2)$ and $\rho \in (0,R)$ then

(2) $$\mu(B_\rho(x) \sim \{y : \operatorname{dist}(y,T_xV) < \varepsilon\rho\}) < \frac{\omega_n}{2}\alpha^n\rho^n \Rightarrow$$

$$\operatorname{spt}\mu \cap B_{\rho/2}(x) \subset \{y : \operatorname{dist}(y,T_xV) < (\varepsilon+\alpha)\rho\} .$$

Indeed if $\xi \in \{y : \operatorname{dist}(y,T_xV) \geq (\varepsilon+\alpha)\rho\} \cap B_{\rho/2}(x)$, then $B_{\alpha\rho}(\xi) \subset B_\rho(x) \sim \{y : \operatorname{dist}(y,T_xV) < \varepsilon\rho\}$ and hence the hypothesis of (2) implies $\mu(B_{\alpha\rho}(\xi)) < \frac{1}{2}\omega_n\alpha^n\rho^n$. On the other hand (1) implies $\mu(B_{\alpha\rho}(\xi)) \geq \frac{1}{2}\omega_n\alpha^n\rho^n$, so we have a contradiction. Thus (2) is proved, and (2) evidently leads immediately to the required result.

§18. POINCARÉ AND SOBOLEV INEQUALITIES (*)

In this section we continue to assume that $V = \underline{\underline{v}}(M,\theta)$ has generalized mean curvature $\underline{\underline{H}}$ in U , and we again write μ for μ_V . We shall also assume $\theta \geq 1$ μ - a.e. $x \in U$ (so that (by 17.9) $\Theta^n(\mu,x) \geq 1$ everywhere in $\operatorname{spt}\mu \cap U$ if $\underline{\underline{H}} \in L^p_{loc}(\mu)$ for some $p > n$).

We begin by considering the possibility of repeating the argument of the previous section, but with $X_x = h(x)\gamma(r)(x-\xi)$ (rather than $X_x = \gamma(r)(x-\xi)$ as before), where h is a non-negative function in $C^1(U)$. In computing $\operatorname{div}_M X$ we will get the additional term $\gamma(r)(x-\xi)\cdot\nabla^M h$, and other terms will be as before with an additional factor $h(x)$ everywhere. Thus in place of 17.2 we get

18.1 $$\frac{\partial}{\partial\rho}(\rho^{-n}\tilde{I}(\rho)) = \rho^{-n}\frac{\partial}{\partial\rho}\int |(Dr)^\perp|^2 h\,\phi(r/\rho)\,d\mu + \rho^{-n-1}\int (x-\xi)\cdot[\nabla^M h + \underline{\underline{H}}h]\phi(r/\rho)\,d\mu$$

where now $\tilde{I}(\rho) = \int \phi(r/\rho)h\,d\mu$.

Thus

$$\frac{\partial}{\partial\rho}[\rho^{-n}\tilde{I}(\rho)] \geq \rho^{-n-1}\int (x-\xi)\cdot(\nabla^M h + \underline{\underline{H}}h)\phi(r/\rho)\,d\mu$$
$$\equiv R \text{ say .}$$

(*) The results of this section are not needed in the sequel.

We can estimate the right-side R here in two ways: if $|\underline{\underline{H}}| \le \Lambda$ we have

(*) $$R \ge -\rho^{-n-1}\int r|\nabla^M h|\ \phi(r/\rho)\,d\mu - (\Lambda\rho)\rho^{-n}\tilde{I}(\rho) .$$

Alternatively, without any assumption on $\underline{\underline{H}}$ we can clearly estimate

(**) $$R \ge -\rho^{-n-1}\int r(|\nabla^M h| + h|\underline{\underline{H}}|)\phi(r/\rho)\,d\mu .$$

If we use (*) in 18.1 and integrate (making use of 17.10) we obtain (after letting ϕ increase to the characteristic function of $(-\infty,1)$ as before)

18.2 $$\frac{\int_{B_\sigma(\xi)} h\ d\mu}{\omega_n\sigma^n} \le e^{\Lambda\rho}\left[\frac{\int_{B_\rho(\xi)} h\ d\mu}{\omega_n\rho^n} + \frac{1}{n\omega_n}\int_{B_\rho(\xi)}\frac{|\nabla^M h|}{|x-\xi|^{n-1}}\,d\mu\right] ,$$

provided $B_\rho(\xi) \subset U$ and $0 < \sigma < \rho$.

If instead we use (**) then we similarly get

$$\frac{\int_{B_\sigma(\xi)} h\ d\mu}{\omega_n\sigma^n} \le \frac{\int_{B_\rho(\xi)} h\ d\mu}{\omega_n\rho^n} + \omega_n^{-1}\int_\sigma^\rho \tau^{-n-1}\int_{B_\tau(\xi)} r(|\nabla^M h| + h|\underline{\underline{H}}|)\,d\mu d\tau .$$

and hence (by 17.10 again)

18.3 $$\frac{\int_{B_\sigma(\xi)} h\ d\mu}{\omega_n\sigma^n} \le \frac{\int_{B_\rho(\xi)} h\ d\mu}{\omega_n\rho^n} + (n\omega_n)^{-1}\int_{B_\rho(\xi)}\frac{(|\nabla^M h| + h|\underline{\underline{H}}|)}{|x-\xi|^{n-1}}\,d\mu$$

provided $B_\rho(\xi) \subset U$ and $0 < \sigma < \rho$.

If we let $\sigma \downarrow 0$ in 18.2 then we get (since $\Theta(\mu,\xi) \ge 1$ for $\xi \in \operatorname{spt}\mu$)

$$h(\xi) \le e^{\Lambda\rho}\left(\frac{\int_{B_\rho(\xi)} h\ d\mu}{\omega_n\rho^n} + \frac{1}{n\omega_n}\int_{B_\rho(\xi)}\frac{|\nabla^M h|}{|x-\xi|^{n-1}}\right), \ \xi \in \operatorname{spt}\mu ,\ B_\rho(\xi) \subset U .$$

We now state our Poincaré-type inequality.

18.4 THEOREM *Suppose* $h \in C^1(U)$, $h \geq 0$, $B_{2\rho}(\xi) \subset U$, $|\underline{\underline{H}}| \leq \Lambda$, $\theta \geq 1$ μ-a.e. *in* U *and*

(i) $$\mu\{x \in B_\rho(\xi) : h(x) > 0\} \leq (1-\alpha)\omega_n\rho^n , \quad e^{\Lambda\rho} \leq 1+\alpha$$

for some $\alpha \in (0,1)$. *Suppose also that*

(ii) $$\mu(B_{2\rho}(\xi)) \leq \Gamma \rho^n , \quad \Gamma > 0 .$$

Then there are constants $\beta = \beta(n,\alpha,\Gamma) \in (0,1/2)$ *and* $c = c(n,\alpha,\Gamma) > 0$ *such that*

$$\int_{B_{\beta\rho}(\xi)} h \, d\mu \leq c\rho \int_{B_\rho(\xi)} |\nabla^M h| \, d\mu .$$

Proof To begin we take β to be an arbitrary parameter in $(0,1/2)$ and apply 18.2 with $\eta \in B_{\beta\rho}(\xi) \cap \operatorname{spt}\mu$ in place of ξ . This gives

(1) $$h(\eta) \leq e^{\Lambda(1-\beta)\rho}\left(\frac{\int_{B_{(1-\beta)\rho}(\eta)} h \, d\mu}{\omega_n((1-\beta)\rho)^n} + \frac{1}{n\omega_n}\int_{B_{(1-\beta)\rho}(\xi)} \frac{|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu \right)$$

$$\leq e^{\Lambda\rho}\left((1-\beta)^{-n} \frac{\int_{B_\rho(\xi)} h \, d\mu}{\omega_n \rho^n} + \frac{1}{n\omega_n}\int_{B_\rho(\xi)} \frac{|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu \right) .$$

Now let γ be a fixed C^1 non-decreasing function on $\mathbb{R}$ with $\gamma(t) = 0$ for $t \leq 0$ and $\gamma(t) \leq 1$ everywhere, and apply (1) with $\gamma(h-t)$ in place of h , where $t \geq 0$ is fixed. Then by (1)

$$\gamma(h(\eta)-t) \leq \frac{1+\alpha}{n\omega_n}\int_{B_\rho(\xi)} \frac{\gamma'(h-t)|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu + (1-\alpha^2)(1-\beta)^{-n} .$$

Selecting β small enough so that $(1-\beta)^{-n}(1-\alpha^2) \le 1-\alpha^2/2$, we thus get

$$(2) \qquad \frac{\alpha^2}{2} \le \frac{1+\alpha}{n\omega_n} \int_{B_\rho(\xi)} \frac{\gamma'(h-t)|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu$$

for any $\eta \in B_{\beta\rho}(\xi) \cap \operatorname{spt}\mu$ such that $\gamma(h(\eta)-t) \ge 1$. Now let $\varepsilon > 0$ and choose γ such that $\gamma(t) \equiv 1$ for $t \ge 1+\varepsilon$. Then (2) implies

$$(3) \qquad 1 \le c \int_{B_\rho(\xi)} \frac{\gamma'(h-t)|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu \, , \quad \eta \in B_{\beta\rho}(\xi) \cap A_{t+\varepsilon} \, ,$$

where $A_\tau = \{y \in \operatorname{spt}\mu : h(y) > \tau\}$. Integrating over $A_{t+\varepsilon} \cap B_{\beta\rho}(\xi)$ we thus get (after interchanging the order of integration on the right)

$$(A_{t+\varepsilon} \cap B_{\beta\rho}(\xi)) \le c \int_{B_\rho(\xi)} \gamma'(h(x)-t)|\nabla^M h(x)| \left(\int_{B_{\beta\rho}(\xi)} \frac{1}{|x-\eta|^{n-1}} \, d\mu(\eta) \right) d\mu(x)$$

$$\le c\Gamma\rho \int_{B_\rho(\xi)} \gamma'(h-t)|\nabla^M h| \, d\mu$$

by hypothesis (ii) and Remark 17.9(3). Since $\gamma'(h(x)-t) = -\frac{\partial}{\partial t}\gamma(h(x)-t)$ we can now integrate over $t \in (0,\infty)$ to obtain (from 17.10) that

$$\int_{A_\varepsilon \cap B_{\beta\rho}(\xi)} (h-\varepsilon) \le c\Gamma\rho \int_{B_\rho(\xi)} |\nabla^M h| \, d\mu \, .$$

Letting $\varepsilon \downarrow 0$, we have the required inequality.

18.5 REMARK If we drop the assumption that $\theta \ge 1$, then the above argument still yields

$$\int_{\{x:\theta(x)\ge 1\} \cap B_{\beta\rho}(\xi)} h \, d\mu \le c\rho \int_{B_\rho(\xi)} |\nabla^M h| \, d\mu \, .$$

We can also prove a Sobolev inequality as follows.

18.6 THEOREM *Suppose* $h \in C_0^1(U)$, $h \geq 0$, *and* $\theta \geq 1$ μ - a.e. *in* U . *Then*

$$\left(\int h^{\frac{n}{n-1}}\, d\mu\right)^{\frac{n-1}{n}} \leq c\int (|\nabla^M h| + h|\underline{\underline{H}}|)\, d\mu\ , \quad c = c(n)\ .$$

<u>Note</u>: c does not depend on k .

Proof In the proof we shall need the following simple calculus lemma.

18.7 LEMMA *Suppose* f , g *are bounded and non-decreasing on* $(0,\infty)$ *and*

$$(1) \qquad 1 \leq \sigma^{-n} f(\sigma) \leq \rho^{-n} f(\rho) + \int_0^\rho \tau^{-n} g(\tau)\, d\tau\ , \quad 0 < \sigma < \rho < \infty\ .$$

then $\exists\ \rho$ *with* $0 < \rho < \rho_0 \equiv 2(f(\infty))^{1/n}$ $(f(\infty) = \lim_{\rho\uparrow\infty} f(\rho))$ *such that*

$$(2) \qquad f(5\rho) \leq \frac{1}{2}\ 5^n \rho_0\quad g(\rho)\ .$$

Proof of Lemma Suppose (2) is false for each $\rho \in (0,\rho_0)$. Then (1) $\Rightarrow$

$$1 \leq \sup_{0<\sigma<\rho_0} \sigma^{-n} f(\sigma) \leq \rho_0^{-n} f(\rho_0) + \frac{2.5^{-n}}{\rho_0}\int_0^{\rho_0} \rho^{-n} f(5\rho)\, d\rho$$

$$\equiv \rho_0^{-n} f(\rho_0) + \frac{2}{5\rho_0}\int_0^{5\rho_0} \rho^{-n} f(\rho)\, d\rho$$

$$\equiv \rho_0^{-n} f(\rho_0) + \frac{2}{5\rho_0}\left(\int_0^{\rho_0} \rho^{-n} f(\rho)\, d\rho + \int_{\rho_0}^{5\rho_0} \rho^{-n} f(\rho)\, d\rho\right)$$

$$\leq \rho_0^{-n} f(\infty) + \frac{2}{5}\sup_{0<\rho<\rho_0} \rho^{-n} f(\rho) + \frac{2}{5(n-1)}\ \rho_0^{-n} f(\infty)\ .$$

Thus

$$\frac{1}{2} \leq \frac{1}{2}\sup_{0<\sigma<\rho_0} \sigma^{-n} f(\sigma) < 2\rho_0^{-n} f(\infty) = 2^{-n}\ ,$$ which is a

contradiction.

Continuation of the proof of Theorem 18.6

First note that because h has compact support in U, the formula 18.3 is actually valid here for all $0 < \sigma < \rho < \infty$. Hence we can apply the above lemma with the choices

$$f(\rho) = \omega_n^{-1} \int_{B_\rho(\xi)} h \, d\mu ,$$

$$g(\rho) = \omega_n^{-1} \int_{B_\rho(\xi)} (|\nabla^M h| + h|\underline{\underline{H}}|) \, d\mu ,$$

provided that $\xi \in \operatorname{spt}\mu$ and $h(\xi) \geq 1$.

Thus for each $\xi \in \{x \in \operatorname{spt}\mu : h(x) \geq 1\}$ we have $\rho < 2(\omega_n^{-1} \int_M h \, d\mu)^{1/n}$ such that

$$(1) \qquad \int_{B_{5\rho}(\xi)} h \, d\mu \leq 5^n (\omega_n^{-1} \int_M h \, d\mu)^{1/n} \int_{B_\rho(\xi)} (|\nabla^M h| + h|\underline{\underline{H}}|) \, d\mu .$$

Using the covering Lemma (Theorem 3.3) we can select disjoint balls $B_{\rho_1}(\xi_1)$, $B_{\rho_2}(\xi_2)$, ..., $\xi_i \in \{\xi \in \operatorname{spt}\mu : h(\xi) \geq 1\}$ such that $\{\xi \in M : h(\xi) \geq 1\} \subset \bigcup_{j=1}^{\infty} B_{5\rho_j}(\xi_j)$. Then applying (1) and summing over j we have

$$\int_{\{x \in \operatorname{spt}\mu : h(x) \geq 1\}} h \, d\mu \leq 5^n \left(\omega_n^{-1} \int_M h \, d\mu\right)^{1/n} \int_M (|\nabla^M h| + h|\underline{\underline{H}}|) \, d\mu .$$

Next let γ be a non-decreasing $C^1(\mathbb{R})$ function such that $\gamma(t) \equiv 1$ for $t > \varepsilon$ and $\gamma(t) \equiv 0$ for $t < 0$, and use this with $\gamma(h-t)$, $t \geq 0$, in place of h. This gives

$$\mu(M_{t+\varepsilon}) \leq 5^n(\omega_n \quad (\mu(M_t))^{1/n} \int_M (\gamma'(h-t)|\nabla^M h| + \gamma(h-t)|\underline{\underline{H}}|) \, d\mu ,$$

where

$$M_\alpha = \{x \in M : h(x) > \alpha\} , \quad \alpha \geq 0 .$$

Multiplying this inequality by $(t+\varepsilon)^{\frac{1}{n-1}}$ and using the trivial inequality $(t+\varepsilon)^{\frac{1}{n-1}}\mu(M_t) \le \int_{M_t}(h+\varepsilon)^{\frac{1}{n-1}}\,d\mu$ on the right, we then get

$$(t+\varepsilon)^{\frac{1}{n-1}}\mu(M_{t+\varepsilon}) \le 5^n\omega_n^{-1/n}\left(\int_M (h+\varepsilon)^{\frac{n}{n-1}}\,d\mu\right)^{1/n}\left(-\frac{d}{dt}\int_M \gamma(\xi-t)|\nabla^M h| + \int_{M_t}|\underline{\underline{H}}|\,d\mu\right).$$

Now integrate of $t \in (0,\infty)$ and use 17.10. This then gives

$$\int_{M_\varepsilon}\left(h^{\frac{n}{n-1}} - \varepsilon^{\frac{n}{n-1}}\right)d\mu \le 5^{n+1}\omega_n^{-1/n}\left(\int_M (h+\varepsilon)^{\frac{n}{n-1}}\right)^{1/n}\int_M (|\nabla^M h| + h|\underline{\underline{H}}|)\,d\mu\ .$$

The theorem (with $c = 5^{n+1}\omega_n^{-1/n}$) now follows by letting $\varepsilon \downarrow 0$.

18.8 REMARK Note that the inequality of Theorem 18.6 is valid without any boundedness hypothesis on $\underline{\underline{H}}$: it suffices that $\underline{\underline{H}}$ is merely in $L^1_{loc}(\mu)$.

§19. MISCELLANEOUS ADDITIONAL CONSEQUENCES OF THE MONOTONICITY FORMULAE

Here $V = \underline{\underline{v}}(M,\theta)$ is a rectifiable n-varifold in $\mathbb{R}^{n+k}$ and we continue to assume V has an $L^1_{loc}(\mu_V)$ mean curvature $\underline{\underline{H}}$ in U , U open in $\mathbb{R}^{n+k}$.

We first want to derive convex hull properties for V in case $\underline{\underline{H}}$ is bounded.

19.1 LEMMA *Suppose* $U = \mathbb{R}^{n+k} \sim \bar{B}_R(\xi)$ *and* $n^{-1}|\underline{\underline{H}}(x)\cdot(x-\xi)| < 1$ μ_V - a.e. $x \in U$, *and suppose* spt V *is compact. Then*

$$\operatorname{spt} V \subset \bar{B}_R(\xi)\ .$$

(i.e. $V \llcorner U = 0$.)

Proof Since spt V is compact it is easily checked that the formulae (see §17)

$$n\int \gamma(r)\,d\mu_V + \int r\gamma'(r)(1-|D^{\perp}r|^2)\,d\mu_V = -\int \underline{\underline{H}}(x)\cdot(x-\xi)\gamma(r)\,d\mu_V(x)$$

(where $r = |x-\xi|$) actually holds for any non-negative non-decreasing $C^1(\mathbb{R})$ function γ with $\gamma(t) = 0$ for $t \le R+\varepsilon$. ($\varepsilon > 0$ arbitrary.) We see this as in §17, by substituting $X(x) = \psi(x)\ \gamma(r)(x-\xi)$, where $\psi \equiv 1$ in a neighbourhood of spt V. Since $1 - |D^{\perp}r|^2 \ge 0$ and $|\underline{\underline{H}}\cdot(x-\xi)| < n$ μ_V-a.e., we thus deduce $\int \gamma(r)\,d\mu_V = 0$ for any such γ. Since we may select γ so that $\gamma(t) > 0$ for $t > R+\varepsilon$, we thus conclude spt V ($\equiv$ spt μ_V) $\subset \bar{B}_{R+\varepsilon}(\xi)$. Because $\varepsilon > 0$ was arbitrary, this proves the lemma.

19.2 THEOREM (Convex hull property for stationary varifolds)

Suppose spt V *is compact and* V *is stationary in* $\mathbb{R}^{n+k} \sim K$, K *compact. Then*

$$\text{spt } V \subset \text{convex hull of } K\ .$$

Proof The convex hull of K can be written as the intersection of all balls $B_R(\xi)$ with $K \subset B_R(\xi)$. Hence the result follows immediately from 19.1.

Next we want to derive a rather important fact concerning existence of "tangent cones" for V in U. We will actually derive much more general theorems of this type later (in Chapter 10); the present simple result suffices for our applications to minimizing currents in Chapter 7.

The main idea here is to consider the possibility of getting a cone (or a plane) as the limit when we take a sequence of enlargements near a given point $\xi \in U$. Specifically, we use the transformation $\eta_{\xi,\lambda} : x \mapsto \lambda^{-1}(x-\xi)$, and we consider the sequence $V_j = \eta_{\xi,\lambda_j\#} V$ (see 15.6 for notation) of "enlargements" of V centred at ξ for a sequence $\lambda_j \downarrow 0$.

19.3 THEOREM *Suppose* $\xi \in U$, $\Theta^n(\mu_V,\xi) = \lim_{\rho\downarrow 0} \dfrac{\mu_V(\bar{B}_\rho(\xi))}{\omega_n \rho^n}$ *exists, and, with* $V_j = \eta_{\xi,\lambda_j\#} V$ *as above, suppose* $\mu_{V_j} \to \mu_W$ *in the sense of Radon measures in* $\mathbb{R}^{n+k}$, *where* W *is a rectifiable* n*-varifold which is stationary in all of* $\mathbb{R}^{n+k}$. *Then* W *is a* *cone,* *in the sense that* $W = \underline{\underline{v}}(C,\psi)$, *where* C *is a countably* n*-rectifiable set invariant under all homotheties* $x \to \lambda^{-1}x$, $\lambda > 0$, *and* ψ *is a positive locally* $\mathcal{H}^n$*-integrable function on* C *with* $\psi(x) \equiv \psi(\lambda^{-1}x)$ *for* $x \in C$, $\lambda > 0$.

19.4 REMARK We do not need to assume V has a generalized mean curvature here. However note that (by 17.8) generalized mean curvature in $L^p_{loc}(\mu_V)$, $p > n$, guarantees the hypothesis that $\Theta^n(\mu,x)$ exists. Furthermore, in later applications the fact that the limit varifold W is stationary will often be a *consequence* of the fact that V has a generalized mean curvature which satisfies suitable restrictions near ξ.

Proof of 19.3 Whenever $\mu_W(\partial B_\sigma(0)) = 0$ (which is true except possibly for countably many σ) we have

(1) $$\sigma^{-n}\mu_W(B_\sigma(0)) = \lim_{j\to\infty} \sigma^{-n}\mu_{V_j}(\bar{B}_\sigma(0))$$

$$= \lim_{j\to\infty} (\lambda_j\sigma)^{-n}\, \mu_V(\bar{B}_{\lambda_j\sigma}(\xi)) \text{ (by definition of } V_j)$$

$$= \omega_n \Theta^n(\mu_V,\xi),$$

independent of σ.

On the other hand since W is stationary in $\mathbb{R}^{n+k}$ we know by 17.5 that (with $r = |x|$)

$$\sigma^{-n}\mu_W(B_\sigma(0)) = \rho^{-n}\mu_W(B_\rho(0)) - \int_{B_\rho(0)\sim B_\sigma(0)} \frac{|D^\perp r|^2}{r^n}\, d\mu_W ,$$

so that from (1) we deduce

(2) $$|D^\perp r|^2 = 0 \quad \mu_W\text{-a.e.}$$

But recall that (letting grad denote gradient taken in $\mathbb{R}^{n+k}$)

$$D^\perp r(x) = q_x(\text{grad } r(x)) \quad (\equiv r^{-1}q_x(x)) , \quad \mu_W\text{-a.e. } x ,$$

where q_x denotes the orthogonal projection of $\mathbb{R}^{n+k}$ onto $(T_xW)^\perp$, T_xW the tangent space of W at x (see §15). Therefore (2) implies

$$q_x(x) = 0 , \quad \mu_W\text{-a.e. } x ;$$

in other words

(3) $$x \in T_xW \quad \mu_W\text{-a.e. } x .$$

Next note that if h is a $C^1(\mathbb{R}^{n+k}\sim\{0\})$ homogeneous function of degree zero, so that $h(x) \equiv h(\frac{x}{|x|})$, then $x\cdot\text{grad } h(x) = 0$, $x \neq 0$, and so, for such a function h, (3) implies

(4) $$x \cdot \nabla^W h = 0$$

$(\nabla^W h(x) = p_{T_xW}(\text{grad } h(x)))$.

Thus for any homogeneous degree zero function h we see from (2), (4) and 18.1 that

(5) $$\rho^{-n}\int_{B_\rho(0)} h\, d\mu_W = \text{const. (independent of } \rho\text{)}.$$

(Notice the fact that it is valid to substitute h in 18.1, even though h is not C^1 at 0, is a consequence of a simple approximation argument, using the fact that $\sigma^{-n}\,\mu(B_\sigma(0))$ is constant.)

It is easy to check that (5) (for arbitrary non-negative $C^1(\mathbb{R}^{n+k}\sim\{0\})$ homogeneous degree zero functions) implies that μ_W is invariant under homotheties in the sense that $\lambda^{-n}\,\mu_W(\lambda A) = \mu_W(A)$ for any subset $A \subset \mathbb{R}^{n+k}$.

Thus the theorem is proved by taking

$$C = \{x : \Theta^n(\mu_W,x) > 0\},$$

$$\psi(x) \equiv \Theta^n(\mu_W,x).$$

Finally we wish to prove a technical lemma concerning densities which we shall need in the next chapter.

19.5 LEMMA *Suppose* $0 < \ell,\beta < 1$, $R > 0$, $\bar{B}_R(0) \subset U$, $p > n$,

$$(*) \qquad \left(\omega_n^{-1}\int_{B_R(0)} |\underline{\underline{H}}|^p d\mu_V\right)^{1/p} \le (1-n/p)\Gamma, \quad \Gamma R^{1-n/p} \le 1/2$$

and suppose $y,z \in B_{\beta R}(0)$ *with* $|y-z| \ge \beta R/4$, $\Theta^n(\mu_V,y)$, $\Theta^n(\mu_V,z) \ge 1$, *and* $|q(y-z)| \ge \ell|y-z|$, *where* q *is the orthogonal projection of* $\mathbb{R}^{n+k}$ *onto* $\mathbb{R}^k$. *Then*

$$\Theta^n(\mu_V,y) + \Theta^n(\mu_V,z) \le (1+c(\ell\beta)^{-n}\Gamma R^{1-n/p})(1-\beta)^{-n}\omega_n^{-1}R^{-n}\mu_V(B_R(0))$$

$$+\, c(\ell\beta)^{-n-1}R^{-n}\int_{B_R(0)} \|p-p_x\|\,d\mu_V,$$

where $c = c(n,k,p)$, $p = p_{\mathbb{R}^n}$, $p_x = p_{T_xV}(\equiv p_{T_xM}\ \mu_V$ a.e. $x)$.

19.6 REMARK By (*) and Remark 17.9(2) we can use the monotonicity formulae 17.6 with $\Lambda = 2\Gamma R^{-n/p}$, $\alpha = 1-n/p$, and $\xi = y$ or z. Notice that in fact the quantity $\Lambda R^{1-\alpha}\rho^\alpha$ is then just $2\Gamma\rho^{1-n/p}$ and, since $e^t \le 1+2t$ for

$t \le 1$, we have by 17.6(1) that

$$(**) \qquad \omega_n^{-1}\tau^{-n}\mu(B_\tau(\xi)) \le (1+4\Gamma\sigma^{1-n/p})\omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\xi))$$

whenever $B_\sigma(\xi) \subset B_R(0)$, $0 < \tau < \sigma$, and $\Theta^n(\mu,\xi) \ge 1$, where we write μ for μ_V .

Proof of 19.5 First note that by 18.3 we have

$$\sigma^{-n}\int_{B_\sigma(\xi)} h\,d\mu \le \rho^{-n}\int_{B_\rho(\xi)} h\,d\mu + \int_\sigma^\rho \tau^{-n}\int_{B_\tau(\xi)} (|\nabla^M h| + |\underline{\underline{H}}|h)\,d\mu\,d\tau$$

for any non-negative $C^1(\mathbb{R}^{n+k})$ function h , provided $0<\sigma<\rho<(1-\beta)R$ and $\xi = y$ or z . We make a special choice of h such that $h = f(|q(x-\xi)|)$, where f is $C^1(\mathbb{R})$ with:

$f(t) \equiv 1$ for $|t| < \ell\beta R/16, f(t) \equiv 0$ for $|t| > \ell\beta R/8$, $|f'(t)| \le 3(\ell\beta R)^{-1}$ and

$0 \le f(t) \le 1 \quad \forall\ t$.

Then, since $|\nabla_j^M(q(x-\xi))| \le |p_x \circ q| \equiv |(p_x - p)\circ q| \le |p_x - p| \le \sqrt{n+k}\,\|p_x - p\|$ for $j = 1,\dots,n+k$ (where $\nabla_j^M = e_j\cdot\nabla^M$ as in §12), we deduce, with $\sigma \le \ell\beta R/2$, $\rho \le (1-\beta)R$

$$\begin{aligned}(1) \qquad \omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\xi)) \le\ & \omega_n^{-1}\rho^{-n}\mu(B_\rho(\xi)\cap\{x : |q(x-\xi)| \le \ell\beta R/8\}) \\ & + c\,\sigma^{-n}(\ell\beta R)^{-1}\rho\int_{B_\rho(\xi)} \|p_x - p\|\,d\mu \\ & + c\,\sigma^{-n}\rho\int_{B_\rho(\xi)} |\underline{\underline{H}}|\,d\mu\ .\end{aligned}$$

Now (see 17.9(2)) from (*) we have

$$(2) \qquad \int_{B_\rho(\xi)} |\underline{\underline{H}}|\,d\mu \le 2\Gamma\rho^{-n/p}\mu(B_\rho(\xi))\ .$$

Taking alternately $\xi = y$, $\xi = z$ and adding the resultant inequalities in (1), (2) and 19.6 (**), we deduce the required result (upon letting $\tau \downarrow 0$ in 19.6 (**) and taking $\sigma = \ell\beta R/8$ and $\rho = (1-\beta)R$ in all inequalities).