APPENDIX I

DISCRETE SPECTRAL VALUES

This appendix is a supplement to Section 7. We give necessary and sufficient conditions for an isolated spectral value of $T \in BL(X)$ to be a *pole* of the resolvent operator $R(z)$. These conditions do not involve the knowledge of $R(z)$ for z near λ. They, in turn, give conditions for λ to be in the discrete spectrum of T. We show that a spectral value λ of T belongs to the discrete spectrum of T if and only if some commuting compact perturbation of T dislodges λ from the spectrum.

Let A be a linear operator on X. Consider the ascending chain of subspaces of X :

$$\{0\} \subset Z(A) \subset Z(A^2) \subset \ldots$$

and also the descending chain

$$X \supset R(A) \supset R(A^2) \supset \ldots .$$

As we have seen in Remark 7.2, if equality holds at any of the inclusions in the above two chains, then it persists at all later inclusions. This property allows us to define the following concepts. As usual, $A^0 = I$, the identity operator.

The <u>ascent</u> of A is

$$\alpha(A) = \begin{cases} 0 , & \text{if } Z(A) = \{0\} , \\ p , & \text{if } Z(A^{p-1}) \neq Z(A^p) = Z(A^{p+1}) , \ 1 \leq p < \infty , \\ \infty , & \text{otherwise.} \end{cases}$$

The descent of A is

$$\delta(A) = \begin{cases} 0 , & \text{if } R(A) = X , \\ p , & \text{if } R(A^{p-1}) \neq R(A^p) = R(A^{p+1}) , \ 1 \leq p < \infty , \\ \infty , & \text{otherwise.} \end{cases}$$

LEMMA 1 Let $A \in BL(X)$ with $\alpha(A) < \infty$ and $\delta(A) < \infty$. Then

(1) $$\alpha(A) = \delta(A) = \alpha \text{ , say.}$$

The subspaces $Y = R(A^\alpha)$ and $Z = Z(A^\alpha)$ are closed in X and

(2) $$X = R(A^\alpha) \oplus Z(A^\alpha) .$$

Further, the operator A is decomposed by (Y,Z) ; $A|_Y$ is invertible and A_Z is nilpotent: $(A_Z)^\alpha = 0$.

Proof Let $\alpha(A) = \alpha < \infty$ and $\delta(A) = \delta < \infty$. We claim that

(3) $$R(A^\alpha) \cap Z(A^{\delta+1}) = \{0\} ,$$

(4) $$R(A^{\alpha+1}) + Z(A^\delta) = X .$$

For, if $y \in R(A^\alpha) \cap Z(A^{\delta+1})$, then $y = A^\alpha x$, $x \in X$, and

$$0 = A^{\delta+1}y = A^{\alpha+\delta+1}x ,$$

so that $x \in Z(A^{\alpha+\delta+1}) = Z(A^\alpha)$ since $\alpha(A) = \alpha$. Thus, $y = A^\alpha x = 0$, proving (3). Next, for $x \in X$, we have $A^\delta x \in R(A^\delta) = R(A^{\delta+\alpha+1})$ since $\delta(A) = \delta$. Then $A^\delta x = A^{\delta+\alpha+1}u$ for some, $u \in X$, so that $A^\delta(x-A^{\alpha+1}u) = 0$. Now,

$$x = A^{\alpha+1}u + (x-A^{\alpha+1}u)$$

with $A^{\alpha+1}u \in R(A^{\alpha+1})$ and $x - A^{\alpha+1}u \in Z(A^\delta)$, proving (4).

Since $Z(A^{\delta+1}) \supset Z(A^{\delta})$ and $R(A^{\alpha+1}) \subset R(A^{\alpha})$, (3) and (4) give

$$(5) \qquad R(A^{\alpha}) \cap Z(A^{\delta}) = \{0\} \text{ and } R(A^{\alpha}) + Z(A^{\delta}) = X .$$

To show $\alpha \leq \delta$, we note that if $x \in Z(A^{\delta+1})$, then by (5), we have $x = x_1 + x_2$, with $x_1 \in R(A^{\alpha})$ and $x_2 \in Z(A^{\delta})$. Since $x_1 = x-x_2 \in Z(A^{\delta+1})$ also, we see that (3) implies $x_1 = 0$, or $x = x_2 \in Z(A^{\delta})$. Thus, $Z(A^{\delta+1}) = Z(A^{\delta})$, showing $\alpha \leq \delta$.

Similarly, to show $\delta \leq \alpha$, we note that if $x \in R(A^{\alpha})$, then by (4), $x = x_1 + x_2$ with $x_1 \in R(A^{\alpha+1})$ and $x_2 \in Z(A^{\delta})$. But $R(A^{\alpha+1}) \subset R(A^{\alpha})$, and (5) shows that $x_2 = 0$, or $x = x_1 \in R(A^{\alpha+1})$. Thus, $R(A^{\alpha}) = R(A^{\alpha+1})$, proving $\delta \leq \alpha$, and consequently $\alpha = \delta$. Thus, (5) implies (2).

The subspace $Z = Z(A^{\alpha})$ is closed in X because A^{α} is continuous. We now show that $Y = R(A^{\alpha})$ is also closed in X. Consider the space $W = X \times Z$ with $\|(x,z)\| = \|x\| + \|z\|$, and the continuous linear map $B : W \to X$ given by

$$B(x,z) = A^{\alpha}x + z , \quad (x,z) \in W .$$

Since $\delta = \alpha$, we see from (5) that B maps W onto X, and hence it is an open map, i.e., there exists $\epsilon > 0$ such that if $\|Bw\| < \epsilon$, then $Bw = Bw'$ for some $w' \in W$ with $\|w'\| < 1$. Thus, for $w \in W$,

$$\epsilon \operatorname{dist}(w,Z(B)) < \|Bw\| .$$

For $x \in X$, let $w = (x,0)$. Then the above inequality reduces to

$$\epsilon \operatorname{dist}(x,Z(A^{\alpha})) < \|A^{\alpha}x\| .$$

Considering the induced quotient map from X/Z onto X, we conclude from this inequality that the range Y of A^{α} is closed in X.

Finally, it is clear that $A(Y) = R(A^{\alpha+1}) = R(A^{\alpha}) = Y$, and $A(Z) \subset Z(A^{\alpha}) = Z$, so that A is decomposed by the pair (Y,Z) . The restriction map $A_Y : Y \to Y$ is onto since $R(A^{\alpha}) \subset R(A^{\alpha+1})$, and it is one to one since $Z(A) \subset Z$. Hence A_Y is invertible in $BL(Y)$. Also, for every $z \in Z = Z(A^{\alpha})$, we have $(A_Z)^{\alpha} z = 0$, proving that A_Z is nilpotent. //

We remark that if $\lambda \in \sigma(T)$ but is not an eigenvalue of T , then $\delta(T-\lambda I) = \infty$, by Lemma 1. For, otherwise $\delta(T-\lambda I) = \alpha(T-\lambda I) = 0$, and $T - \lambda I$ would be one to one and onto, contradicting $\lambda \in \sigma(T)$.

THEOREM 2 Let $T \in BL(X)$ and $\lambda \in \sigma(T)$. Then λ is pole of $R(z)$ if and only if

$$\alpha(T-\lambda I) < \infty \quad \text{and} \quad \delta(T-\lambda I) < \infty . \tag{6}$$

In that case, the order of the pole at λ is $\alpha(T-\lambda I) = \delta(T-\lambda I)$.

Proof Let λ be a pole of $R(z)$ of order ℓ , $1 \le \ell < \infty$, and let P_λ be the corresponding spectral projection. Then by (7.9) and (7.10),

$$Z((T-\lambda I)^{\ell}) \subset Z((T-\lambda I)^{\ell+1}) \subset R(P_\lambda) ,$$

$$R((T-\lambda I)^{\ell}) \supset R((T-\lambda I)^{\ell+1}) \supset Z(P_\lambda) .$$

By Lemma 7.1(b) , $R(P_\lambda) = Z((T-\lambda I)^{\ell})$, $Z(P_\lambda) = R((T-\lambda I)^{\ell})$, so that

$$Z((T-\lambda I)^{\ell}) = Z((T-\lambda I)^{\ell+1}) ,$$
$$R((T-\lambda I)^{\ell}) = R((T-\lambda I)^{\ell+1}) ,$$

and ℓ is the smallest such positive integer, i.e., $\alpha(T-\lambda I) = \delta(T-\lambda I) = \ell < \infty$.

Conversely, assume that (6) holds. By Lemma 1,

$$\alpha(T-\lambda I) = \delta(T-\lambda I) = \alpha < \infty .$$

Since $\lambda \in \sigma(T)$, we see that $\alpha > 0$. If we let

$$Y = R((T-\lambda I)^{\alpha}) , \quad Z = Z((T-\lambda I)^{\alpha}) ,$$

then $Z \neq \{0\}$, and

$$X = Y \oplus Z ,$$

$$T - \lambda I = (T_Y-\lambda I_Y) \oplus (T_Z-\lambda I_Z) .$$

Also, $\lambda \in \rho(T_Y)$, since $T_Y-\lambda I_Y$ is invertible. Since $\rho(T_Y)$ is an open subset of $\mathbb{C}$ (Theorem 5.1), there is a neighbourhood U of λ contained in $\rho(T_Y)$.

Again by Lemma 1, $(T_Z-\lambda I_Z)^{\alpha} = 0$. Hence $\sigma(T_Z) = \{\lambda\}$. (Note that $Z \neq \{0\}$.) Thus, every $z \neq \lambda$ in U belongs to $\rho(T_Y) \cap \rho(T_Z) = \rho(T)$, showing that λ is an isolated point of $\sigma(T)$.

Let Γ be a simple closed rectifiable curve in U which encloses λ. For every $z \in \Gamma$, the operator $R(T,z)$ is decomposed by (Y,Z). Consider $y \in Y$. Since

$$R(T,z)y = R(T_Y,z)y , \quad z \in \Gamma ,$$

we have

$$\begin{aligned} P_\lambda y &= \left[-\frac{1}{2\pi i}\int_\Gamma R(T,z)dz)\right]y \\ &= \frac{-1}{2\pi i}\int_\Gamma R(T,z)y\ dz \\ &= \frac{-1}{2\pi i}\int_\Gamma R(T_Y,z)y\ dz \\ &= \left[-\frac{1}{2\pi i}\int_\Gamma R(T_Y,z)dz\right]y . \end{aligned}$$

But since $\Gamma \subset U \subset \rho(T_Y)$, we see that

$$\int_\Gamma R(T_Y,z)dz = 0 ,$$

by Cauchy's theorem (Theorem 4.5(b)). Thus, $P_\lambda y = 0$ for every $y \in Y = R((T-\lambda I)^\alpha)$. This shows that

$$D_\lambda^\alpha = P_\lambda (T-\lambda I)^\alpha = 0 \ .$$

Hence λ is a pole of $R(T,z)$ of order $\leq \alpha$. But by the first part, we see that the order of the pole must be α . //

Let us remark that in the course of the above proof we have shown that if $\lambda \in \sigma(T)$, $\alpha(T-\lambda I) < \infty$ and $\delta(T-\lambda I) < \infty$, then λ is an isolated point of $\sigma(T)$. Conversely, it can be proved that if λ is an isolated point of $\sigma(T)$ and $\delta(T-\lambda I) < \infty$, then $\alpha(T-\lambda I) < \infty$. (See [T], Theorem 10.4) However, for an isolated point λ of $\sigma(T)$ the condition $\alpha(T-\lambda I) < \infty$ does not imply $\delta(T-\lambda I) < \infty$. For example, let λ be an isolated spectral value of T which is not an eigenvalue of T . Then by the remark made after Lemma 1, we must have $\delta(T-\lambda I) = \infty$ while $\alpha(T-\lambda I) = 0$.

COROLLARY 3 Let $\lambda \in \sigma(T)$, $T \in BL(X)$. Then $\lambda \in \sigma_d(T)$ if and only if

$$\alpha(T-\lambda I) < \infty \ , \quad \delta(T-\lambda I) < \infty \quad \text{and} \quad \dim Z(T-\lambda I) < \infty \ .$$

Proof The result follows from Theorem 7.5 and Theorem 2 above. //

The above Corollary says that a spectral value λ is an isolated point of $\sigma(T)$ and is an eigenvalue of finite algebraic multiplicity if and only if the ascent and the descent of $(T-\lambda I)$ as well as the geometric multiplicity of λ are finite.

Finally, we give another interesting characterization of a discrete spectral value.

THEOREM 4 Let $T \in BL(X)$ and $\lambda \in \sigma(T)$. Then $\lambda \in \sigma_d(T)$ if and only if there exists a compact operator K on X such that $TK = KT$ and $\lambda \in \rho(T+K)$. Thus,

$$\sigma(T) \setminus \sigma_d(T) = \cap \{\sigma(T+K) : K \in BL(X) \text{ compact}, TK = KT\} . \tag{7}$$

Proof Let $\lambda \in \sigma_d(T)$, and P_λ be the corresponding spectral projection. Consider some $z_0 \in \rho(T)$, and let

$$K = (\lambda - z_0)P_\lambda .$$

Then K is compact since P_λ is of finite rank, and K commutes with T since every spectral projection associated with T commutes with T (Proposition 6.2(a)). Let

$$\tilde{T} = T + K - \lambda I ,$$

$$Y = R(P_\lambda) , \quad \text{and} \quad Z = Z(P_\lambda) .$$

Since P_λ commutes with $\tilde{T}$, it follows by Proposition 2.1 that $\tilde{T}$ is decomposed by (Y,Z). Also,

$$\tilde{T}_Y = T_Y - z_0 I_Y ,$$

which is invertible in $BL(Y)$ since $z_0 \in \rho(T_Y)$, and

$$\tilde{T}_Z = T_Z - \lambda I_Z ,$$

which is invertible in $BL(Z)$ by (6.11). Hence $\tilde{T}$ is invertible, i.e., $\lambda \in \rho(T+K)$.

Conversely, let K be a compact operator on X with $TK = KT$ and $\lambda \in \rho(T+K)$. Then

$$(T+K-\lambda I)^{-1}(T+K-\lambda I) = I .$$

If we let

$$\tilde{K} = (T+K-\lambda I)^{-1}K ,$$

then we see that

$$T - \lambda I = (T+K-\lambda I)(I-\tilde{K}) .$$

Now, K and $\tilde{K}$ commute with T, and $(T+K-\lambda I)$ is bijective. Hence it follows that for each $n = 1,2,\ldots,$

$$R((T-\lambda I)^n) = R((I-\tilde{K})^n) ,$$
$$Z((T-\lambda I)^n) = Z((I-\tilde{K})^n) .$$

But since $\tilde{K}$ is compact, either $-1 \in \rho(\tilde{K})$ or $-1 \in \sigma_d(\tilde{K})$. Hence by Theorem 2,

$$\alpha(I-\tilde{K}) < \infty , \quad \delta(I-\tilde{K}) < \infty , \quad \text{and} \quad \dim Z(I-\tilde{K}) < \infty ,$$

so that

$$\alpha(T-\lambda I) < \infty , \quad \delta(T-\lambda I) < \infty , \quad \text{and} \quad \dim Z(T-\lambda I) < \infty .$$

By Theorem 2, $\lambda \in \sigma_d(T)$ and the relation (7) follows easily. //

In view of the relation (7) in the above theorem, the complement of $\sigma_d(T)$ in $\sigma(T)$ is sometimes called the <u>essential spectrum of</u> T, being the part of $\sigma(T)$ which is 'stable' under all compact perturbations of T which commute with T.