15. METHODS RELATED TO PROJECTIONS

In this and the next section we describe some concrete practical ways of constructing sequences of operators which approximate a compact operator T in the norm, or in the collectively compact manner. As such, they give resolvent operator approximations of T. The spectral considerations of the previous section are then applicable.

In the present section we consider a group of methods which arise from a sequence of (bounded) projections $\pi_n : X \to X$. For $T \in BL(X)$ and $n = 1,2,\ldots,$ we say that the operators

$$(15.1) \qquad T_n^P = \pi_n T \;, \quad T_n^S = T\pi_n \quad \text{and} \quad T_n^G = \pi_n T \pi_n$$

give the <u>projection method</u>, the <u>Sloan method</u> and the <u>Galerkin method</u> for approximating T, respectively. If each $\pi_n(X)$ is finite dimensional, then the above operators are of finite rank. We now consider the convergence of these approximation methods.

THEOREM 15.1 Let $\pi_n \xrightarrow{p} I$, the identity operator on X. Then (T_n^P), (T_n^S) and (T_n^G) are pointwise approximations of T.

If T is compact, then $T_n^P \xrightarrow{\|\ \|} T$, while $T_n^S \xrightarrow{cc} T$ and $T_n^G \xrightarrow{cc} T$.

If, in addition, $\pi_n^* \xrightarrow{p} I$, then $T_n^S \xrightarrow{\|\ \|} T$ and $T_n^G \xrightarrow{\|\ \|} T$. In particular, this is the case when X is a Hilbert space and each π_n is an orthogonal projection.

Proof It is easy to see that $T_n^P \xrightarrow{p} T$ and $T_n^S \xrightarrow{p} T$. Also, for $x \in X$,

$$\|T_n^G x - Tx\| \le \|\pi_n\| \, \|T_n^S x - Tx\| + \|T_n^P x - Tx\| .$$

Since $\pi_n \xrightarrow{p} I$, we see by the uniform boundedness principle that $(\|\pi_n\|)$ is a bounded sequence. Hence $T_n^G \xrightarrow{p} T$.

Let, now, T be compact. Then the pointwise convergence of π_n to I is uniform on the totally bounded set $\{Tx : x \in X , \|x\| \leq 1\}$ ([L], 9.3(b)). Thus,

$$\|T_n^P - T\| = \|\pi_n T - T\| \to 0 .$$

Next, by letting $A_n = \pi_n$, $A = I$ and $B_n = B = T$ in (13.4), we see that

$$T_n^S = T\pi_n = B_n A_n \xrightarrow{cc} BA = T .$$

Again, letting $A_n = \pi_n$, $A = I$, $B_n = T\pi_n$ and $B = T$ in (13.4), we have

$$T_n^G = \pi_n T\pi_n = A_n B_n \xrightarrow{cc} AB = T .$$

Finally, let $\pi_n^* \xrightarrow{p} I$, in addition. Then

$$(T_n^S)^* = \pi_n^* T^* \xrightarrow{p} T^* \quad \text{and} \quad (T_n^G)^* = \pi_n^* T^* \pi_n^* \xrightarrow{p} T^* ,$$

as before. Hence by Theorem 13.5(b),

$$T_n^S \xrightarrow{\|\ \|} T \quad \text{and} \quad T_n^G \xrightarrow{\|\ \|} T . \qquad //$$

We remark that the condition $\pi_n \xrightarrow{p} I$ is not really needed for concluding $T_n^P \xrightarrow{p} T$ or $T_n^P \xrightarrow{\|\ \|} T$; it is enough to have $\pi_n x \to x$ for every x in the range of T . In fact, we shall later give examples to show that the projecions π_n need not even be defined on the whole of X . We shall also give an example to show that $T_n^S \xrightarrow{cc} T$ is possible without having $\pi_n \xrightarrow{p} I$.

We consider some necessary and sufficient conditions for $\pi_n \xrightarrow{p} I$.

PROPOSITION 15.2 Let (π_n) be a sequence of (bounded) projections defined on X , and let Y be a dense subspace of X . Then the following conditions are equivalent:

(i) $\pi_n \xrightarrow{p} I$

(ii) $(\|\pi_n\|)$ is a bounded sequence and $\pi_n x \to x$ for every $x \in Y$

(iii) $(\|\pi_n\|)$ is a bounded sequence and for every $x \in Y$, we have $\mathrm{dist}(x,\pi_n(X)) \to 0$.

Proof We have for every $x \in X$,

$$\mathrm{dist}(x,\pi_n(X)) \leq \|x-\pi_n x\| , \tag{15.2}$$

and on the other hand, for all $y \in \pi_n(X)$,

$$\|x-\pi_n x\| = \|(I-\pi_n)(x-y)\| \leq \|I-\pi_n\| \, \|x-y\| ,$$

so that

$$\|x-\pi_n x\| \leq \|I-\pi_n\| \, \mathrm{dist}(x,\pi_n(X)) \leq (1+\|\pi_n\|)\mathrm{dist}(x,\pi_n(X)) , \tag{15.3}$$

by taking infimum over all $y \in \pi_n(X)$.

Let $\pi_n \xrightarrow{p} I$. Then $(\|\pi_n\|)$ is bounded, and by (15.2) , $\mathrm{dist}(x,\pi_n(X)) \to 0$ for every $x \in X$. Hence the conditions (ii) and (iii) follow.

Let, now, $(\|\pi_n\|)$ be bounded. Then (15.2) and (15.3) show that for every $x \in Y$,

$$\pi_n x \to x \quad \text{if and only if} \quad \mathrm{dist}(x,\pi_n(X)) \to 0 ,$$

and in that case, the denseness of Y in X implies that $\pi_n x \to x$ for every $x \in X$, i.e., the condition (i) holds. //

We now give several constructions of bounded projections on X and examine their pointwise convergence to the identity operator.

Truncations of a Schauder expansion

Let X be a (separable) Banach space with a Schauder basis $\{x_k : k = 1,2,\ldots\}$, i.e., $x_k \in X$, $\|x_k\| = 1$, and for every $x \in X$

$$x = \sum_{k=1}^{\infty} a_k(x) x_k$$

for some unique $a_k(x) \in \mathbb{C}$. Define

$$\pi_n x = \sum_{k=1}^{n} a_k(x) x_k . \tag{15.4}$$

Since each linear functional $x \mapsto a_k(x)$ is bounded ([L], 11.6), we see that each π_n is a bounded projection. Also, by the very definition of a Schauder basis, we have $\pi_n \xrightarrow{p} I$. Note that each π_n is of finite rank.

As a special case, let X be a (separable) Hilbert space and let $\{x_k : k = 1,2,\ldots\}$ be an orthonormal basis for X. Then $a_k(x) = \langle x, x_k \rangle$, so that

$$\pi_n x = \sum_{k=1}^{n} \langle x, x_k \rangle x_k .$$

Then each π_n is an orthogonal projection and $\|\pi_n\| = 1$.

We consider some concrete examples.

(i) Let $X = \ell^p$, $1 \leq p < \infty$, and for $k = 1,2,\ldots$

$$x_k = [0,\ldots,0,1,0,0,\ldots]^t ,$$

where 1 occurs only in the k-th place.

(ii) Let $X = C([0,1])$. For $t \in \mathbb{R}$, let $x_0(t) = t$, $x_1(t) = 1 - t$,

$$x_2(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 1/2 \\ 2-2t, & \text{if } 1/2 \leq t \leq 1 \\ 0, & \text{if } t < 0 \text{ or } t > 1, \end{cases}$$

and for $n = 1,2,\ldots$, $j = 1,\ldots,2^n$, let

$$x_{2^n+j}(t) = x_2(2^n t-j+1) .$$

Then $\{x_{k|[0,1]} : k = 1,2,\ldots\}$ is a Schauder basis of X consisting of saw-tooth functions ([L], p.69).

(iii) Let $X = L^2([0,1])$. For $t \in [0,1]$, let $x_{0,0}(t) = 1$,

$$x_{1,0}(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1/2 \\ -1, & \text{if } 1/2 < t \leq 1 \\ 0, & \text{if } t = 1/2 \end{cases},$$

and for $n = 1,2,\ldots$, $j = 1,\ldots,2^n$, let

$$x_{n,j}(t) = \begin{cases} \sqrt{2^n}, & \text{if } (j-1)/2^n \leq t \leq (2j-1)/2^{n+1} \\ -\sqrt{2^n}, & \text{if } (2j-1)/2^{n+1} \leq t \leq j/2^n \\ 0, & \text{otherwise.} \end{cases}$$

Then the <u>Haar system</u> $\{x_{n,j}\}$ is an orthonormal basis of X consisting of piecewise constant functions ([L], p.198).

(iv) Let $X = L^2([-\pi,\pi])$. The functions $x_k(t) = e^{ikt}/\sqrt{2\pi}$, $k = 0,\pm1,\pm2,\ldots$ form an orthonormal basis of X , consisting of trigonometric functions ([L], p.194). If $X = L^2([0,\pi])$, then $x_k(t) = \sin kt$, $k = 1,2,\ldots$, or $x_k(t) = \cos kt$, $k = 0,1,2,\ldots$ also form orthonormal bases of X .

(v) Let $X = L^2([-1,1])$. If we orthonormalize the set $\{1,t,t^2,\dots\}$ by the Gram-Schmidt process ([L], p.187) , then we obtain the orthonormal basis of X consisting of Legendre polynomials x_k of degree $k = 0,1,2,\dots$. Note that

$$x_0(t) = 1/\sqrt{2}\ , \quad x_1(t) = \sqrt{3/2}\ t\ , \quad x_2(t) = (3/4)\sqrt{10}\ (t^2-13)\ , \quad \text{etc.}$$

Again, if we orthonormalize the same set with respect to the weight function $w(t) = 1/\sqrt{1-t^2}$ (resp. $\sqrt{1-t^2}$) , $t \in (-1,1)$, then we obtain the Tchebychev polynomials of the first kind (resp., second kind) (cf.[L], p.189).

Orthogonal projections onto piecewise polynomials

Let $X = L^2([a,b])$. For $n = 2,3,\dots$, consider a partition

$$a = t_0^{(n)} < t_1^{(n)} < \dots < t_{n-1}^{n} < t_n^{(n)} = b$$

of $[a,b]$. Let $h_n = \max\{t_i^{(n)}-t_{i-1}^{(n)} : i = 1,\dots,n\}$ be the mesh of this partition. For a fixed integer $k \geq 0$, let $\mathbb{P}_k$ denote the set of all polynomials of degree less than or equal to k , and let

$$\mathbb{P}_{k,n} = \left\{x : [a,b] \to \mathbb{C}\ ;\ x_{|(t_{i-1}^{(n)},t_i^{(n)})} \in \mathbb{P}_k \quad \text{for} \quad i = 1,\dots,n\right\} .$$

If we identify functions on $[a,b]$ which equal almost everywhere, $\mathbb{P}_{k,n}$ becomes a closed subspace of $L^2([a,b])$. Let $\pi_i^{(n)}$ denote the orthogonal projection from $L^2([t_{i-1}^{(n)},t_i^{(n)}])$ onto the space of all polynomials of degree $\leq k$ on $[t_{i-1}^{(n)},t_i^{(n)}]$. Define $\pi_n : L^2([a,b]) \to L^2([a,b])$ by

$$(15.5) \qquad \pi_n x(t) = \pi_i^{(n)} x(t)\ ,\ x \in L^2([a,b])\ ,\ t_{i-1}^{(n)} < t < t_i^{(n)} .$$

It is clear that π_n is a projection onto the set $\mathbb{P}_{k,n}$ of piecewise polynomials. Further, π_n is orthogonal. To see this, let $x_i^{(n)} = x_{|[t_{i-1}^{(n)},t_i^{(n)}]}$ for $x \in L^2([a,b])$, and consider $y \in R(\pi_n)$, $z \in Z(\pi_n)$. Then

$$\langle y,z\rangle = \sum_{i=1}^{n} \langle y_i^{(n)},z_i^{(n)}\rangle = 0 ,$$

since $y_i^{(n)} \in R(\pi_i^{(n)})$ and $z_i^{(n)} \in Z(\pi_i^{(n)})$, where the projection $\pi_i^{(n)}$ is orthogonal. Thus, π_n is an orthogonal projection, and as such $\|\pi_n\| = 1$.

Let $h_n \to 0$. We show that $\pi_n \xrightarrow{p} I$. Since π_n is orthogonal, $\|I-\pi_n\| = 1$. Hence by (15.2) nad (15.3),

$$\begin{aligned}\|\pi_n x - x\|_2 &= \min\{\|y-x\|_2 : y \in \mathbb{P}_{k,n}\} \\ &\leq \min\{\|y-x\|_2 : y \in \mathbb{P}_{0,n}\} .\end{aligned}$$

since $\mathbb{P}_{0,n} \subset \mathbb{P}_{k,n}$. This shows that it is enough to consider the case $k = 0$, i.e., when π_n is the orthogonal projection onto piecewise constant functions. By Proposition 15.2(ii), we need only prove that $\pi_n x \to x$ for every $x \in C([a,b])$, since $C([a,b])$ is dense in $L^2([a,b])$. Let $x \in C([a,b])$ and $\epsilon > 0$. Find $\delta > 0$ such that $|t-s| < \delta$ implies $|x(s)-x(t)| < \epsilon$, and choose n_0 so large that $n \geq n_0$ implies $h_n < \delta$. Now,

$$\|\pi_n x - x\|_2^2 = \sum_{i=1}^{n} \|\pi_i^{(n)} x_i^{(n)} - x_i^{(n)}\|_2^2 .$$

But since $k = 0$, we see that

$$\pi_i^{(n)} x_i^{(n)} = \langle x_i^{(n)},c_i^{(n)}\rangle c_i^{(n)} ,$$

where $c_i^{(n)}$ is the constant function defined by

$$c_i^{(n)}(t) = 1 / (t_i^{(n)}-t_{i-1}^{(n)})^{1/2} , \quad t_{i-1}^{(n)} < t < t_i^{(n)} .$$

Hence

$$\|\pi_i^{(n)}x_i^{(n)} - x_i^{(n)}\|_2^2 = \int_{[t_{i-1}^{(n)},t_i^{(n)}]} |\pi_i^{(n)}x_i^{(n)}(t) - x(t)|^2 dt$$

$$\leq \epsilon^2(t_i^{(n)}-t_{i-1}^{(n)}) ,$$

since

$$\pi_i^{(n)}x_i^{(n)}(t) - x(t) = \frac{1}{t_i^{(n)}-t_{i-1}^{(n)}}\int_{[t_{i-1}^{(n)},t_i^{(n)}]} [x(s)-x(t)]ds .$$

Thus, we have

$$\|\pi_n x - x\|_2 \leq \left[\epsilon^2 \sum_{i=1}^{n} (t_i^{(n)}-t_{i-1}^{(n)})\right]^{1/2} = \epsilon\sqrt{b-a} ,$$

for $n \geq n_0$. This completes the proof of $\pi_n \xrightarrow{p} I$.

Interpolatory projections

Let $X = C([a,b])$ with the supremum norm. For $n = 1,2,\ldots$, consider the n <u>nodes</u> $t_1^{(n)},\ldots,t_n^{(n)}$ in $[a,b]$:

$$a = t_0^{(n)} \leq t_1^{(n)} < \ldots < t_n^{(n)} \leq t_{n+1}^{(n)} = b .$$

Let $u_i^{(n)} \in C([a,b])$ satisfy

$$u_i^{(n)}(t_j^{(n)}) = \delta_{i,j} , \; i,j = 1,\ldots,n .$$

For $x \in C([a,b])$, let

$$\pi_n x(t) = \sum_{i=1}^{n} x(t_i^{(n)})u_i^{(n)}(t) .$$

Since $\pi_n x(t_i^{(n)}) = x(t_i^{(n)})$, i.e., $\pi_n x$ *interpolates* x at $t_i^{(n)}$, we say that π_n is <u>an interpolatory projection</u>. Note that

$\pi_n(X) = \text{span}\{u_1^{(n)},\ldots,u_n^{(n)}\}$, and hence π_n is of rank n . We show

$$\|\pi_n\| = \sup_{t\in[a,b]} \sum_{i=1}^{n} |u_i^{(n)}(t)| . \tag{15.6}$$

It is clear that $\|\pi_n\|$ does not exceed the right hand side. Now, since $[a,b]$ is compact, let $t_0 \in [a,b]$ be such that the right hand side equals $\sum_{i=1}^{n} |u_i^{(n)}(t_0)|$. Choose $x_0 \in C([a,b])$ such that

$$x_0(a) = x(t_1^{(n)}) , \quad x_0(b) = x(t_n^{(n)}) ,$$

$$x_0(t_i^{(n)}) = \begin{cases} 0 , & \text{if } u_i^{(n)}(t_0) = 0 \\ |u_i^{(n)}(t_0)| \,/\, u_i^{(n)}(t_0) , & \text{otherwise} \end{cases}$$

and x_0 is linear on $[t_i^{(n)}, t_{i+1}^{(n)}]$, $i = 0,\ldots,n$. Then

$$\pi_n x_0(t_0) = \sum_{i=1}^{n} |u_i^{(n)}(t_0)| = \sup_{t\in[a,b]} \sum_{i=1}^{n} |u_i^{(n)}(t)| .$$

This completes the proof of (15.6).

Methods related to interpolatory projections are known as <u>collocation methods</u>. Now we consider several specific choices of the functions $u_i^{(n)}$, $i = 1,\ldots,n$.

(i) **Lagrange interpolation.** In this case the function $u_i^{(n)}$ is chosen to be the polynomial $\ell_i^{(n)}$ of degree $(n-1)$. In fact, we have

$$\ell_i^{(n)}(t) = \prod_{\substack{j=1 \\ j\neq i}}^{n} (t-t_j^{(n)}) \,/\, \prod_{\substack{j=1 \\ j\neq i}}^{n} (t_i^{(n)}-t_j^{(n)}) . \tag{15.7}$$

It is clear that $\ell_i^{(n)}$ vanishes precisely at $t_j^{(n)}$, $j = 1,\ldots,n$, $j \neq i$. Hence the support of $\ell_i^{(n)}$ is the whole interval $[a,b]$. This usually creates problems in convergence and numerical stability of the computations.

Let L_n denote the interpolatory projection corresponding to $\ell_1^{(n)},\ldots,\ell_n^{(n)}$; it is known as the Lagrange interpolation. A result of Kharshiladze and Lozinski says that if π_n is a (bounded) projection of $C([a,b])$ onto $\mathbb{P}_n$, $n = 1,2,\ldots$, then there is $x \in C([a,b])$ such that the sequence $(\|x - \pi_n x\|_\infty)$ is unbounded ([CN], p.214). In particular, we do *not* have $L_n \xrightarrow{p} I$.

A variation of the Lagrange interpolation is the Fejér-Hermite interpolation. Here the function $u_i^{(n)}$ is chosen to be the polynomial $f_i^{(n)}$ of degree $(2n-1)$ whose derivative is zero at all $t_1^{(n)},\ldots,t_n^{(n)}$. In fact,

$$f_i^{(n)}(t) = \left[1 - 2(t-t_i^{(n)})(\ell_i^{(n)})'(t_i^{(n)})\right] \left[\ell_i^{(n)}(t)\right]^2 .$$

Let F_n denote the interpolatory projection corresponding to $f_1^{(n)},\ldots,f_n^{(n)}$. If the nodes are the n roots of the Tchebychev polynomial p_{n-1} of the first kind, then we have

$$F_n(x)(t) = \frac{1}{n^2} p_{n-1}^2(t) \sum_{i=1}^{n} x(t_i^{(n)})(1-t_i^{(n)}t)/(t-t_i^{(n)})^2$$

for $x \in C([-1,1])$. (See [CN], p.70.) It follows by Korovkin's theorem ([L], 3.18) that $F_n \xrightarrow{p} I$.

Although $L_n \xrightarrow{p} I$ does not hold, we show that the projection and the Sloan methods defined with the help of the L_n's can converge.

Let w be a continuous positive function on (a,b) . If we orthonormalize the set $\{1,t,t^2,\ldots\}$ with respect to the weight function w , then we obtain polynomials $p_0,p_1,\ldots$, which satisfy

$$\int_a^b p_i(t)\overline{p_j(t)}w(t)dt = \delta_{i,j} , \quad i,j = 0,1,\ldots .$$

Note that the degree of p_i is i . These polynomials are known as the orthogonal polynomials with respect to the weight function w . Let $L_w^2([a,b])$ denote the set of all Lebesgue measurable functions x on $[a,b]$ satisfying

$$\|x\|_{2,w} = \left[\int_a^b |x(t)|^2 w(t)dt\right]^{1/2} < \infty ,$$

where we identify functions which are equal almost everywhere. Then $L_w^2([a,b])$ is a Hilbert space with the inner product

$$\langle x,y\rangle_w = \int_a^b x(t)\overline{y(t)}w(t)dt \ , \ x,y \in L_w^2([a,b]) .$$

We now state an interesting result.

THEOREM 15.3 (Erdös-Turan) Let $p_0,p_1,\ldots$ be the orthogonal polynomials on $[a,b]$ with respect to the weight function w . Let the nodes $t_1^{(n)},\ldots,t_n^{(n)}$ be the roots of the polynomial p_n . If L_n denotes the Lagrange projection, then

$$\|L_n x - x\|_{2,w} \to 0 ,$$

for every $x \in C([a,b])$.

We refer the reader to [CN], p.137 for a proof. For $n = 1,2,\ldots$, let

$$\|L_n\|' = \sup\{\|L_n x\|_{2,w} : \|x\|_\infty \le 1\} .$$

Then by the uniform boundedness principle, we see that $\|L_n\|' \le \alpha$ for some constant α and $n = 1,2,\ldots$. It can then be seen that

$$(15.8) \qquad \|L_n x - x\|_{2,w} \le \left[\alpha + \left[\int_a^b w(t)dt\right]^2\right] \operatorname{dist}(x,\mathbb{P}_{n-1}) ,$$

since for every $y \in \mathbb{P}_{n-1}$, we have

$$\begin{aligned} \|L_n x - x\|_{2,w} &\le \|L_n(x-y) - (x-y)\|_{2,w} \\ &\le \|L_n(x-y)\|_{2,w} + \|x-y\|_{2,w} \\ &\le \alpha\|x-y\|_\infty + \left[\int_a^b w(t)dt\right]^{1/2}\|x-y\|_\infty . \end{aligned}$$

THEOREM 15.4 For $n = 1,2,\ldots$, let L_n be as in Theorem 15.3.

(a) (Vainikko) Let $T : L_w^2([a,b]) \to L_w^2([a,b])$ be a linear operator with $R(T) \subset C([a,b])$. Then

$$T_n^P = L_n T \xrightarrow{p} T .$$

If, in addition, T is compact, then $T_n^P \xrightarrow{\|\ \|} T$.

(b) (Sloan-Burn) Let the weight function w satisfy

$$\int_a^b [1/w(t)]dt < \infty .$$

Let $T : C([a,b]) \to C([a,b])$ be defined by

$$Tx(s) = \int_a^b k(s,t)x(t)dt \ , \ x \in C([a,b]) \ , \ s \in [a,b] \ ,$$

where $k(s,t)$ is a continuous complex-valued function for $s,t \in [a,b]$. Then, with respect to the sup norm,

$$T_n^S = TL_n \xrightarrow{cc} T .$$

Proof (a) By Theorem 15.3, we have $T_n^P x = L_n Tx \to Tx$ for every $x \in L_w^2([a,b])$, since $Tx \in C([a,b])$. If, in addition, T is compact, then the pointwise convergence of L_n to I is uniform on the totally bounded set $\{Tx : x \in L_w^2([a,b]) \ , \|x\|_{2,w} \le 1\}$. Hence $T_n^P \xrightarrow{\|\ \|} T$.

(b) Let $x \in C([a,b])$. For $s \in [a,b]$,

$$|TL_n x(s) - Tx(s)|^2 = \left|\int_a^b k(s,t)[L_n x(t) - x(t)]dt\right|^2$$
$$\leq \left[\int_a^b \left|\frac{k(s,t)}{w(t)}\right|^2 w(t)dt\right] \times$$
$$\left[\int_a^b |L_n x(t) - x(t)|^2 w(t)dt\right],$$

by the Hölder inequality for the space $L^2_w([a,b])$. Hence

$$|TL_n x(s) - Tx(s)| \leq \|k\|_\infty \left[\int_a^b \frac{dt}{w(t)}\right]^{1/2} \|L_n x - x\|_{2,w}.$$

Again, by Theorem 15.3 we see that $TL_n x(s)$ converges to $Tx(s)$, uniformly for $s \in [a,b]$. Hence $TL_n \xrightarrow{p} T$ in $C([a,b])$. To conclude $TL_n \xrightarrow{cc} T$, it is enough to show that the set

$$E = \bigcup_{n=1}^{\infty} \{TL_n x : x \in C([a,b]),\ \|x\|_\infty \leq 1\}$$

is totally bounded, since T itself is a compact operator. For this purpose, we show that the set E is uniformly bounded and equicontinuous. Let $x \in C([a,b])$ and $\|x\|_\infty \leq 1$. Then for all $x \in [a,b]$,

$$|TL_n x(s)| \leq \|TL_n x\|_\infty \leq \|TL_n\| \leq \beta,$$

by the uniform boundedness principle. Also, for all s_1, s_2 in $[a,b]$, we have

$$|TL_n x(s_1) - TL_n x(s_2)|^2 = \left|\int_a^b [k(s_1,t) - k(s_2,t)]L_n x(t)dt\right|^2$$
$$\leq \left[\int_a^b \left|k(s_1,t) - k(s_2,t)\right|^2 \frac{dt}{w(t)}\right] \times$$
$$\left[\int_a^b \left|L_n x(t)\right|^2 w(t)dt\right],$$

as before. Let $\epsilon > 0$, and find $\delta > 0$ such that $|s_1-s_2| < \delta$ implies $|k(s_1,t) - k(s_2,t)| < \epsilon$ for all $t \in [a,b]$. Then

$$|TL_n x(s_1) - TL_n x(s_2)| \le \epsilon \left[\int_a^b \frac{dt}{w(t)}\right]^{1/2} \|L_n\|' \le \epsilon\alpha \left[\int_a^b \frac{dt}{w(t)}\right]^{1/2} ,$$

by (15.8). Thus, Ascoli's theorem ([L],3.17) shows that the set E is totally bounded in $C([a,b])$, and the proof is complete. //

We remark that the projections L_n in part (a) of the above theorem are not even defined on the entire space $X = L^2_w([a,b])$; yet we have the norm convergence of the projection method. Similarly, in part (b), we have $TL_n \xrightarrow{cc} T$ without having $L_n \xrightarrow{p} I$.

(ii) **Piecewise linear interpolation.** In this case the functions $u_i^{(n)}$ are chosen to be the functions $e_i^{(n)}$ which are linear on each of the subintervals $[t_i^{(n)}, t_{i+1}^{(n)}]$, $i = 0,\ldots,n$ and satisfy

$$e_1^{(n)}(a) = 1 = e_n^{(n)}(b) ,$$
$$e_i^{(n)}(a) = 0 \quad \text{for} \quad i = 2,\ldots,n , \quad e_i^{(n)}(b) = 0 \quad \text{for} \quad i = 1,\ldots,n-1 .$$

Thus, $e_1^{(n)},\ldots,e_n^{(n)}$ are the hat functions introduced in Example (iii) of Section 3. We shall make use of the properties of these functions discussed there. Let, as usual,

$$\pi_n x(t) = \sum_{i=1}^{n} x(t_i^{(n)}) e_i^{(n)} , \quad x \in C([a,b]) .$$

Since $e_i^{(n)}(t) \ge 0$ and $\sum_{i=1}^{n} e_i^{(n)}(t) = 1$ for all $t \in [a,b]$, it is easy to see from (15.6) that $\|\pi_n\| = 1$. Also, $\pi_n x(t_i^{(n)}) = x(t_i^{(n)})$ for $i = 1,\ldots,n$ and $\pi_n x(a) = x(t_i^{(n)})$, $\pi_n x(b) = x(t_i^{(n)})$. Hence

(15.9)

$$\pi_n x(t) = \begin{cases} x(t_1^{(n)}) \ , & \text{if} \quad t < t_{1,n} \\ x(t_{i-1}^{(n)}) + \dfrac{x(t_i^{(n)}) - x(t_{i-1}^{(n)})}{t_i^{(n)} - t_{i-1}^{(n)}} (t-t_{i-1}^{(n)}) \ , & \text{if} \quad t_{i-1}^{(n)} \le t \le t_i^{(n)} \ , \\ & \quad i = 2,\dots,n-1 \\ x(t_n^{(n)}) \ , & \text{if} \quad t > t_n^{(n)} \end{cases}$$

We show graphically some $x \in C([a,b])$ and $\pi_n(x)$:

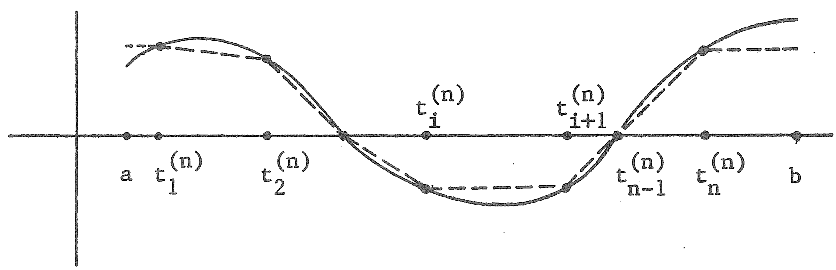

Figure 15.1

Let $h_n = \max\{t_i^{(n)}-t_{i-1}^{(n)} : i = 1,\dots,n+1\}$ be the mesh of the partition, and assume that $h_n \to 0$. We show $\pi_n \xrightarrow{p} I$. Let $x \in C([a,b])$ and $\epsilon > 0$. Find $\delta > 0$ such that $|s-t| < \delta$ implies $|x(s)-x(t)| < \epsilon$, and choose n_0 such that $n \ge n_0$ implies $h_n < \delta$. Then

$$|\pi_n x(t)-x(t)| = \begin{cases} |x(t_1^{(n)})-x(t)| < \epsilon & \text{for} \quad t < t_1^{(n)} \\ |x(t_n^{(n)})-x(t)| < \epsilon & \text{for} \quad t > t_n^{(n)} \ , \end{cases}$$

and for $t_{i-1}^{(n)} \le t \le t_i^{(n)}$, we have

$$|\pi_n x(t)-x(t)| \le \frac{\left|[x(t_{i-1}^{(n)})-x(t)](t_i^{(n)}-t_{i-1}^{(n)}) + [x(t_i^{(n)})-x(t_{i-1}^{(n)})](t-t_{i-1}^{(n)})\right|}{t_i^{(n)} - t_{i-1}^{(n)}}$$

$$= \frac{\left|[x(t_{i-1}^{(n)})-x(t)](t_i^{(n)}-t) + [x(t_i^{(n)})-x(t)](t-t_{i-1}^{(n)})\right|}{t_i^{(n)} - t_{i-1}^{(n)}}$$

$$< [\epsilon(t_i^{(n)}-t) + \epsilon(t-t_{i-1}^{(n)})] \,/\, (t_i^{(n)}-t_{i-1}^{(n)})$$

$$= \epsilon \ .$$

Thus, $\|\pi_n x - x\|_\infty \leq \epsilon$, and we see that $\pi_n \xrightarrow{p} I$. If $x \in C^1([a,b])$, i.e., x is continuously differentiable on $[a,b]$, then the above argument shows that $\|\pi_n x - x\|_\infty \leq \|x'\|_\infty h_n$, by the mean value theorem.

We consider some special choices of the nodes $t_i^{(n)}$ in $[0,1]$.

1. $t_i^{(n)} = \frac{i}{n}$, $i = 1,\ldots,n$.

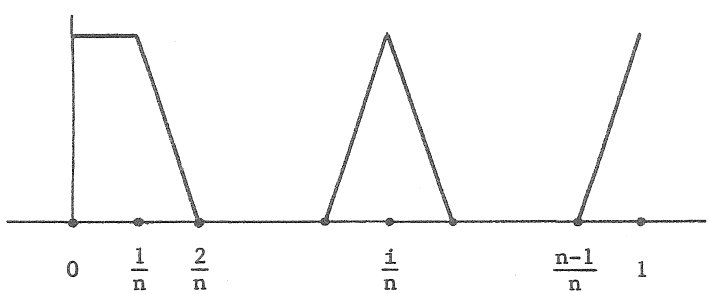

Figure 15.2

Similarly, $t_i^{(n)} = \frac{i-1}{n}$, $i = 1,\ldots,n$,

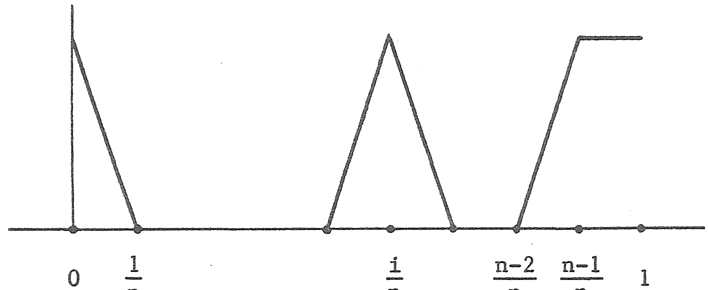

Figure 15.3

2. $t_i^{(n)} = \frac{2i-1}{2n}$, $i = 1,\ldots,n$.

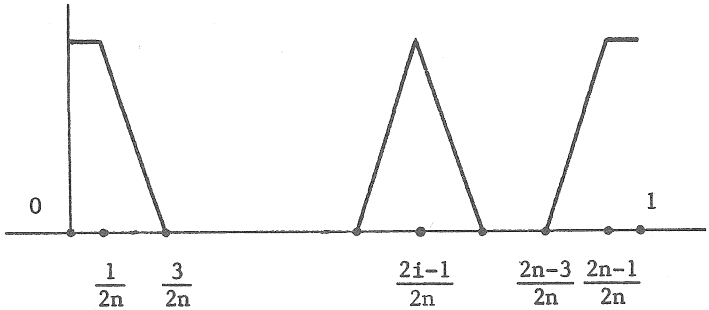

Figure 15.4

3. $t_i^{(n)} = \frac{i-1}{n-1}$, $i = 1,\ldots,n$, $n = 2,3\ldots$

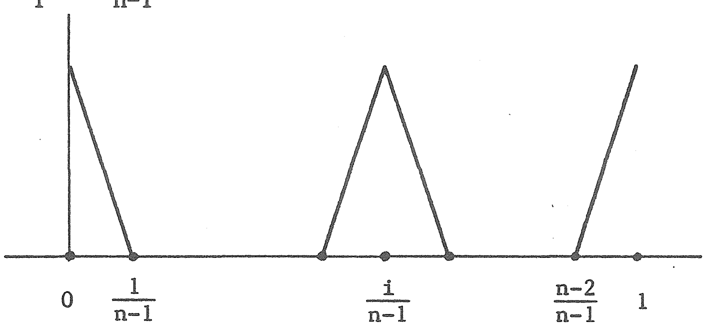

Figure 15.5

4. Compound two point rules. Let n be even, $n = 2m$. Let r_1 and r_2 be such that $-1 < r_1 < r_2 < 1$, and consider the nodes $\frac{2j-1+r_1}{n}$ and $\frac{2j-1+r_2}{n}$ in the interval $\left[\frac{2j-2}{n}, \frac{2j}{n}\right]$, $1 \leq j \leq m$. Thus

$$t_i^{(n)} = \begin{cases} (i+r_1)/n , & \text{if } i = 1,3,\ldots,n-1 , \\ (i-1+r_2)/n , & \text{if } i = 2,4,\ldots,n . \end{cases}$$

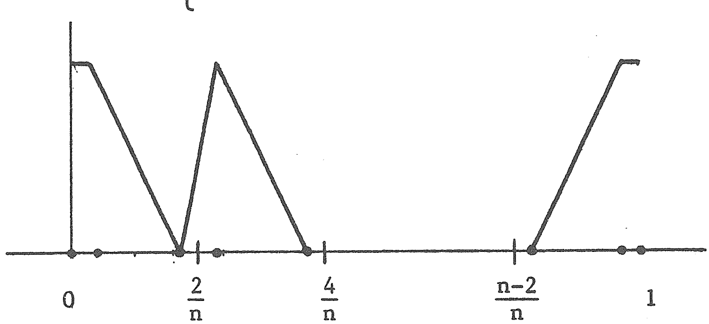

Figure 15.6

Some specific cases are worth mentioning. We have the compound Gauss two point rule when r_1 and r_2 are the roots of the Legendre polynomial $\frac{3}{4}\sqrt{10}(t^2-\frac{1}{3})$ of degree 2, i.e., $r_1 = -1/\sqrt{3}$ and $r_2 = 1/\sqrt{3}$. Next, if r_1 and r_2 are the roots of the Tchebychev polynomial of the first kind $\frac{2}{\sqrt{2\pi}}(2t^2-1)$ of degree 2, then $r_1 = -1/\sqrt{2}$ and $r_2 = 1/\sqrt{2}$, and we have the compound Tchebychev two point rule.

Similar examples can be given for 3 point and 4 point rules. These repeated quadrature rules give, in general, better approximations than ordinary quadrature rules.

(iii) **Cubic spline interpolation.** Consider the partition

$$0 = t_1^{(n)} < t_2^{(n)} < \ldots < t_{n-1}^{(n)} < t_n^{(n)} = 1$$

of $[0,1]$, and let

$$C_n = \left\{ x \in C^2([0,1]) : x_{|[t_i^{(n)}, t_{i+1}^{(n)}]} \text{ is a polynomial of degree } \leq 3 \text{ , } i = 1,\ldots,n-1 \right\} .$$

C_n is called the set of cubic spline functions on the given partition. The dimension of the subspace C_n of $C([0,1])$ is $n+2$, as can be verified by noting that a cubic polynomial on each of the $(n-1)$ intervals has 4 degrees of freedom, which are constrained by 3 continuity conditions at the $(n-2)$ points $t_2^{(n)},\ldots,t_{n-1}^{(n)}$. In fact, it can be shown that for $i = 1,\ldots,n$, there is unique cubic spline function $x_i^{(n)} \in C_n$ such that $x_i^{(n)}(t_j^{(n)}) = \delta_{i,j}$ and which has zero derivatives at 0 and 1. For $x \in C([0,1])$, let, as usual,

$$\pi_n x = \sum_{i=1}^{n} x(t_i^{(n)}) x_i^{(n)} .$$

If $t_i^{(n)} = \frac{i-1}{n-1}$, $i = 1,\ldots,n$, then for $t \in \left[\frac{i-1}{n-1}, \frac{i}{n-1}\right]$, we have, in fact

$$(15.10)\quad \pi_n x(t) = \frac{1}{6(n-1)}\left[a_{i+1}[t - \tfrac{i-1}{n-1}]^3 + a_i[\tfrac{i}{n-1} - t]^3\right]$$
$$+ \frac{1}{(n-1)}\left[x(\tfrac{i}{n-1})[t - \tfrac{i-1}{n-1}] + x(\tfrac{i-1}{n-1})[\tfrac{i}{n-1} - t]\right]$$
$$- \frac{(n-1)}{6}\left[a_{i+1}[t - \tfrac{i-1}{n-1}] + a_i[\tfrac{i}{n-1} - t]\right],$$

where $a_1,\ldots,a_n$ satisfy

$$\begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & 0 & \\ & & \ddots & & \\ & 0 & 1 & 4 & 1 \\ & & & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ \\ \vdots \\ \\ a_n \end{bmatrix} = \frac{1}{(n-1)^2} \begin{bmatrix} x(\frac{1}{n-1}) - x(0) \\ \vdots \\ x(\frac{i}{n-1}) - 2x(\frac{i-1}{n-1}) + x(\frac{i-2}{n-1}) \\ \vdots \\ -[x(1) - x(\frac{n-2}{n-1})] \end{bmatrix}$$

THEOREM 15.5 Let $h_n = \max\{t_i^{(n)}-t_{i-1}^{(n)} : i = 1,\ldots,n\}$, $\tilde{h}_n = \min\{t_i^{(n)}-t_{i-1}^{(n)} : i = 2,\ldots,n\}$, and $r_n = h_n/\tilde{h}_n$. Then

$$\|\pi_n\|_\infty \leq 8r_n^2 + 1,$$

and if x is continuously differentiable on $[0,1]$, then

$$\|\pi_n x - x\|_\infty \leq 4(r_n+1)\ \|x'\|_\infty\ h_n .$$

In particular, if $h_n \to 0$ and (r_n) is bounded, then $\pi_n \xrightarrow{p} I$.

For a proof, we refer the reader to p.144 and Problem 5.26 of [CR].

Other end conditions such as $x''(0) = 0 = x''(1)$ for $x \in C_n$ can also be used to define pointwise convergent interpolatory projections using the cubic spline functions. (See [LS], p.169).

Problems

15.1 Let $T \in BL(X)$, and (π_n) , $(\tilde{\pi}_n)$ be sequences of projections in $BL(X)$ such that $\pi_n \xrightarrow{p} I$ and $\tilde{\pi}_n \xrightarrow{p} I$. Let $T_n = \pi_n T \tilde{\pi}_n$. Then $T_n \xrightarrow{p} T$. If T is compact, then $T_n \xrightarrow{cc} T$, and if, in addition, $\pi_n^* \xrightarrow{p} I$ as well as $\tilde{\pi}_n^* \xrightarrow{p} I$, then $T_n \xrightarrow{\| \|} T$.

15.2 Let $t_1,\ldots,t_n$ be distinct points in $[a,b]$, and let $x_1,\ldots,x_n \in C([a,b])$ be such that $\det(x_i(t_j)) \neq 0$. Then there exist unique $u_1,\ldots,u_n \in \text{span}\{x_1,\ldots,x_n\}$ such that $u_i(t_j) = \delta_{i,j}$, $i,j = 1,\ldots,n$.

15.3 Let $a = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = b$, and h_n denote the mesh of this partition. If $X = L^\infty([a,b])$, the <u>averaging projection</u> $\pi_n : X \to X$ is defined by

$$\pi_n x(t) = \int_{t_{i-1}^{(n)}}^{t_i^{(n)}} x(s)ds \,/\, (t_i^{(n)}-t_{i-1}^{(n)}) \ , \quad t_{i-1}^{(n)} < t \leq t_i^{(n)} \ , \quad i = 1,\ldots,n \ .$$

If X denotes the set of all bounded complex-valued functions on $[a,b]$ with the sup norm, and for $i = 1,\ldots,n$, $s_i^{(n)} \in (t_{i-1}^{(n)}, t_i^{(n)}]$, then the <u>piecewise constant interpolatory projection</u> $\pi_n : X \to X$ with nodes at a and $s_i^{(n)}$, $i = 1,\ldots,n$, is defined by

$$\pi_n x(a) = x(a) \ , \ \pi_n x(t) = x(s_i^{(n)}) \ , \quad t_{i-1}^{(n)} < t \leq t_i^{(n)} \ , i = 1,\ldots,n \ .$$

Then for every $x \in C^1([a,b])$, $\|\pi_n x - x\|_\infty \leq \|x'\|_\infty h_n$. If $h_n \to 0$, then for every $x \in C([a,b])$, $\|\pi_n x - x\|_\infty \to 0$. Is this true for every $x \in X$?

Let $T \in BL(X)$ be such that $R(T) \subset C([a,b])$. Then $T_n^P = \pi_n T \xrightarrow{p} T$, and if T is compact, then $T_n^P \xrightarrow{\| \|} T$, provided $h_n \to 0$ ..

15.4 Let $X = C([a,b])$. For $i = 1,\dots,n$, let $s_i^{(n)} \in (t_{i-1}^{(n)}, t_i^{(n)})$, where $a = t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)} = b$. Consider the piecewise quadratic interpolatory projection $\pi_n : X \to X$, where $\pi_n x|[t_{i-1}^{(n)}, t_i^{(n)}]$ is the unique quadratic polynomial which agrees with x at $t_{i-1}^{(n)}$, $s_i^{(n)}$ and $t_i^{(n)}$, $1 \le i \le n$. Then $\pi_n \xrightarrow{p} I$ need not hold even if $h_n \to 0$, where $h_n = \max\{t_i^{(n)} - t_{i-1}^{(n)} : i = 1,\dots,n\}$. However, if there exist constants α and β such that

$$0 < \alpha \le (t_i^{(n)} - s_i^{(n)}) / (s_i^{(n)} - t_{i-1}^{(n)}) \le 1/\beta$$

for all $n = 1,2,\dots$ and $i = 1,\dots,n$, and if $h_n \to 0$, then $\pi_n \xrightarrow{p} I$.

15.5 Let $0 = t_1^{(n)} < \dots < t_n^{(n)} = 1$. For $i = 1,\dots,n$, there is a unique cubic spline $\tilde{x}_i^{(n)} \in C_n$ such that $\tilde{x}_i^{(n)}(t_j^{(n)}) = \delta_{i,j}$, $j = 1,\dots,n$, and which has zero second derivatives at 0 and 1 . For $x \in C([a,b])$, let $\tilde{\pi}_n x = \sum_{i=1}^{n} x(t_i^{(n)})\tilde{x}_i^{(n)}$. If $t_i^{(n)} = \frac{i-1}{n-1}$, $i = 1,\dots,n$, then $\tilde{\pi}_n x$ has the same expression as $\pi_n(x)$ of (15.10), except that $a_1 = 0 = a_n$, while $a_2,\dots,a_{n-1}$ are determined by

$$a_{i-1} + 4a_i + a_{i+1} = [x(\tfrac{i}{n-1}) - 2x(\tfrac{i-1}{n-1}) + x(\tfrac{i-2}{n-1})]/(n-1)^2$$

as before.