ON THE DIFFERENTIABILITY OF CONVEX FUNCTIONS

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ABSTRACT. Let X be a Banach space and C be a closed, convex subset of X such that N(C), the subset of its non support points, is non empty. We investigate differentiability properties of convex functions defined on N(C) and recover many results known to be true in the case N(C) = int(C).

0. Notation. Let X be a Banach space, X^* be its topological dual and $C \subset X$ be a closed convex set. We shall denote by S(C) the set of all support points of C and by N(C) the set of all non-support points of C. For $x \in C$ let $C_x = \{y \in X;\ x+ty \in C \text{ for some } t>0\}$ be the cone generated by C from x.

Recall that N(C), if non-empty, is a convex, dense G_δ subset of C, in fact a Baire space. If C has interior points, then N(C) is exactly the interior of C. Also $x \in N(C)$ iff $cl(C_x) = X$.

1. THEOREM. Let C be a closed, convex set of X with $N(C) \neq \emptyset$ and A be a relatively open subset of N(C). Let $f: N(C) \to R$ be convex and such that $f|A$ is locally Lipschitz. Then

(i) $\partial f(x) \neq \emptyset$ for all $x \in A$;

(ii) $\partial f(x)$ is a weak* compact subset of X^*;

1980 Mathematics Subject Classification (1985 Revision). 49A51

(iii) the subdifferential map $\partial f: A \to (2^{X^*}, \text{weak}^*)$ is usco and locally bounded.

Proof. The first two assertions are slightly more general than the corresponding assertions in Theorem 1 of [V]; the proof given there is also valid in the actual context. The third assertion was noticed in [R1].

Remark. As a matter of fact, the first assertion is true for A relatively open in C and $f: C \to R$ convex and locally Lipschitz on A (see [N]). The proof given in [V] can be used to obtain this result too.

Conversely, assume that C is a convex subset of X, $f: C \to R$ is convex, $\partial f(x) \neq \emptyset$ for all $x \in C$ and $\partial f: C \to 2^{X^*}$ is locally bounded. Then it is easily seen that f is locally Lipschitz on C. As a matter of fact, as noticed by D. Noll [N], the following result holds true: if $f: C \to R$ is convex and $\partial f(x) \neq \emptyset$ for x in a Baire space which is dense in C, then f is locally Lipschitz at the points of a dense, relatively open subset of C. A proof of a slightly more general result can also be found in [R2]. In what follows we shall present a different, more direct proof of the same result.

2. PROPOSITION. Let C be a convex subset of a Banach space X and $f: C \to R$ be convex. Let A be a Baire space contained in C such that $\partial f(x) \neq \emptyset$ for $x \in A$. Then there exists a dense in A set D such that the restriction of f to A is locally Lipschitz at each point of D. If in addition f is lsc on $cl_C(A)$, the closure of A in C, then the restriction of f to $cl_C(A)$ is locally Lipschitz at each point of D.

<u>Proof.</u> Notice first that f is lsc on A (since $\partial f(x) \neq \emptyset$ for $x \in A$). For each $n \geq 1$ let

$$F_n = \{ x \in A;\ \partial f(x) \cap nB^* \neq \emptyset \}$$

(B^* is the closed unit ball in X^*). Clearly $A = \bigcup F_n$.

STEP I. F_n **is closed in A.** Indeed let (x_k) be a sequence in F_n converging to $x \in A$. For each k choose $h_k \in \partial f(x_k) \cap nB^*$. Since nB^* is bw^* compact, there exists $h \in nB^*$, a bw^* cluster point of the sequence (h_k). Let $y \in C$ and $\varepsilon > 0$; then there exists k such that $|h_k(y)-h(y)| \leq \varepsilon$, $|h_k(x_i)-h(x_i)| \leq \varepsilon$, $i \geq 1$, $|h_k(x)-h(x)| \leq \varepsilon$, $\|x_k - x\| < \varepsilon / \|h\|$, $f(x_k) > f(x) - \varepsilon$. We have :

$$h(y-x) = h_k(y-x_k) + h(y) - h_k(y) + h_k(x_k) - h(x_k) + h(x_k) - h(x)$$
$$\leq f(y) - f(x_k) + \varepsilon + \varepsilon + \|h\| \cdot \|x_k - x\| \leq f(y) - f(x) + 4\varepsilon .$$

Since ε is arbitrary, $h(y-x) \leq f(y) - f(x)$, showing that h is a subgradient of f at x. By construction $h \in nB^*$, so $h \in F_n$.

STEP II. f **is Lipschitz on** F_n **with Lipschitz constant n.** Let $x, y \in F_n$. For $h \in \partial f(x) \cap nB^*$, we obtain $h(y-x) \leq f(y) - f(x)$. So

$$f(x) - f(y) \leq h(x-y) \leq n\|x-y\| .$$

By symmetry, $f(y) - f(x) \leq n\|x-y\|$, proving the assertion.

STEP III. **Construction of** D. Let G_n be the interior of F_n in A and let $D = \bigcup G_n$. Since A is Baire, D is dense in A. The first assertion is proved.

Assume now that f is lsc on $cl_C(A)$.

STEP IV. $f|cl_C(A)$ **is locally Lipschitz at each point of** D. Let $z \in D$. There exists $\delta>0$ such that $B(z,\delta)\cap A \subset F_n$ for some n. Let $x,y \in B(z,\delta)\cap cl_C(A)$ and let $\varepsilon>0$. Since f is lsc at x there exists $x' \in B(z,\delta)\cap B(x,\varepsilon)\cap A$ such that

$$f(x') > f(x) - \varepsilon.$$

Next pick $y' \in B(z,\delta)\cap B(y,\varepsilon)\cap A$ and $h \in \partial f(y')$. We have

$$h(y-y') \leq f(y) - f(y').$$

Combining the last two inequalities we get

$$f(x)-f(y)=f(x)-f(x')+f(x')-f(y')+f(y')-f(y) \leq \varepsilon+n\|x'-y'\| + h(y'-y)$$
$$\leq \varepsilon + n(2\varepsilon+\|x-y\|)+n\varepsilon.$$

Since ε is arbitrary, $f(x) - f(y) \leq n\|x-y\|$, which proves the last assertion.

It is natural now to investigate the Gâteaux and Fréchet differentiability of such functions. A first result in this direction was obtained in [V]where, under the assumption that X is separable, a generalization of Mazur's theorem was given. That result was extended in [R1] to a larger class of Banach spaces. In what follows we shall extend some results of Stegall [S1,S2] to our context and then use them to reobtain the results in [R1].

Let B be a subset of a Banach space Y. Let $T_x(B) \subset Y$ consist of those $v \in Y$ with the following property: there exists a sequence (t_n) of positive real numbers, decreasing to 0 and such that $x+t_n v \in B$ for each n. If B is convex, $T_x(B) = B_x$.

DEFINITIONS. Let X, Y be Banach spaces, $B \subset Y$.

(1) A function $h:B \to X$ is called Gâteaux differentiable at $x \in B$ if there exists $h_x: Y \to X$ linear and continuous such

$$h_x(v) = \lim_{t \downarrow 0} (h(x+tv)-h(x))/t, \quad \text{for all } v \in T_x(B).$$

(2) A function $h:B \to X$ is called Fréchet differentiable at $x \in B$ if there exists $h_x: Y \to X$ linear and continuous such that the function $O_{h,x}:B \to X$ defined by

$$O_{h,x}(y) = \begin{cases} (h(y)-h(x)-h_x(y-x))/\|y-x\| & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}$$

is $(\| \|,\| \|)$ continuous at x.

Observe that in the above definitions the linear continuous map h_x is in general not unique. However if cl aff $T_x(B) = Y$, (for example when B is convex and $N(B) \neq \emptyset$) then h_x is unique. If B is open we recover the usual definitions.

3. THEOREM. Let X, Y be Banach spaces, where X is Asplund (resp. $X \in$ class S (see [S1, S2]). Let $B \subset Y$ be a Baire space, $C \subset X$ be a closed convex set with $N(C) \neq \emptyset$ and U be a relatively open subset of N(C). Let $h:B \to U$ be continuous on B and Fréchet (resp. Gâteaux) differentiable on a dense G_δ subset of B and $f: N(C) \to R$ be convex and locally Lipschitz on U. Then $f \circ h$ is Fréchet (resp. Gâteaux) differentiable on a dense G_δ subset of B.

<u>Proof</u>. We shall prove the assertion about Fréchet differentiability. The other one can be proved similarly. By Theorem 1 it follows that $\partial f:U \to (2^{X^*},\text{weak}^*)$ is usco and locally bounded. Then,

the set valued map $G: B \to (2^{X^*}, \text{weak}^*)$ defined by $G(x) = \partial f(h(x))$ is usco and locally bounded. Since X is Asplund, it follows from Lemma 6.12 and Proposition 6.3 (b) in [P] that there exist a selection $\sigma : B \to X^*$ for G and a dense G_δ subset D_1 of B such that σ is $(\| \|, \| \|)$ continuous at each point of D_1. Let D_2 be the dense G_δ subset of B on which h is Fréchet differentiable. Then $D = D_1 \cap D_2$ is a dense G_δ subset of B. For $x \in D$ define $F_x : Y \to R$ by $F_x = \sigma(x) \circ h_x$, ($h_x$ is the Fréchet differential of h at x). Clearly F_x is linear and continuous. Let $x, y \in B$; then $\sigma(x) \in \partial f(h(x))$, $\sigma(y) \in \partial f(h(y))$ and

$$0 \leq f \circ h(y) - f \circ h(x) - \sigma(x)(h(y)-h(x)) \leq (\sigma(y)-\sigma(x))(h(y)-h(x)).$$

Using the Fréchet differentiability of h at x, we get

$$0 \leq f \circ h(y) - f \circ h(x) - \sigma(x)(h_x(y-x) + \|y-x\| \cdot 0_{h,x}(y))$$

$$\leq (\sigma(y)-\sigma(x))(h_x(y-x) + \|y-x\| \cdot 0_{h,x}(y))$$

or

$$\sigma(x) 0_{h,x}(y) \leq (f \circ h(y) - f \circ h(x) - \sigma(x)(h_x(y-x)))/\|y-x\|$$

$$\leq \sigma(y)(0_{h,x}(y)) + ((\sigma(y)-\sigma(x))(h_x(y-x)))/\|y-x\|$$

$$\leq \sigma(y)(0_{h,x}(y)) + \|\sigma(y)-\sigma(x)\| \cdot \|h_x\| \;;$$

since $\sigma(x) \circ h_x(y-x) = F_x(y-x)$, we get

$$\sigma(x) 0_{h,x}(y) \leq 0_{f \circ h, x}(y) \leq \sigma(y) 0_{h,x}(y) + \|\sigma(y)-\sigma(x)\| \cdot \|h_x\|.$$

Since σ is $(\| \|, \| \|)$ continuous at x (hence bounded on a neighborhood of x) and $O_{h,x}$ is continuous at x, $O_{f\circ h,x}$ is continuous at x which proves the theorem.

Note. Stegall ([4] ,[5]) proved the above results in the case when B is open and U = X.

Taking B = U and h = Id we obtain the following corollary.

4. COROLLARY [R1]. Let X be a Banach space of class S (resp. Asplund), $C \subset X$ be closed and convex with non-empty N(C) and f: N(C) $\to$ R be convex and locally Lipschitz on a dense, relatively open subset of N(C). Then f is Gâteaux (resp. Fréchet) differentiable on a dense G_δ subset of N(C).

Note. In view of Proposition 3, in the preceeding Corollary one can replace the locally Lipschitz assumption by: "f is lsc on N(C) and $\partial f(x) \neq \emptyset$ for all x in a Baire, dense subset of N(C)". The Fréchet differentiability part in the above corollary was also proved in [N] for convex, locally Lipschitz functions defined on Baire convex sets.

Another result that is true in this context is Kenderov's theorem: In a Banach space X, for every convex locally Lipschitz function f on N(C) there exists a dense G_δ subset of N(C) at each point x of which $\partial f(x)$ lies in a face of a sphere of X^*. This can be proved as in [G, p.135]. Using this, one can proceed as in [G, Theorem 15, p.137] and obtain that the Gâteaux differentiability assertion in Corollary 4 is true if X can be equivalently renormed such that X^* is rotund.

Aknowledgement. I would like to thank A. Verona for very helpful discussions on this subject.

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