"STRONG DENSITY OF FINITE RANK OPERATORS IN SUBALGEBRAS OF B(X)."

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Abstract: An open problem in operator theory asks whether for a complete atomic Boolean subspace lattice $\mathcal{L}$ the rank one subalgebra of Alg$\mathcal{L}$ is strong operator dense in Alg$\mathcal{L}$. A very special case of this problem turns out to be equivalent to an open problem in the Theory of Bases. Here various related questions are surveyed and some positive results are given.

§0 Introduction The first part of this paper is to survey certain density results and open problems in Operator Theory. It turns out that a special case of the main open problem is equivalent to an old standing problem in the Theory of Bases. This perhaps unexpected link between an Operator Theory version and a Basis Theory version of the same open problem is explored in the second part. The third part of the paper gives certain new results along these lines.

The link between Operator Theory and Basis Theory here is provided by a result in [1] which is under preparation. To avoid the overlap however we shall only report a brief (but sufficiently long) summary of the proof.

The author wishes to thank his co-authors of [1], S. Argyros ar

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§1. Subspace Lattices. In the following H will denote a Hilbert space over the real or complex scalars. The cases when H is required to be restricted to be a separable complex Hilbert space over $\mathbb{C}$, will be specified whenever required. The letter X will denote a Banach space, again over $\mathbb{R}$ or $\mathbb{C}$, and X^* is its dual. The set of (bounded linear) operators on X is denoted by $\mathcal{B}(X)$ and the rank one operator $x \mapsto e^*(x)f$ for fixed $f \in X$ and $e^* \in X^*$ is denoted by $e^* \otimes f$. By $\mathcal{C}(X)$ we denote the lattice of closed subspaces of X, and a collection $\mathcal{L} \subseteq \mathcal{C}(X)$ is called a subspace lattice if it contains the two extremes, X and the zero subspace (0), and is complete with respect to taking arbitrary closed linear spans and intersections. That is, whenever $L_i \in \mathcal{L}$ $(i \in I)$ for some indexing set I, then also $\bigvee_I L_i$ and $\bigcap_I Li$ belong to $\mathcal{L}$. If $\mathcal{L} \subseteq \mathcal{C}(X)$ then Alg$\mathcal{L}$ denotes the set $\{A \in \mathcal{B}(X) \mid A(L) \subseteq L \text{ for all } L \in \mathcal{L}\}$. That is, Alg$\mathcal{L}$ is the set of all operators leaving the elements of $\mathcal{L}$ invariant. It is easy to see that Alg$\mathcal{L}$ is an algebra which is closed in the weak operator (and hence strong operator and norm) topology. Dually, if $\mathcal{A} \subseteq \mathcal{B}(X)$ then Lat$\mathcal{A}$ denotes the set $\{L \in \mathcal{C}(X) \mid A(L) \subseteq L \text{ all } A \text{ in } \mathcal{A}\}$. For any $\mathcal{A}$ the set Lat$\mathcal{A}$ is a subspace lattice. Following Halmos' by now standard terminology, a subspace lattice is called reflexive if LatAlg$\mathcal{L} = \mathcal{L}$. Note that the inclusion LatAlg$\mathcal{L} \supseteq \mathcal{L}$ is always true. It is easy to see that a necessary and sufficient condition for $\mathcal{L}$ to be reflexive is that $\mathcal{L} =$ Lat$\mathcal{A}$ for some $\mathcal{A}$ (necessarily $\mathcal{A} \subseteq$ Alg$\mathcal{L}$). For further discussion on the topic of invariant subspaces we refer to [22].

The question of characterizing reflexive (necessarily subspace) lattices has the following partial answers. "Most" of the known examples of reflexive lattices are distributive in the sense that if L,M,N, are elements of the lattice then $(L \vee M) \wedge N = (L \wedge N) \vee (M \wedge N)$ and its dual hold (see [3] for standard terminology on Lattice Theory.) Indeed, in finite dimensional Banach spaces we have the following characterization of R. E. Johnson.

Theorem 1 ([12]) In a finite dimensional vector space a finite lattice is reflexive if and only if it is distributive.

The first reflexivity result in infinite dimensional spaces is due to Ringrose who studied certain generalizations of subspace lattices considered by Kadison and Singer in their seminal paper [13]. Ringroses' Theorem is stated and proved in [23] for Hilbert spaces but the proof can be adapted (we shall omit this here) to Banach (in fact just normed) spaces.

Theorem 2 ([23]) Any totally ordered subspace lattice $\mathcal{L}$ of subspaces of a Banach space is reflexive. In fact $\mathcal{L}$= Lat$\mathcal{R}$ where $\mathcal{R}$ is the set of rank one operators of Alg$\mathcal{L}$.

To fix one more notational symbol, for a given subspace lattice $\mathcal{L}$, the set of finite sums of rank one operators of Alg$\mathcal{L}$ will be denoted by $\mathcal{R}$. This may be empty, and will be called the rank one subalgebra of Alg$\mathcal{L}$. The $\mathcal{R}$ in the Theorem 2 can be replaced by this, new, $\mathcal{R}$.

The next infinite dimensional reflexivity result is due to Halmos : Again this is proved for Hilbert spaces but is also valid for Banach spaces.

Theorem 3 ([9]) Any atomic Boolean subspace lattice $\mathcal{L}$ of subspaces of a Banach space satisfies $\mathcal{L}$= Lat$\mathcal{R}$ and hence is reflexive.

(The conclusion $\mathcal{L}$= Lat$\mathcal{R}$ is not stated in [9] but it is implicit in the proof). Recall that Boolean lattices are distributive. Related to these is the result of Harrison [10] who showed, again in the presence of a certain (strong) distributivity condition, a reflexivity result. Specifically an infinitely distributive subspace lattice $\mathcal{L}$ in which each non-zero sub-space is the join of completely-join-irreducible subspaces in $\mathcal{L}$ is reflexive. All of the last three results are a special case of the result of Longstaff (which we state for Banach spaces instead of the original Hilbert space version).

<u>Theorem 4</u> ([18]). Every completely distributive subspace lattice $\mathcal{L}$ of subspaces of a Banach space satisfies $\mathcal{L} = \mathrm{Lat}\,\mathcal{R}$ and hence is reflexive.

We shall not attempt to define complete distributivity here (see [18], [14]). We only mention that it is equivalent to the identity

$$L = \bigcap \{ M_- \mid M \in \mathcal{L},\ M \not\leq L \}$$

holding for all L in $\mathcal{L}$, where M_- is defined as

$$M_- = \bigvee \{ K \in \mathcal{L} \mid M \not\leq K \}.$$

A most interesting family of reflexive lattices that has been the object of very active research in the past few years are, in complex Hilbert spaces, the commutative subspace lattices of Arveson [2]. A subspace lattice is called <u>commutative</u> if the corresponding projections commute. For example totally ordered subspace lattices have this property. In the pioneering paper [2] we have (for separable Hilbert spaces but extended to general ones by Davidson in [6]):

<u>Theorem 5</u> ([2], [6]). In a complex Hilbert space every commutative subspace lattice is reflexive.

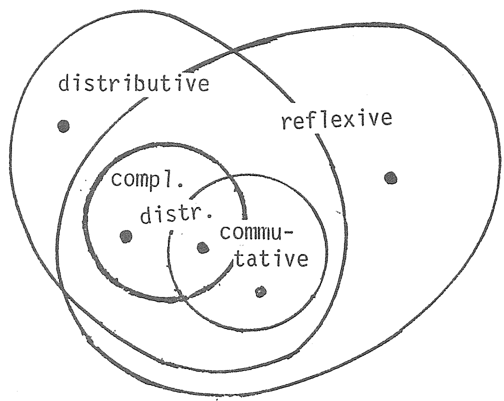
It is easy to see that commutative subspace lattices are distributive, but not conversely.(For example two non-orthogonal quasi-complemented subspaces). An example of T. Trent in [11] (Example 4) shows that there exist commutative subspace lattices which are not completely distributive. A necessary and sufficient condition for complete distributivity of a commutative subspace lattice is given in [11], drawing upon deep results of Arveson in [2].

Not all examples of reflexive lattices are distributive. For example $\mathcal{C}(X)$ is such, but a non-trivial example (a pentagonal lattice) is given by Halmos in [9]. This result is extended by Longstaff [18] to lattices satisfying the condition

$$\dim \left(\bigcap \{ M_- \mid M \in \mathcal{L},\ L \not\leq M \} \ominus L \right) \leq 1$$

for all L in $\mathcal{L}$. Notice that if this dimension is zero, we have complete distributivity.

In infinite dimensional spaces the other direction of Johnson's finite dimensional characterization, Theorem 1 above, also fails. Conway [4] gave an example of a non-reflexive Boolean (and thus distributive) lattice. To summarize, the above results can be pictured in the following diagram, where each dot signifies that the space concerned is non-empty. Below we produce an example showing that the remaining space is also non-empty. That is, we produce a reflexive distributive subspace lattice which is neither completely distributive nor commutative (In order not to distract from the survey, we postpone the example till §3).



The Ringrose, Halmos and Longstaff reflexivity results above showed the presence of sufficiently many rank one operators to describe the lattice. A pertinent question is whether $\mathcal{R}$ is large enough to describe the algebra : Is it true that the strong operator closure of $\mathcal{R}$ is the whole of Alg$\mathcal{L}$? For example the algebra $\mathcal{B}(X)$ falls into this category, and note that $\mathcal{B}(X)=$ Alg$\mathcal{L}$ for the trivial subspace lattice $\mathcal{L}=\{(0), X\}$. It turns out that complete distributivity is the right context for strong density of $\mathcal{R}$. Infact the following characterization, which for obvious reasons we call the 1-density, is valid. The necessity is from [19] and the sufficiency from [15].

<u>Theorem 6</u> ([19], [15]) Let $\mathcal{L}$ be a subspace lattice on a Banach space X. Then a necessary and sufficient condition for $\mathcal{R}$ to be completely distribu-

tive is that for each $x \in X$, $\varepsilon > 0$ and $A \in \mathrm{Alg}\mathcal{L}$ there exists an element R of $\mathcal{R}$, the set of finite sums of rank one operators of $\mathrm{Alg}\mathcal{L}$, such that $\|Ax - Rx\| < \varepsilon$.

Observe that the condition in this theorem need only be verified for $A = I$, the identity on X, since the set $\mathcal{R}$ is an ideal of $\mathrm{Alg}\mathcal{L}$.

Is strong density a conclusion in the above theorem? That is, if we can approximate at any one given vector x within epsilon, can we do the same for any given finite set of vectors? This is an open problem and we state:

Question 1 Let $\mathcal{L}$ be a completely distributive subspace lattice. Is the set of finite sums of rank one operators of $\mathrm{Alg}\mathcal{L}$ strong operator dense in $\mathrm{Alg}\mathcal{L}$?

The question is equivalent to asking weak operator density of $\mathcal{R}$ in $\mathrm{Alg}\mathcal{L}$, since on convex sets the two closures coincide. For Hilbert spaces it is also of interest to know (the harder) density in the ultraweak and ultrastrong topology or to know whether density can be uniformly bounded or sequential. Finally whether, at least, the various density properties hold for sets between $\mathcal{R}$ and $\mathrm{Alg}\mathcal{L}$, such as the trace class or Hilbert- Schmidt or even the compact operators of $\mathrm{Alg}\mathcal{L}$.

In several special cases the above question is known to have an affirmative answer. Perhaps the best result in this context is the Erdos density theorem which not only concludes strong density of $\mathcal{R}$ in the case of totally ordered $\mathcal{L}$ in separable Hilbert spaces, but the following Kaplansky type unit ball density theorem holds.

Theorem 7 ([7]) Let $\mathcal{L}$ be a totally ordered subspace lattice on a complex separable Hilbert space. Then for any given A in the unit ball of $\mathrm{Alg}\mathcal{L}$, and given $x_1, x_2, \ldots, x_n$ in X, $\varepsilon > 0$, there is an R in the unit ball of $\mathcal{R}$ such that $\|Ax_i - Rx_i\| < \varepsilon$ $(i = 1, 2, .., n)$.

Totally ordered subspace lattices are both commutative and completely distributive. In the presence of commutativity we have an affirmative answer to Question 1. The following is putting together results from [11] and [17] which used [2] and [16].

Theorem 8 ([11], [17]). Let $\mathcal{L}$ be a commutative subspace lattice. Then $\mathcal{L}$ is completely distributive if and only if the Hilbert- Schmidt operators of Alg$\mathcal{L}$ are strong operator dense in Alg$\mathcal{L}$ if and only if $\mathcal{R}$ is dense in Alg$\mathcal{L}$ in any of the strong, weak, ultrastrong or ultraweak operator topologies.

In their theorem [11] the authors give yet another characterization of complete distributivity (in the commutative case.) This is a measure theoretic characterization which was used by T. Trent in his example (see above) to show that a commutative subspace lattice may have trivial $\mathcal{R}$, and hence may be non-completely distributive.

In the unpublished [1] the special case of a complete atomic Boolean subspace lattice with two atoms is settled affirmatively. Moreover a result of Harrison to appear in [1] settles the unit ball question when the underlying Hilbert space is separable and complex.

Theorem 9 ([1] and Harrison reported in [1]) (i). Let L and M be quasi-complemented subspaces of a Banach space X. Then in Alg$\mathcal{L}$, where $\mathcal{L} = \{(0), L, M, X\}$, $\mathcal{R}$ is strongly dense in Alg$\mathcal{L}$. Moreover

(ii) In the case of complex separable Hilbert spaces, the conclusion of Theorem 7 holds.

Another special case of complete atomic Boolean subspace lattices on a Banach space are the ones in the other extreme, namely those with one dimensional atoms. Those proved [1] to be intimately related to a generalization of a Schauder basis, and a special case of Question 1 turned out to be an open problem in the Theory of Bases. We discuss this in the next section.

§ 2 Strong M-bases. One of the generalisations of a Schauder basis on a Banach space studied in [26] are the strong M- bases. Recall, a strong M- basis is an M- basis (complete and total biorthogonal family) (f_n, f_n^*) with the additional property that

$$\bigcap_I \operatorname{Ker} f_n^* = \bigvee_{I^c} f_n \quad \text{(for every } I \subseteq \mathbb{N}\text{)}$$

These bases were introduced previously by various authors under a variety of names or equivalent (as it was later proved) definitions. For example the strongly complete bases of Markus [20] and the 1- series summable bases of Ruckle [25], are identical to the strong M- bases. The following is more or less from [1] and connects the notion of strong M- bases to completely distributive subspace lattices.

Theorem 10 Let (f_n, f_n^*) be an M- basis. Then the following are equivalent

(i) The set $\mathcal{L} = \{\bigvee_I f_i \mid I \subseteq \mathbb{N}\}$ is a complete atomic Boolean subspace lattice (with one dimensional atoms the $\langle f_i \rangle$).

(ii) For each I and J contained in $\mathbb{N}$ we have $(\bigvee_I f_i) \cap (\bigvee_J f_j) = \bigvee_{I \cap J} f_i$.

(iii) (f_n, f_n^*) is a strong M- basis.

(iv) For any x in X and $\varepsilon > 0$ there exists a finite rank operator of the form $R = \sum \lambda_i f_i^* \otimes f_i$ such that $\| x - Rx \| < \varepsilon$.

Briefly [1] : The hardest part of the proof is (ii) $\Rightarrow$ (i). Assuming (ii) clearly $\mathcal{L}$ is complemented and distributive. The difficulty is to prove completeness. This requires us to prove that the condition in (ii) extends to arbitrary intersections:

$$\bigcap_{\lambda \in \Lambda} \bigvee_{I_\lambda} f_i = \bigvee_{\cap I_\lambda} f_i \quad (\text{ all } I_\lambda, \lambda \in \Lambda)$$

The one inclusion being obvious let x be in the left hand side. We can construct inductively using (ii) a sequence I_n of finite sets of indices such that x is within 1/n of the linear span of $\{f_i \mid i \in I_n\}$ and such that $I_{n+1} \cap (\bigcup_{m=1}^{n} I_m) \subseteq \bigcap I_\lambda$. Then if $I = \bigcup_n I_{2n+1}$, $J = \bigcup_n I_{2n}$, it is easy to see that x belongs to both $\bigvee_I f_i$ and $\bigvee_J f_i$. Again by (ii) it follows that $x \in \bigvee_{I \cap J} f_i \subseteq \bigvee_{\cap I_\lambda} f_i$, completing the summary of the proof. The other parts are included in one way or another in the literature.

The crucial observation in the above theorem is that the operators appearing in (iv) are in Alg$\mathcal{L}$ for $\mathcal{L}$ the complete atomic Boolean subspace lattice described in (i). By Lemma 3.1 of [18] these operators exhaust $\mathcal{R}$. So (iv) is simply the conclusion of Theorem 6 which is valid for the more general case of completely distributive subspace lattices. (Recall that complete atomic Boolean lattices are completely distributive by a result of Tarski [3]). On the other hand, strong density of $\mathcal{R}$ is not known even in the (very) special case of complete atomic Boolean subspace lattices with one dimensional atoms considered here. This question was raised, in a different language, by Ruckle in [25] and we state it in an equivalent but Operator Theory context.

Question 2 If $\mathcal{L}$ is a complete atomic Boolean subspace lattice with one dimensional atoms, is $\mathcal{R}$ dense in Alg$\mathcal{L}$ in the strong operator topology? Equivalently, if (f_i, f_i^*) is a strong M- basis, is it true that for given $x_1, \ldots, x_N$ in X and $\varepsilon > 0$, there exist scalars $\lambda_1, \ldots, \lambda_M$ such that $\| x_i - (\sum_{1}^{m} \lambda_n f_n^* \otimes f_n) x_i \| < \varepsilon$?

The corresponding question for sequential density of $\mathcal{R}$ was raised in [25] and was proved false in [5], so unit ball density as in Erdos' Theorem 7 above fails here. Also notice that for Schauder bases, the λ_n's above could be taken as 1's. In [21] Menshov gives an example where the λ's cannot always be replaced by 1's in the Fourier trigonometric functions (which by Fejers theorem and the equivalence of (iii) and (iv) above , form a strong M- basis).

If we call the approximation at any given vectors described in Question 2 as the n- density property of $\mathcal{R}$, the above question can also be rephrased as : Does 1- density imply n- density for each n? By analogy to the Jacobson n- fold transitivity theorem (see [J] or [22] Chapter 8) we feel that the following question is more proper.

Question 2' If $\mathcal{L}$ is a complete atomic Boolean subspace lattice with the 2- density property (notice that the 1- density property is automatic), does it follow that it will have the n- density property?

Partial answers to relevant questions postponed above are given in the section following.

§3 Some new results

It was mentioned in §1 that there exists a reflexive and distributive subspace lattice which is neither completely distributive nor commutative. The example is as follows:

Example Let $e_0, e_1, e_2 \dots$ be an orthonormal basis of the Hilbert space $H = \ell_2$. Consider the following subspaces of ℓ_2

$$L_n = \bigvee_{k\geq 0}^{n} e_k \qquad (n = 0, 1, 2\dots)$$

$$M_0 = (0),\ M_n = \bigvee_{k\geq 1}^{n} (e_0+e_k) \qquad (n = 1, 2, \dots)$$

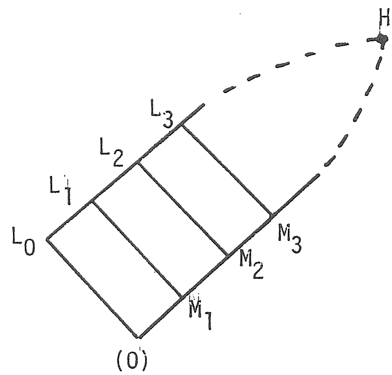
so that $L_0 \subset L_1 \subset L_2, \dots, M_0 \subset M_1 \subset M_2 \subset \dots, M_n \subseteq L_n\ (n \geq 0)$, $L_n \vee M_{n+1} = L_{n+1}\ (n \geq 0)$, and, because of the linear independence of $e_{n+1}\ (n \geq 0)$ from the previous vectors, $L_n \cap M_{n+1} = M_n\ (n \geq 0)$. These relations show that $L_n \vee M_n = L_{\max(n,m)}\ L_n \cap M_m = M_{\min(n,m)}$ and hence that $\mathcal{L} = H \cup \{L_n \mid n \geq 0\} \cup \{M_n \mid n \geq 0\}$ is a (not necessarily complete) lattice. We proceed to show that it is also complete and therefore a subspace lattice. A moments reflection, because of the above properties, shows that we only need to show

$$\bigvee_I L_n = H = \bigvee_I M_n$$

for any infinite set $I \subseteq \mathbb{N}$ of indexes. Because of the inclusions $L_n \supseteq L_{n-1}$, $M_n \supseteq M_{n-1}$ $(n \geq 1)$ we only need to show $\bigvee_{\mathbb{N}} L_n = H = \bigvee_{\mathbb{N}} M_n$. Of these the first is clear and the second sollows from the following: since

$$\| e_0 - \sum_{k=1}^{N} \frac{1}{N}(e_0 + e_k) \| = \frac{1}{N} \| \sum_{k=1}^{N} e_k \| = \frac{1}{\sqrt{N}} \longrightarrow 0 \qquad \text{—— } (*)$$

we have that $e_0 \in \bigvee_{\mathbb{N}} M_n$. Thus $e_k = (e_k + e_0) - e_0$ are also in $\bigvee_{\mathbb{N}} M_n$, showing the required property $\bigvee_{\mathbb{N}} M_n = H$. The following Hasse diagram summerizes the above.



Easy direct calculations show that $\mathfrak{L}$ is distributive (alternatively , since clearly $\mathfrak{L}$ contains neither a pentagon nor a double triangle it is distributive).

On the other hand complete distributivity fails since

$$L_0 \cap (\bigvee_{n \geq 1} M_n) = L_0 \quad \text{yet} \quad \bigvee_{n \geq 1} (L_0 \cap M_n) = (0).$$

Finally commutativity fails since the orthogonal projections onto the one dimensional subspaces L_0 and M_1, namely $e_0 \otimes e_0$ and $\frac{1}{2}(e_0 + e_1) \otimes (e_0 + e_1)$, do not commute.

It remains to show that $\mathfrak{L}$ is reflexive. Although it is possible to describe all of Alg$\mathfrak{L}$, we only describe sufficiently many operators to guarantee reflexivity. We define $R_{mn} = e_m \otimes (e_0 + e_n)$ for $m \geq n \geq 1$, and show that they leave invariant all of $\mathfrak{L}$. This is so because

$$R_{mn}(L_k) = \{0\} \quad (0 \leq k \leq n-1), \qquad R_{mn}(L_k) \subseteq \langle e_0 + e_n \rangle \subseteq L_k \quad (k \geq n)$$

$$R_{mn}(M_k) = \{0\} \quad (0 \leq k \leq n-1), \quad R_{mn}(M_k) \subseteq \langle e_0 + e_n \rangle \subseteq M_k \quad (k \geq n)$$

Let then $(0) \neq L \in$ LatAlg$\mathfrak{L}$, so that $R_{mn}(L) \subseteq L$. We are to show that $L \in \mathfrak{L}$. The first step is to show that if $x = (x_0, x_1, \ldots)$ is a non- zero vector in L, and if for some $n>0$ the entry x_n is non- zero, then L also contains the vectors $e_0+e_1,\dots,e_0+e_n$. Indeed, for $1\leq k\leq n$ we have

$$e_0+e_k = \frac{1}{x_n} R_{nk}x \in R_{nk}(L) \subseteq L.$$

We now distinguish two cases, according as $e_0 \notin L$ or $e_0 \in L$.

Case 1, $e_0 \in L$. Suppose that the vectors in L all have a zero entry from some N onwards, and let N be the least such N. In this case we have $L \subseteq \bigvee_{k=0}^{N} e_k$. By the first step each $e_k = (e_0+e_k) - e_0$ $(1\leq k\leq N)$ is also in L and thus $\bigvee_{k=0}^{N} e_k \subseteq L$, showing that $L = \bigvee_{k=0}^{N} e_k = L_N \in \mathcal{L}$.

If on the other hand the vectors in L do not have all zero entries from any N onwards, the first step shows that L contains all e_k $(k\geq 0)$ so that $L = H \in \mathcal{L}$.

Case 2, $e_0 \notin L$. In this case we shall show $L = M_n$ for some n. Clearly the vectors of L must be zero from some co- ordinate onwards: otherwise, by the first step, L would contain all e_0+e_n and hence e_0 by (*). So let N be the least integer with the entries of each vector in L having zero co- ordinate x_n for $n\geq N+1$. So each vector x of L is of the form $x = (x_0,x_1,\dots,x_N,0,0,\dots)$. We show that we must have $x_0 = \sum_1^N x_i$. Indeed, the vectors e_0+e_k $(1\leq k\leq N)$ are all in L and hence so is the vector

$$x - \sum_1^N x_i(e_0+e_i) = (x_0 - \sum_1^N x_i)\, e_0$$

As we have assumed that $e_0 \notin L$ we must have $x_0 - \sum_1^N x_i = 0$, as required. Hence the vector x of L can be written as $x = \sum_1^N x_i(e_0+e_i)$ showing that it is in M_N. As $M_N \subseteq L$ by the first step we have $L = M_N \in \mathcal{L}$ concluding the proof of the reflexivity of $\mathcal{L}$. ∎

We remark that we could base the above reflexivity proof on Corollary3.2.1 of [18]. We have not done so because we can slightly modify the above example (replacing L_n by $\bigvee_{k\geq 1}^{2n+1} e_k$ and M_n by $\bigvee_{k\geq 1}^{n}(e_0+e_{2k}) \vee \bigvee_{k\geq 1}^{n}(e_{1}+e_{2k+1}))$ to arrange that this $\mathcal{L}$ is not covered by the said corollary. In this case it can be shown that with the notation in [18] $\dim(M_{n*} \ominus M_n) = 2$. On the other hand the proof of reflexivity is a trivial modification of the one given here. We now turn to density related results.

Before stating the next theorem, some remarks are in order. The property $\overline{\mathcal{R}} = \mathrm{Alg}\mathcal{L}$ for a subspace lattice $\mathcal{L}$, where $^{-}$ denotes closure under some operator topology, (strong, weak etc.), is equivalent by the Hahn- Banach theorem and the ideal property of $\mathcal{R}$ in $\mathrm{Alg}\mathcal{L}$ to the property that for each linear functional φ continuous in $\mathcal{B}(X)$ with respect to this topology we have

$$(\varphi(R) = 0 \text{ for all } R \text{ in } \mathcal{R}) \Rightarrow \varphi(I) = 0.$$

In the case of the ultraweak operator topology on a Hilbert space, the continuous linear functionals are given by $\varphi_T(\cdot) = \mathrm{tr}\,(T\cdot)$ Where T is a trace class operator and tr denotes trace. So to show $\overline{\mathcal{R}}^{uw} = \mathrm{Alg}\,\mathcal{L}$ it is equivalent to showing that if $\mathrm{tr}\,(TR) = 0$ for all R in $\mathcal{R}$ then $\mathrm{tr}\,(T) = 0$. Recall that the trace of T is given by $\sum \langle Te_n, e_n\rangle$ where $\{e_n\}$ is any orthonormal basis. In the following we show under certain assumptions on a trace class operator T, there exists a closed linear transformation and an orthonormal basis such that $\langle TAe_n, A^{*-1} e_n\rangle = 0$. Note that formally (but not exactly) this says that $A^{-1}TA$ has its diagonal consisting entirely of zeros and a fortiori has zero trace. If for example we knew that A were a bounded invertible operator it would follow that $A^{-1}TA$ and hence T itself would have zero trace.

<u>Theorem 11</u> Let (f_i, f_i^*) be a strong M- basis (equivalently the $\langle f_i\rangle$ generate as atoms a complete atomic Boolean subspace lattice) on a separable Hilbert space and let $\mathrm{tr}(TR) = 0$ for all R in $\mathcal{R}$,where T is a trace class operator. Then there exists a densely defined injective linear transformation A with dense range and an orthonormal basis $(e_n)_{\mathbb{N}}$ such that each e_n is in the domain of both A and A^{*-1} and such that $\langle TAe_n, A^{*-1}e_n\rangle = 0\ (n \in \mathbb{N})$. We

may even take A to be positive.

<u>Proof</u> Let $(e_n)_{\mathbb{N}}$ be any orthonormal basis in H and let A_0 be defined on the linear span of the (e_n) by $A_0 e_n = f_n$. The completeness and totality assumptions show that A_0 is well defined, is injective, its domain is dense and so is its range. We show that this A_0 is closable. Let then $x_n \in \mathfrak{D}(A_0)$, $x_n \to 0$ and $Ax_n \to y$. If $x_n = \sum_i \lambda_i^{(n)} e_i$ for some finite sum, we have $\lambda_i^{(n)} \to 0$ for each fixed i. But $Ax_n = \sum_i \lambda_i^{(n)} Ae_i = \sum_i \lambda_i^{(n)} f_i$ so that taking inner product with f_i^* and using $Ax_n \to y$ we get $\lambda_i^{(n)} \to \langle y, f_i^* \rangle$, so that $\langle y, f_i^* \rangle = 0$ By the totality of the f_i^* we conclude that $y = 0$, showing that A_0 is closable.

Let A be the closed extension of A with $G(A) = \overline{G(A_0)}$. We have $f_n = Ae_n$ and claim that $e_n \in \mathfrak{D}(A^{*-1})$ and $A^{*-1} e_n = f_n^*$. Equivalently we are to show that $f_n^* \in \mathfrak{D}(A^*)$ and $A^* f_n^* = e_n$. Indeed, for $x \in \mathfrak{D}(A_0)$ (not $\mathfrak{D}(A)$)we have, for some finite sum, $x = \sum \mu_i e_i$ so for any fixed n

$|\langle f_n^*, A_0 x \rangle| = |\langle f_n^*, A_0 \sum \mu_i e_i \rangle| = |\langle f_n^*, \sum \mu_i f_i \rangle|$.

This last expression is either zero or $|\mu_n|$ according to whether n exceeds or not the largest index in the summation. In any case the last expression is less that or equal to $(\sum |\mu_i|^2)^{\frac{1}{2}} = ||x||$.

Hence the map $x \mapsto \langle f_n^*, A_0 x \rangle$ is continous on $\mathfrak{D}(A_0)$ showing that $f_n^* \in \mathfrak{D}(A_0^*)$. But as $\overline{\mathfrak{D}(A_0)} = H$ we have $\mathfrak{D}(A^*) = \mathfrak{D}(A_0^*)$ (see [8] page 54) proving that $f_n \in \mathfrak{D}(A^*)$. Now for fixed m,n we have

$\langle A^* f_n^*, e_m \rangle = \langle f_n^*, Ae_m \rangle = \langle f_n^*, f_m \rangle = \delta_{nm} = \langle e_n, e_m \rangle$ concluding that $A^* f_n^* = e_n$, as required.

Let now T be a trace class operator such that tr (TR) = 0 for all R in $\mathcal{R}$. In particular for the elements $f_i^* \otimes f_i$ of $\mathcal{R}$ we have that

$$\langle Tf_i, f_i^* \rangle = tr(T(f_i^* \otimes f_i)) = 0$$

so that $\langle TAe_i, A^{*-1} e_i \rangle = 0$, as claimed.

Now, A is injective with dense range and hence so is the closed linear transformation A^*.

By the closed linear transformation version of the polar decomposition theorem ([24] page 297) applied to A^* we have $A^* = U\,|AA^*|^{\frac{1}{2}} = UB$ say, with B closed self- adjoint positive, $\mathfrak{D}(A^*) = \mathfrak{D}(B)$, and U a partial isometry. Because A^* is injective with dense range the partial isometry U is actually a unitary and so we have $A^{*-1} = B^{-1}U^*$ and $A = B^*U^* = BU^*$.

Hence $$0 = \langle TAe_n, A^{*-1}e_n\rangle = \langle TBU^*e_n, B^{-1}U^*e_n\rangle$$

and we may replace A by B and (e_n) by (U^*e_n) which is also an orthonormal basis. ∎

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