REPRESENTING MONOTONE OPERATORS BY CONVEX FUNCTIONS

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Abstract We transform the representation of monotone operators due to Krauss to get a representation of monotone operators in terms of the subdifferentials of convex functions on the product of the space and its dual. The convex functions representing maximal monotone operators satisfy a minimality condition.

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1. Introduction. The representation of monotone operators on a space E in terms of the subdifferentials of saddle functions on $E\times E$ was accomplished by Krauss [Kr]. In this paper we develop the representation of monotone operators on E in terms of the subdifferentials of convex functions on $E\times E^*$. These are actually transforms of Krauss' saddle functions. However the results we obtain have quite a different flavour than those of Krauss.

We originally tried this approach in attempting to solve a problem we stated in [F-C]: if a monotone operator on Banach space E has domain E and range E^* then must E be reflexive? However it seems that convex analysis on $E\times E^*$ does not help answer that question.

Throughout this paper E is a Hausdorff locally convex space and E^* is its dual with the weak* topology. We recall some definitions. A mapping T of E into subsets of E^* is a *monotone operator* provided for each $x^*\in Tx$ and $y^*\in Ty$ we have $\langle x^* - y^*, x - y\rangle \geq 0$. The *domain* of T is the set $D(T):=\{x\in E \mid Tx\neq\emptyset\}$, the *range* of T is the set $R(T):=\{x^*\in E^* \mid x^*\in Tx \text{ for some } x\in E\}$ and the *graph* of T is the set $G(T):=\{(x,x^*) \mid x\in D(T), x^*\in Tx\}$. If T is monotone and $G(T)$ is not properly contained in the graph of a monotone operator on E then T is said to be *maximal monotone* .

We adopt the natural duality on $E\times E^*$ identifying $(E\times E^*)^*$ with $E^*\times E$ so that $\langle (y^*,y),(x,x^*)\rangle := \langle y^*,x\rangle + \langle x^*,y\rangle$ for all x and y in E and x^* and y^* in E^*. Our convex functions will be proper, that is, they have values in $]-\infty,\infty]$ and are not identically equal to ∞.

2. Convex functions on E×E*. In this section we define and study a monotone operator on E using a convex function on E×E*.

2.1. Definition For each convex function f on E×E* let

$$T_f x := \{x^* \in E^* \mid (x^*,x) \in \partial f(x,x^*)\}$$

for each $x \in E$. ♦

2.2. Proposition If f is convex on E×E* then T_f is a monotone operator on E.

Proof If $x^* \in T_f x$ and $y^* \in T_f y$ then, since ∂f is monotone, we have

$$\langle x^* - y^*, x - y\rangle = (1/2)\langle (x^*,x) - (y^*,y), (x,x^*) - (y,y^*)\rangle \geq 0.$$ ♦

2.3. Example Let g be a convex function on E and

$$f(x,x^*) := g(x) + g^*(x^*) = \sup\{g(x) - g(y) + \langle x^*,y\rangle \mid y \in E\}.$$

Then $T_f = \partial g$.

Proof If $x^* \in \partial g(x)$ then $x \in \partial g^*(x^*)$ so that $(x^*,x) \in \partial f(x,x^*)$. On the other hand if $(x^*,x) \in \partial f(x,x^*)$ then for $u \in E$ we have

$$\langle x^*,u\rangle = \langle (x^*,x),(u,0)\rangle \leq f(x+u,x^*) - f(x,x^*)$$
$$= g(x+u) + g^*(x^*) - g(x) - g^*(x^*) = g(x+u) - g(x)$$

so that $x^* \in \partial g(x)$ as required. ♦

We will be interested in the case when $f(x,x^*) \geq \langle x^*,x\rangle$ for all $x \in E$ and $x^* \in E^*$. This allows a simple way to guarantee $x^* \in T_f x$.

2.4. Theorem Suppose f is a convex function on E×E* such that $f(x,x^*) \geq \langle x^*,x\rangle$ for all (x,x^*) in some neighbourhood U of (y,y^*). If $f(y,y^*) = \langle y^*,y\rangle$ then $y^* \in T_f y$.

Proof Let $(z,z^*) \in E \times E^*$ and $s > 0$ so that $(y+sz, y^*+sz^*) \in U$. Then

$$f(y+z,y^*+z^*) - f(y,y^*) \geq s^{-1}[f(y+sz,y^*+sz^*) - f(y,y^*)] \geq s^{-1}[\langle y^*+sz^*, y+sz\rangle - \langle y^*,y\rangle]$$
$$= \langle z^*,y\rangle + \langle y^*,z\rangle + s\langle z^*,z\rangle.$$

Letting $s \to 0+$ we have $f(y+z,y^*+z^*) - f(y,y^*) \geq \langle z^*,y\rangle + \langle y^*,z\rangle$ so that $(y^*,y) \in \partial f(y,y^*)$ and $y^* \in T_f y$ as required. ♦

Now denote the x-*section* of f by $f_x(x^*) := f(x,x^*)$ for $x \in E$ and $x^* \in E^*$.

2.5. Theorem Suppose f is a convex function on E×E* and $x \in \partial f_x(x^*)$. If

$$\sup\{\langle y^*,x\rangle - f(x,y^*) \mid y^* \in E^*\} = 0 \qquad (1)$$

then $f(x,x^*) = \langle x^*,x\rangle$.

Proof If $u^* \in E^*$ then $\langle u^*,x\rangle \leq f_x(x^*+u^*) - f_x(x^*)$ so we have

$$\langle x^*+u^*,x\rangle - f(x,x^*+u^*) \leq \langle x^*,x\rangle - f(x,x^*).$$

Taking the supremum over u^* we see from (1) that $0 \leq \langle x^*,x\rangle - f(x,x^*)$. However putting $y^*=x^*$ in (1) we get $\langle x^*,x\rangle - f(x,x^*) \leq 0$ so $f(x,x^*) = \langle x^*,x\rangle$. ♦

2.6. Corollary Suppose $\sup\{\langle y^*,y\rangle - f(y,y^*) \mid y^*\in E^*\} = 0$ for all $y\in E$. Then $x\in \partial f_x(x^*)$ if and only if $x^*\in T_f x$.
Proof Combine Theorems 2.4 and 2.5. ♦

3. Convex functions from monotone operators. In this section we define and study a convex function on $E\times E^*$ using a monotone operator on E.

3.1. Definition Let T be a monotone operator on E. For $x\in E$ and $x^*\in E^*$ let

$$L_T(x,x^*) := \sup\{\langle x^*,y\rangle + \langle y^*,x - y\rangle \mid (y,y^*)\in G(T)\}.$$ ♦

The first result is immediate from the definition.

3.2. Proposition If $D(T)\neq\emptyset$ then the function L_T is lower semicontinuous and convex on $E\times E^*$. ♦

As a start to examining ∂L_T we have the following result.

3.3. Lemma If T is a monotone operator on E and $(y,y^*)\in G(T)$ and for some $x\in E$ and $x^*\in E^*$ we have

$$L_T(x,x^*) = \langle y^*,x - y\rangle + \langle x^*,y\rangle$$

then $(y,y^*) \in \partial L_T(x,x^*)$.
Proof For each $u\in E$ and $u^*\in E^*$ we have

$$\begin{aligned} &L_T(x+u,x^*+u^*) - L_T(x,x^*) \\ &\qquad= \sup\{\langle x^*+u^*,v\rangle + \langle v^*,x+u\rangle - \langle v^*,v\rangle \mid (v,v^*) \in G(T)\} - L_T(x,x^*) \\ &\qquad\geq \langle x^*+u^*,y\rangle + \langle y^*,x+u\rangle - \langle y^*,y\rangle - \langle x^*,y\rangle - \langle y^*,x-y\rangle \\ &\qquad= \langle y^*,u\rangle + \langle u^*,y\rangle \end{aligned}$$

so we have $(y,y^*) \in \partial L_T(x,x^*)$. ♦

3.4. Theorem If T is a monotone operator on E and $(x,x^*) \in G(T)$ then $L_T(x,x^*) = \langle x^*,x\rangle$ and $(x^*,x) \in \partial L_T(x,x^*)$.
Proof By monotonicity, for all $(y,y^*) \in G(T)$ we have $\langle x^*,x\rangle \geq \langle x^*,y\rangle + \langle y^*,x-y\rangle$ so that $L_T(x,x^*) \leq \langle x^*,x\rangle$. On the other hand

$$L_T(x,x^*) \geq \langle x^*,x\rangle + \langle x^*,x-x\rangle = \langle x^*,x\rangle$$

and now Lemma 2.4 shows that $(x^*,x) \in \partial L_T(x,x^*)$. ♦

3.5. Corollary For each monotone operator T on E we have $Tx \subseteq T_{L_T}x$ for all $x\in E$. If T is maximal monotone then $T = T_{L_T}$. ♦

We note that $T = T_{L_T}$ for some monotone operators T which are not maximal monotone. For example if T is the monotone operator whose graph is just $\{(0,0)\}$ then L_T is identically equal to 0 and $T = T_{L_T}$.

The situation for L_{T_f} is not so clear since one can add a constant to f without changing ∂f. To make progress we need to assume that $f(x,x^*) \geq \langle x^*,x\rangle$ for certain x and x^*.

3.6. Theorem Let f be a convex function on $E\times E^*$ and suppose $f(x,x^*) \geq \langle x^*,x\rangle$ for all x and x^* such that $(x^*,x)\in\partial f(x,x^*)$. Then $L_{T_f} \leq f$.

Proof Let $y\in E$ and $y^*\in E^*$. Then

$$\begin{aligned} L_{T_f}(y,y^*) - f(y,y^*) &= \sup\{\langle y^*,x\rangle + \langle x^*,y-x\rangle - f(y,y^*) \mid (x^*,x)\in\partial f(x,x^*)\} \\ &= \sup\{\langle y^*-x^*,x\rangle + \langle x^*,y-x\rangle + \langle x^*,x\rangle - f(y,y^*) \mid (x^*,x)\in\partial f(x,x^*)\} \\ &\leq \sup\{f(y,y^*) - f(x,x^*) + \langle x^*,x\rangle - f(y,y^*) \mid (x^*,x)\in\partial f(x,x^*)\} \\ &= \sup\{\langle x^*,x\rangle - f(x,x^*) \mid (x^*,x)\in\partial f(x,x^*)\} \\ &= 0 \end{aligned}$$

by our assumption on f. ◆

Next we show a minimality property of L_T.

3.7. Theorem Let T be a monotone operator on E. If f is a convex function on $E\times E^*$ with $f(x,x^*) \geq \langle x^*,x\rangle$ for all $x\in E$ and $x^*\in E^*$ and if $f(y,y^*) = \langle y^*,y\rangle$ for all $(y,y^*)\in G(T)$ then $L_T \leq f$.

Proof By Theorem 2.4, if $y^*\in Ty$ then $y^*\in T_f y$. Thus for all $x\in E$ and $x^*\in E^*$ we have

$$\begin{aligned} L_T(x,x^*) &= \sup\{\langle x^*,y\rangle + \langle y^*,x-y\rangle \mid (y,y^*)\in G(T)\} \\ &\leq \sup\{\langle x^*,y\rangle + \langle y^*,x-y\rangle \mid (y,y^*)\in G(T_f)\} \\ &= L_{T_f}(x,x^*) \;\leq\; f(x,x^*) \end{aligned}$$

by Theorem 3.6. ◆

However to get $L_T(x,x^*) \geq \langle x^*,x\rangle$ we need maximal monotonicity.

3.8. Theorem If T is a monotone operator on E then T is maximal monotone if and only if $L_T(x,x^*) > \langle x^*,x\rangle$ whenever $x\in E$ and $x^*\in E^*\setminus T(x)$.

Proof If $L_T(x,x^*) \leq \langle x^*,x\rangle$ then we have $\langle x^*,y\rangle + \langle y^*,x-y\rangle \leq \langle x^*,x\rangle$ for all $(y,y^*)\in G(T)$ so $\langle x^*-y^*,x-y\rangle \geq 0$. When T is maximal monotone that implies $x^*\in Tx$. Conversely if T is not maximal monotone then there are $x\in E$ and $x^*\in E^*\setminus T(x)$ such that $\langle x^*-y^*,x-y\rangle \geq 0$ for all $(y,y^*)\in G(T)$. It follows that $L_T(x,x^*) \leq \langle x^*,x\rangle$. ◆

3.9. Corollary Let T be a maximal monotone operator on E. Then $L_T(x,x^*) \geq \langle x^*,x\rangle$ for all $x\in E$ and $x^*\in E^*$, and $L_T(x,x^*) = \langle x^*,x\rangle$ if and only if $x^*\in Tx$.

Proof Use Theorems 3.4 and 3.8. ◆

Now for maximal monotone operators we have a nice characterization of L_T.

3.10. Theorem If T is a maximal monotone operator on E then L_T is the minimal convex function f on $E \times E^*$ such that $f(x,x^*) \geq \langle x^*,x \rangle$ for all $x \in E$ and $x^* \in E^*$ and $f(y,y^*) = \langle y^*,y \rangle$ for all $(y,y^*) \in G(T)$.
Proof We have $L_T \leq f$ for any such function f by Theorem 3.7. However L_T has the required properties by Corollary 3.9. ♦

Recall that a monotone operator T on E is *angle-bounded* provided there is $\alpha > 0$ such that $\langle x^*-y^*,y-z \rangle \leq \alpha \langle x^*-z^*,x-z \rangle$ whenever (x,x^*), (y,y^*) and (z,z^*) are in $G(T)$.

3.11. Theorem If T is an angle-bounded monotone operator on E and $x \in D(T)$ and $z^* \in R(T)$ then $L_T(x,z^*) < \infty$.
Proof Let (x,x^*) and (z,z^*) belong to $G(T)$. For all $(y,y^*) \in G(T)$ we have

$$\begin{aligned} \langle z^*,y \rangle + \langle y^*,x-y \rangle &= \langle z^*-y^*,y-x \rangle + \langle z^*,x \rangle \\ &\leq \alpha \langle z^*-x^*,z-x \rangle + \langle z^*,x \rangle \end{aligned}$$

so $L_T(x,z^*) \leq \alpha \langle z^*-x^*,z-x \rangle + \langle z^*,x \rangle < \infty$. ♦

3.12. Corollary If T is angle-bounded then L_T is finite on $\operatorname{conv} D(T) \times \operatorname{conv} R(T)$.
Proof Since L_T is convex and is finite on $D(T) \times R(T)$ we see that L_T is finite on $\operatorname{conv}(D(T) \times R(T)) = \operatorname{conv} D(T) \times \operatorname{conv} R(T)$. ♦

3.13. Corollary If g is a lower semicontinuous convex function on E then $L_{\partial g}$ is finite on $\operatorname{conv} D(\partial g) \times \operatorname{conv} R(\partial g)$.
Proof The monotone operator ∂g is angle-bounded with $\alpha = 1$. ♦

4. Duality results. For each monotone operator T on E let

$$h_T(x,x^*) := \begin{cases} \langle x^*,x \rangle & \text{if } (x,x^*) \in G(T) \\ \infty & \text{otherwise.} \end{cases}$$

4.1. Proposition If T is a monotone operator on E then $L_T(x,x^*) = h_T^*(x^*,x)$ and $L_T^*(x^*,x) = h_T^{**}(x,x^*)$ for all $x \in E$ and $x^* \in E^*$.
Proof These statements are immediate from the definitions. ♦

4.2. Proposition If T is a monotone operator on E then $L_T(x,x^*) \leq L_T^*(x^*,x) \leq h_T(x,x^*)$ for all $x \in E$ and $x^* \in E^*$. For all $(y,y^*) \in G(T)$ we have $L_T^*(y^*,y) = \langle y^*,y \rangle$.
Proof If $(y,y^*) \in G(T)$ then $L_T(y,y^*) = \langle y^*,y \rangle$ which shows $L_T \leq h_T$. Thus $L_T^* \geq h_T^*$, so $L_T^*(x^*,x) \geq L_T(x,x^*)$ for all $x \in E$ and $x^* \in E^*$. Since $h_T^{**}(x,x^*) \leq h_T(x,x^*)$ we

have $L_T^*(x^*,x) \le h_T(x,x^*)$. Now if $(y,y^*) \in G(T)$ then $\langle y^*,y\rangle = L_T(y,y^*) \le L_T^*(y^*,y) \le h_T(y,y^*) = \langle y^*,y\rangle$ and we see $L_T^*(y^*,y) = \langle y^*,y\rangle$. ♦

4.3. Theorem If T is a monotone operator on E then $L_T(x,x^*) \le L_T^*(x^*,x) < \infty$ for all $(x,x^*) \in \operatorname{conv} G(T)$ and $L_T^*(x^*,x) = \infty$ for all $(x,x^*) \notin \operatorname{cl\,conv} G(T)$.

Proof Since $L_T^*(x^*,x) = \langle x^*,x\rangle < \infty$ for $(x,x^*) \in G(T)$ and L_T^* is convex we have $L_T^*(x^*,x) < \infty$ for all $(x,x^*) \in \operatorname{conv} G(T)$. If (x,x^*) is not in $\operatorname{cl\,conv} G(T)$ then $h_T^{**}(x,x^*) = \infty$ (because h_T^{**} is the lower semicontinuous convex closure of h), that is, $L_T^*(x^*,x) = \infty$. ♦

4.4. Proposition Suppose T is a monotone operator on E and $(y,y^*) \in G(T)$. If $x \in E$ and $x^* \in E^*$ are such that $L_T(x,x^*) = \langle y^*,x - y\rangle + \langle x^*,y\rangle$ then (x,x^*) and (y,y^*) are in $\partial L_T^*(y^*,y)$.

Proof By Lemma 3.3 we have $(y^*,y) \in \partial L_T(x,x^*)$, so $(x,x^*) \in \partial L_T^*(y^*,y)$. Since $L_T(y,y^*) = \langle y^*,y\rangle$ we also get $(y^*,y) \in \partial L_T(y,y^*)$ and $(y,y^*) \in \partial L_T^*(y^*,y)$. ♦

Next we note the relationship of the saddle function K_T of Krauss [Kr] in duality with L_T.

4.5. Theorem If T is a monotone operator on E and $x \in E$ and $x^* \in E^*$ then

$$L_T(x,x^*) = \sup\{\langle x^*,y\rangle - K_T(x,y) \mid y \in E\}.$$

Proof For each $x \in E$ Krauss defines $K_T(x,\cdot)$ to be the closure of the convex function $H_T(x,\cdot)$ which is defined by

$$H_T(x,y) := \inf\left\{ \sum_{i=1}^{n} \lambda_i \langle y_i^*, y_i - x\rangle \;\middle|\; n \in \mathbb{N},\ \lambda_i \ge 0,\ \sum_{i=1}^{n} \lambda_i = 1,\ \sum_{i=1}^{n} \lambda_i y_i = y,\ y_i^* \in Ty_i \right\}.$$

Thus for $x \in E$ and $x^* \in E^*$ we have

$$\sup\{\langle x^*,y\rangle - K_T(x,y) \mid y \in E\} = \sup\{\langle x^*,y\rangle - H_T(x,y) \mid y \in E\}$$

$$= \sup\left\{ \langle x^*,y\rangle + \sum_{i=1}^{n} \lambda_i \langle y_i^*, x - y_i\rangle \;\middle|\; n \in \mathbb{N},\ \lambda_i \ge 0,\ \sum_{i=1}^{n} \lambda_i = 1,\ (y_i,y_i^*) \in G(T) \right\}$$

$$= \sup\{\langle x^*,y\rangle + \langle y^*,x - y\rangle \mid (y,y^*) \in G(T)\} = L_T(x,x^*).$$ ♦

Similarly one can express K_T in terms of L_T as follows.

4.6. Theorem If T is a monotone operator on E and $x \in E$ and $y \in E$ then

$$K_T(x,y) = \sup\{\langle x^*,y\rangle - L_T(x,x^*) \mid x^* \in E^*\}.$$ ♦

Thus one could translate the results of Krauss [Kr] into our framework. However that procedure seems to yield unwieldly statements and there may be more natural conditions for existence of solutions and for the maximality of sums of monotone operators, which can be expressed in terms of T_f and L_T.

5. Problems. Finally we list some open problems about T_f and L_T.

5.1. Problem For which convex functions f is $L_{T_f} = f$?

5.2. Problem For which monotone operators T is $T_{L_T} = T$?

5.3. Problem For which convex functions f is T_f maximal monotone?

5.4. Problem If S and T are monotone operators characterize L_{S+T}.

5.5. Problem The convex function f in Example 2.3 has $f(x,x^*) = f^*(x^*,x)$. Given a monotone operator T on E find a convex function f on $E \times E^*$ such that $T(x) \subseteq T_f(x)$ and $f(x,x^*) = f^*(x^*,x)$ for all $x \in E$ and $x^* \in E^*$. For which such f is T_f maximal monotone?

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References.

[F-C] Simon Fitzpatrick and Bruce Calvert, "In a nonreflexive space the subdifferential is not onto," *Math. Z.* **189** (1985), 555-560.

[Kr] Eckehard Krauss, "A representation of maximal monotone operators by saddle functions," *Rev. Roumaine Math. Pures Appl.* **30** (1985), 823-837.

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