CONTRACTIVE PROJECTIONS ON BANACH SPACES

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ABSTRACT. Increasing sequences of contractive projections on a reflexive L^p space share an unconditionality property similar to that exhibited by sequences of self-adjoint projections on a Hilbert space. A slight variation of this property is shown to be precisely the correct condition on a reflexive Banach space to ensure that every operator with a contractive AC-functional calculus is scalar-type spectral.

1. Introduction

Our aim in what follows is to discuss a few questions about the behaviour of the contractive projections on Banach spaces which arise from some problems in abstract spectral theory. Let us fix some notation. Throughout, $\mathcal{H}$ will denote a separable complex Hilbert space, X will denote a real or complex Banach space and $Proj_1(X)$ will denote the set of all contractive projections on X, i.e.

$$Proj_1(X) = \{P \in B(X) : P^2 = P \text{ and } \|P\| \leq 1\}.$$

We shall say that a sequence $\{P_j\}$ of contractive projections is *increasing* if, for all i, j,

$$P_i P_j = P_j P_i = P_{\min\{i,j\}}.$$

If the index set is $\mathbb{N}$, we shall employ the convention that $P_0 = 0$.

Our starting point is a simple property of the contractive projections on $\mathcal{H}$. These of course are just the usual orthogonal projections.

THEOREM 1. *If $0 = P_0, P_1, \ldots$ is an increasing sequence of contractive projections on $\mathcal{H}$ and $\{a_j\}_{j=1}^{\infty}$ is a sequence of scalars such that $|a_j| \leq 1$ for all j, then*

$$\left\| \sum_{j=1}^{\infty} a_j (P_j - P_{j-1}) \right\| \leq 1.$$

The series here converges in the strong operator topology.

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An analogue of this holds for the contractive projections on reflexive L^p spaces. This result is much more difficult, depending on a characterisation of the contractive projections on these spaces in terms of conditional expectation operators, and on the fact that martingale transforms are bounded on $L^p[0,1]$, $1 < p < \infty$. For $1 < p < \infty$, let $p^* = \max\{p, p/(p-1)\}$.

THEOREM 2 [DO,PR,D2]. *Suppose that* $1 < p < \infty$ *and that* $(\Omega, \mathcal{A}, \mu)$ *is an arbitrary measure space. Suppose also that* $0 = P_0, P_1, \ldots$ *is an increasing sequence of contractive projections on* $L^p(\Omega, \mathcal{A}, \mu; \mathbb{C})$ *and that* $\{a_j\}_{j=1}^{\infty}$ *is a sequence of scalars such that* $|a_j| \leq 1$ *for all* j. *Then for all* $n \geq 1$

$$\left\| \sum_{j=1}^{n} a_j (P_j - P_{j-1}) \right\| \leq 2(p^* - 1).$$

Theorem 2 was originally proved for real L^p spaces. Burkholder [B] has shown that the constant $p^* - 1$ will suffice in this case and is the smallest constant for which the theorem holds for all measure spaces. The problem of best constants will be discussed in more detail in section 3.

Well-bounded operators on Banach spaces are those which have a norm continuous functional calculus for $AC[a,b]$, the absolutely continuous functions on some compact interval of the real line. Since the polynomials are dense in this Banach algebra, an operator $T \in B(X)$ is well-bounded if there exists a compact interval $[a,b] \subset \mathbb{R}$ and a constant K such that for all polynomials g,

$$\|g(T)\| \leq K \, \|g\|_{AC} = K \left\{ |g(b)| + \int_a^b |g'(t)| dt \right\}.$$

There has been some interest in being able to decide when a well-bounded operator is scalar-type spectral. Scalar-type spectral operators have better behaved spectral expansions and larger functional calculi. Whereas a scalar-type spectral operator has, like a self-adjoint operator, a representation with respect to a countably additive spectral measure, the spectral theorem for well-bounded operators gives only a representation in terms on an increasing family of projections on X^*. On reflexive spaces the situation for well-bounded operators is rather more satisfactory in that a well-bounded operator T on such a space admits a Riemann-Stieltjes type integral representation against a uniformly bounded, strong-operator right-continuous, increasing family $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ of projections on X known as a *spectral family.* In this case we shall write $T = \int_{[a,b]}^{\oplus} \lambda dE(\lambda)$. Precise definitions and the statements of the spectral theorems for scalar-type spectral and well-bounded operators may be found in [**Dow**].

In [**FL**] Fong and Lam showed that a sufficient condition for a well-bounded operator on a Hilbert space to be self-adjoint is that its $AC[a,b]$-functional calculus be contractive.

THEOREM 3 [**FL**, Proposition 2.13,**D1**]. *Suppose that $T \in B(\mathcal{H})$ and that there exist $a \le c \le b \in \mathbb{R}$ such that for all polynomials g,*

$$\|g(T)\| \le \left\{ |g(c)| + \int_a^b |g'(t)|dt \right\}.$$

Then T is self-adjoint.

Fong and Lam used spectral carriers and convexity arguments to prove their result. In [**D2**], Theorem 3 was extended to cover the other reflexive L^p spaces by utilising Theorem 2.

THEOREM 4 [**D2**]. *Suppose that $1 < p < \infty$, that $T \in B(L^p)$ and that there exist $a \le c \le b \in \mathbb{R}$ such that for all polynomials g,*

$$\|g(T)\| \le \left\{ |g(c)| + \int_a^b |g'(t)|dt \right\}.$$

Then T is scalar-type spectral.

Theorem 3 and 4 should be compared to the classical results that:

i) $T \in B(H)$ is self-adjoint if and only if T has an isometric $\mathcal{C}(\sigma(T))$-functional calculus;

ii) $T \in B(X)$ is scalar-type spectral if and only if it has a weakly compact $\mathcal{C}(\sigma(T))$-functional calculus, i.e. for all $x \in X$, the map $f \mapsto f(T)x$ is a weakly compact operator from $\mathcal{C}(\sigma(T))$ into X.

My interest here is in discovering which are the spaces for which an analogue of Theorem 4 holds. It turns out that the condition exhibited by the reflexive L^p spaces in Theorem 2 is almost the "right" one.

2. Unconditionality properties for contractive projections

Definition. 1) A Banach space X is said to have the *uniform unconditionality property for contractive projections (uniform UPCP)*, if there exists $K > 0$ such that for all increasing sequences of contractive projections $\{P_j\}_{j=1}^{\infty}$, all sequences of scalars $\{a_j\}_{j=1}^{\infty}$ with $|a_j| \le 1$ and all $n \ge 1$,

$$\left\| \sum_{j=1}^{n} a_j(P_j - P_{j-1}) \right\| \le K.$$

We shall let $K(X)$ denote the smallest such K.

2) A Banach space X is said to have the *bilateral unconditionality property for contractive projections (bilateral UPCP)*, if given an increasing sequence of contractive projections $\{P_j\}_{j=-\infty}^{\infty}$, there exists $K > 0$ such that for all sequences of scalars $\{a_j\}_{j=-\infty}^{\infty}$ with $|a_j| \leq 1$ and all $n \geq m$,

$$\left\| \sum_{j=m}^{n} a_j(P_j - P_{j-1}) \right\| \leq K.$$

3) A Banach space X is said to have the *unconditionality property for contractive projections (UPCP)*, if given an increasing sequence of contractive projections $\{P_j\}_{j=1}^{\infty}$, there exists $K > 0$ such that for all sequences of scalars $\{a_j\}_{j=1}^{\infty}$ with $|a_j| \leq 1$ and all $n \geq 1$,

$$\left\| \sum_{j=1}^{n} a_j(P_j - P_{j-1}) \right\| \leq K.$$

The following theorem follows directly from the definitions. For those terms not defined above we suggest that the reader consult [**Si1**] and [**Si2**].

THEOREM 5. *Suppose that X is a Banach space. Then*

X has uniform UPCP

$\Downarrow$

X has bilateral UPCP

$\Downarrow$

X has UPCP

$\Downarrow$

All monotone Schauder decompositions of X are unconditional

$\Downarrow$

All monotone Schauder bases of X are unconditional.

We shall say that an operator $T \in B(X)$ has a *contractive AC-functional calculus* if there exist real numbers $a \leq c \leq b$ such that for all polynomials g, $\|g(T)\| \leq |g(c)| + \int_a^b |g'(t)|dt$. The following theorem relates the above unconditionality properties for contractive projection to the property that an analogue of Theorem 4 holds for operators on the space.

THEOREM 6 [**D1**, Theorem 6.3.5]. *Suppose that X is reflexive. Then X has bilateral UPCP if and only if every operator on X with a contractive AC-functional calculus is scalar-type spectral.*

The proof requires two lemmas.

LEMMA 7. *Suppose that X is reflexive and that $\{P_j\}_{j=-\infty}^{\infty}$ is a uniformly bounded increasing sequence of projections on X. Suppose also that $0 < \lambda_j < \lambda_{j+1} < 1$ for $j \in \mathbb{Z}$.*

Define $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ *by*

$$E(\lambda) = \begin{cases} 0, & \lambda < 0; \\ \text{SOT-}\lim_{k\to\infty} P_k, & \lambda = 0; \\ P_j & \lambda \in [\lambda_j, \lambda_{j+1}); \\ I & \lambda \geq 1. \end{cases}$$

Then $\{E(\lambda)\}$ *is a spectral family and*

$$\sup_{\lambda \in \mathbb{R}} \|E(\lambda)\| = \sup_{j \in \mathbb{Z}} \|P_j\| .$$

Proof. The only thing to check is that $\{E(\lambda)\}$ is well defined at $\lambda = 0$ and that it has a strong left-hand limit at $\lambda = 1$. This follows from a result of Lorch (see [**Dow**, Theorem 5.4]). ■

LEMMA 8. *Suppose that* $f : [0,1] \to \mathbb{C}$ *is not of bounded variation. Then there exists a sequence of real numbers* $\{\lambda_j\}_{j=-\infty}^{\infty}$ *such that*

i) $0 \leq \lambda_j \leq \lambda_{j+1} \leq 1$, *for all* $j \in \mathbb{Z}$;

ii) $\sum_{j=-\infty}^{\infty} |f(\lambda_j) - f(\lambda_{j-1})| = \infty$.

The proof of Lemma 8 is left as an exercise. Note however, that the proof requires a little more care than it might appear at first sight.

Proof of theorem. ($\Rightarrow$). Suppose that $T \in B(X)$ admits a contractive AC-functional calculus, i.e.

$$\|g(T)\| \leq \left\{ |g(c)| + \int_a^b |g'(t)| dt \right\}$$

for all polynomials g. As noted above, T is well-bounded and so admits a representation against a spectral family $\{E(\lambda)\}$, say. If T is not scalar-type spectral, then it follows from a result of Berkson and Dowson (see [**Dow**, Theorem 16.16]) that there exist $x \in X$ and $x^* \in X^*$ such that the map $\varphi : \lambda \mapsto < E(\lambda)x, x^* >$ is not of bounded variation on $[a,b]$. Clearly then, either $\varphi|_{[a,c]}$ or $\varphi|_{[c,b]}$ is not of bounded variation.

Suppose that $\varphi|_{[a,c]}$ is not of bounded variation. Then by Lemma 8, there exists an increasing sequence $\{\lambda_j\}_{j=-\infty}^{\infty} \in [a,c]$ such that

$$\sum_{j=-\infty}^{\infty} |\varphi(\lambda_j) - \varphi(\lambda_{j-1})| = \infty.$$

Suppose that $\lambda_j = c$ for some j. In this case let j_0 denote the smallest such j (so that $\lambda_j = c$ for all $j \geq j_0$). Otherwise let $j_0 = \infty$. If we set

$$\mu_j = \begin{cases} \lambda_j & \text{if } j < j_0; \\ \lambda_{j_0-1} & \text{if } j \geq j_0, \end{cases}$$

then it is clear that $\mu_j \in [a, c)$ for all j. A simple calculation shows that

$$\sum_{j=-\infty}^{\infty} |\varphi(\mu_j) - \varphi(\mu_{j-1})| = \infty.$$

For $j \in \mathbb{Z}$, let $P_j = E(\mu_j)$. Then, since $\{E(\lambda)\}$ is a spectral family, $\{P_j\}$ must be an increasing sequence of projections. By [**D2**, Lemma 2.5] we know that $\|E(\lambda)\| \leq 1$ for all $\lambda \in [a, c)$, so all the elements of $\{P_j\}$ must be contractive projections. Now for each $j \in \mathbb{Z}$,

$$\begin{aligned} |\varphi(\mu_j) - \varphi(\mu_{j-1})| &= |< P_j x, x^* > - < P_{j-1} x, x^* >| \\ &= < \alpha_j (P_j - P_{j-1}) x, x^* > \end{aligned}$$

for some unimodular scalar α_j. Thus

$$\left\| \sum_{j=m}^{n} \alpha_j (P_j - P_{j-1}) \right\| \to \infty$$

as $m \to -\infty$ and $n \to \infty$, and so X does not have bilateral UPCP.

Suppose now that $\varphi|_{[c,b]}$ is not of bounded variation. Again by Lemma 8 choose an increasing sequence $\{\lambda_j\}_{j=-\infty}^{\infty} \in [c, b]$ such that

$$\sum_{j=-\infty}^{\infty} |\varphi(\lambda_j) - \varphi(\lambda_{j-1})| = \infty.$$

Now for $j \in \mathbb{Z}$, define $P_j = I - E(\lambda_{-j})$. Again we have that $\{P_j\}$ is an increasing sequence of contractive projections. Also

$$\begin{aligned} |\varphi(\lambda_j) - \varphi(\lambda_{j-1})| &= | < (I - P_{-j})x, x^* > - < (I - P_{1-j})x, x^* > | \\ &= | < (P_{-j} - P_{1-j})x, x^* > | \\ &= < \alpha_k (P_k - P_{k-1})x, x^* > \end{aligned}$$

where $k = -j$ and α_k is some unimodular scalar. The proof is then completed as above.

($\Leftarrow$). Suppose that $\{P_j\}_{j=-\infty}^{\infty}$ is an increasing sequence of contractive projections on X. Construct a spectral family $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$, concentrated on $[0, 1]$ by Lemma 7 and let $T = \int_{[a,b]}^{\oplus} \lambda dE(\lambda)$ be the corresponding well-bounded operator. As is well-known,

$$\|g(T)\| \leq \left\{ |g(b)| + \int_a^b |g'(t)| dt \right\} \sup_{\lambda \in \mathbb{R}} \|E(\lambda)\|.$$

Thus T has a contractive AC-functional calculus and so by hypothesis is scalar-type spectral.

Suppose that $A = (\lambda, \mu] \subset \mathbb{R}$ and denote by χ_A the characteristic function of A. If we denote the resolution of the identity for T by $\mathcal{E}$, then the spectral theorems for well-bounded and scalar-type spectral operators allow us to calculate $\chi_A(T)$ as either $\int_{[0,1]}^{\oplus} \chi_A(\lambda)\,dE(\lambda)$ or $\int_{\sigma(T)} \chi_A(\omega)\mathcal{E}(d\omega)$ (and it is easy to show that these two definitions agree). This implies that

$$\chi_A(T) = \mathcal{E}((\lambda, \mu]) = E(\mu) - E(\lambda).$$

Suppose that $|\alpha_j| \le 1$ for $j = m, \ldots, n$. Then since scalar-type spectral operators possess a functional calculus for the bounded Borel measurable functions under the supremum norm (see [**Dow**, p. 120]),

$$\begin{aligned}\left\| \sum_{j=m}^{n} \alpha_j (P_j - P_{j-1}) \right\| &= \left\| \sum_{j=1}^{n} \alpha_j \mathcal{E}((\lambda_{j-1}, \lambda_j]) \right\| \\ &\le 4 \sup_A \|\mathcal{E}(A)\| \sup_{\omega \in [0,1]} \left| \sum_{j=m}^{n} \alpha_j \chi_{(\lambda_{j-1},\lambda_j]}(\omega) \right| \\ &\le K, \text{ say.}\end{aligned}$$

Note that K does depends only on the sequence of projections $\{P_j\}$ and not on $\{\alpha_j\}, m$ or n, so X must have bilateral UPCP. ■

3. Some questions

The most immediate question is:

Question 9. *Which spaces have (bilateral) UPCP?*

The only spaces we know to have UPCP are the reflexive L^p spaces and spaces of finite dimension. Note that these spaces all have uniform UPCP. It is not difficult to show that many classical non-reflexive spaces do not have UPCP. For example,

i) $L^1[0,1]$, $\mathcal{C}[0,1]$ and c_0 all have conditional monotone bases (see [**Si**, pp. 215, 396 and 634-635]);

ii) $\ell^1 = c_0^*$, $L^\infty[0,1] = L^1[0,1]^*$, etc.;

iii) The space of trace class operators, $\mathcal{C}_1$ has a conditional monotone Schauder decomposition (but no monotone Schauder basis) (see [**D1**,Theorem 6.2.4],[**AF**, §7]).

Note that the UPCP conditions are isometric rather than isomorphic properties. The following theorem allows one to construct examples of reflexive spaces without UPCP.

THEOREM 10 [**D1**, Theorem 6.4.1]. *Suppose that X contains an infinite dimensional complemented subspace with a basis. Then X can be equivalently renormed so that it does not have UPCP.*

Question 11. *If X has UPCP, must X be reflexive?*

Question 12. *Are the three UPCP conditions distinct?*

Question 13. *If X has UPCP, must X^* have UPCP?*

Several classes of spaces have been suggested as good candidates for having UPCP. These include

i) the Lebesgue-Bochner spaces, $L^p([0,1];\mathbb{R}^2)$, for $1 < p < \infty$;

ii) the von Neumann-Schatten p-classes, $\mathcal{C}_p$ for $1 < p < \infty$;

iii) spaces with few contractive projections.

Bosznay and Garay [**BG**] have shown that under many norms (in a sense which we shall not make precise), $\mathbb{R}^n$ (and $\mathbb{C}^n$) admit no contractive projections of rank greater than 1 other than the identity. It seems to be an open question as to whether infinite dimensional Banach spaces an be renormed in this way.

In a slightly different direction, one can try and calculate $K(X)$ for a particular Banach space with uniform UPCP. As was mentioned above, Burkholder has shown that, for $1 < p < \infty$, $K(L^p([0,1];\mathbb{R})) = p^* - 1$. This sharp constant was used in [**D2**] to show that $K(L^p([0,1];\mathbb{C})) \leq 2(p^* - 1)$, but it is not difficult (using the Riesz-Thorin interpolation theorem and the fact that $K(L^p([0,1];\mathbb{C})) = 1$) to show that this cannot be the sharp constant for the complex case. Calculating $K(L^p([0,1];\mathbb{C}))$ is equivalent to finding the basis constant for the Haar basis on $L^p([0,1];\mathbb{C})$. Pełczyński [**P**] has conjectured that this is also $p^* - 1$, but this question is still open.

Even for a two point measure space it seems hard to do very much. It is easy to see that

$$K(\ell^1(2;\mathbb{R})) = K(\ell^1(2;\mathbb{C})) = 3,$$
$$K(\ell^2(2;\mathbb{R})) = K(\ell^2(2;\mathbb{C})) = 1.$$

Using the characterisation of contractive projections on L^p spaces in terms of conditional expectation operators due to Ando [**A**], one can use interpolation to find bounds for $K(\ell^p(2;\mathbb{R}))$ and $K(\ell^p(2;\mathbb{C}))$. Unfortunately these interpolated bounds are also not sharp for $1 < p < 2$. One can show however that

$$K(\ell^p(2;\mathbb{R}))^p$$
$$= \sup_{\mu,\alpha,\beta,\varepsilon} \frac{|\alpha + (\varepsilon - 1)(\alpha\mu + \beta(1-\mu))|^p \mu + |\beta + (\varepsilon - 1)(\alpha\mu + 2(1-\mu))|^p (1-\mu)}{|\alpha|^p \mu + |\beta|^p (1-\mu)}$$

where the supremum is taken over $0 < \mu < 1$, $\alpha, \beta, \varepsilon \in \mathbb{R}$ and $|\varepsilon| \leq 1$. The evaluation of this supremum is left as an exercise for the reader (the author can't do it!). Even showing that the value of the supremum is not increased by allowing α, β and ε to be complex would be interesting.

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