1.8. Convergence of Semigroups

In the preceding sections we examined the existence and construction of various classes of semigroup and next we analyze their stability properties. First we consider convergence properties and use these to extend the foregoing results on semigroup construction.

Let $S^{(n)}$ be a sequence of C_0-semigroups on a Banach space $\mathcal{B}$ and assume that $S_t^{(n)}$ converges strongly to S_t, for each $t \geq 0$. Since the product of strongly convergent sequences is strongly convergent the S_t must satisfy the semigroup property $S_s S_t = S_{s+t}$ for all $s, t \geq 0$, and of course $S_0 = I$. Nevertheless $S = \{S_t\}_{t \geq 0}$ is not necessarily a C_0-semigroup because of a possible lack of continuity. The simplest example of this phenomenon is given by the numerical semigroups $S_t^{(n)} = e^{-nt}$ acting on $\mathbb{C}$. The limit S satisfies $S_0 = I$, and $S_t = 0$ if $t > 0$; it is clearly discontinuous. Thus it is of interest to establish conditions for stability of C_0-semigroups under strong convergence and to identify stability criteria in terms of the generators.

Although the strong limit S of the sequence $S^{(n)}$ often fails to be a C_0-semigroup on the whole Banach space $\mathcal{B}$ it is possible that its restriction to a Banach subspace $\mathcal{B}_0$ is a C_0-semigroup. For example if $\mathcal{B} = \mathcal{B}_0 \oplus \mathbb{C}$ and $S_t^{(n)} = T_t \oplus e^{-nt}$, where T is a fixed C_0-semigroup on $\mathcal{B}_0$, then the limit S is discontinuous for a rather trivial reason; on the subspace $\mathcal{B}_0$ one has continuity, because $S = T$, and the discontinuity only occurs in the extra dimension. Thus it is of some interest to broaden the discussion of stability of convergence by attempting to identify subspaces of continuity for the limit semigroup.

We begin the analysis by first establishing that semigroup convergence is equivalent to convergence of the resolvents of the generators. For simplicity we consider contraction semigroups.

PROPOSITION 1.8.1. *Let* $S_t^{(n)} = \exp\{-tH_n\}$ *be a sequence of* C_0*-semigroups of contractions on the Banach space* $\mathcal{B}$ *and let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup of contractions acting on a Banach subspace* $\mathcal{B}_0 \subseteq \mathcal{B}$.

The following four conditions are equivalent:

1(1')
$$\lim_{n\to\infty} \left\| \left(S_t^{(n)} - S_t\right) a \right\| = 0$$

for all $a \in \mathcal{B}_0$ *and all* $t \geq 0$ *(uniformly for* t *in finite intervals of* $[0, \infty\rangle$*)* ,

2(2')
$$\lim_{n\to\infty} \left\| \left\{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \right\} a \right\| = 0$$

for all $a \in \mathcal{B}_0$ *and for some* $\alpha > 0$ *(uniformly for* α *in finite intervals of* $[0, \infty\rangle$*)* .

Clearly 1' ⇒ 1 and 2' ⇒ 2. The proof that 2 ⇒ 2' involves two arguments. First one uses the Neumann series

$$\left(I+\alpha H_n\right)^{-1} = \left(\frac{\alpha_0}{\alpha}\right) \sum_{n=0}^{\infty} \left(\frac{\alpha-\alpha_0}{\alpha}\right)^n \left(I+\alpha_0 H_n\right)^{-n-1} ,$$

which is convergent for $\alpha > \alpha_0/2$, to prove that resolvent convergence for $\alpha = \alpha_0$ implies resolvent convergence for all $\alpha > \alpha_0/2$, and hence by iteration for all $\alpha > 0$. Second one extimates from the Laplace transform relation

$$(I+\alpha H_n)^{-1}a = \alpha^{-1} \int_0^\infty dt\ e^{-\alpha^{-1}t} S_t^{(n)} a$$

that

$$\left\| \left\{ (I+\alpha_1 H_n)^{-1} - (I+\alpha_2 H_n)^{-1} \right\} a \right\| \le 2(\alpha_1 - \alpha_2)\alpha_2^{-1} \|a\|$$

for $\alpha_1 > \alpha_2 > 0$. Hence the convergence is uniform for α in finite intervals by a standard equicontinuity argument.

Next we argue that $1 \Rightarrow 2 \Rightarrow 1'$.

$1 \Rightarrow 2.$ By Laplace transformation one has

$$\begin{aligned} \left\| \left\{ (I+\alpha H_n)^{-1} - (I+\alpha H)^{-1} \right\} a \right\| &= \left\| \int_0^\infty dt\ e^{-t} \left(S_{\alpha t}^{(n)} - S_{\alpha t} \right) a \right\| \\ &\le \int_0^\infty dt\ e^{-t} \left\| \left(S_{\alpha t}^{(n)} - S_{\alpha t} \right) a \right\| \\ &\xrightarrow[n=\infty]{} 0 \end{aligned}$$

where the last conclusion follows from the Lebesgue dominated convergence theorem.

$2 \Rightarrow 1'.$ Since the semigroups under discussion are all contractive on $\mathcal{B}_0$ it suffices to prove their convergence on a norm dense subspace of $\mathcal{B}_0$. We will repeat the tactic used in the construction of S in Theorem 1.3.1 and work on the norm dense subspace $D(H^2)$. Now $D(H^2) = R((I+\alpha H)^{-2})$ for $\alpha > 0$. Moreover if $a \in \mathcal{B}_0$ then

$$\left(S_t^{(n)} - S_t \right)(I+\alpha H)^{-2} a = A_t^{(n)} + B_t^{(n)} + C_t^{(n)}$$

where

$$A_t^{(n)} = S_t^{(n)}\Big\{(I+\alpha H)^{-1}-\big(I+\alpha H_n\big)^{-1}\Big\}(I+\alpha H)^{-1}a ,$$

$$B_t^{(n)} = \big(I+\alpha H_n\big)^{-1}\big(S^{(n)}-S_t\big)(I+\alpha H)^{-1}a ,$$

and

$$C_t^{(n)} = \Big\{\big(I+\alpha H_n\big)^{-1}-(I+\alpha H)^{-1}\Big\}(I+\alpha H)^{-1}S_t a .$$

Let us estimate each of these terms. First

$$\left\|A_t^{(n)}\right\| \le \left\|\Big\{\big(I+\alpha H_n\big)^{-1}-(I+\alpha H)^{-1}\Big\}(I+\alpha H)^{-1}a\right\|$$

$$\xrightarrow[n=\infty]{} 0 .$$

Second

$$\left\|C_t^{(n)}\right\| = \left\|\Big\{\big(I+\alpha H_n\big)^{-1}-(I+\alpha H)^{-1}\Big\}(I+\alpha H)^{-1}S_t a\right\|$$

$$\xrightarrow[n=\infty]{} 0 .$$

But using $\|\big(I+\alpha H_n\big)^{-1}\| \le 1$ and $\|(I+\alpha H)^{-1}\| \leqq 1$ one readily derives the equicontinuity relation

$$\left\|C_{t_1}^{(n)} - C_{t_2}^{(n)}\right\| \le 2\left\|\big(S_{t_1}-S_{t_2}\big)a\right\|$$

and hence $A^{(n)}$ and $C^{(n)}$ converge to zero uniformly for t in any finite interval of $[0, \infty\rangle$. It remains to examine $B^{(n)}$. For this we use the integral representation

$$B_t^{(n)} = \int_0^t ds\, \frac{d}{ds}\, S_s^{(n)} \left(I+\alpha H_n\right)^{-1} S_{t-s} (I+\alpha H)^{-1} a$$

$$= \alpha^{-1} \int_0^t ds\, S_s^{(n)} \{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \} S_{t-s} a \ .$$

Thus

$$\left\| B_t^{(n)} \right\| \le \alpha^{-1} \int_0^t ds \left\| \{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \} S_{t-s} a \right\|$$

$$\xrightarrow[n=\infty]{} 0$$

where the last conclusion follows from the Lebesgue dominated convergence theorem. But for $t_1 \ge t_2$ one has

$$B_{t_1}^{(n)} - B_{t_2}^{(n)} = \alpha^{-1} \int_{t_2}^{t_1} ds\, S_s^{(n)} \{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \} S_{t_1-s} a$$

$$+ \alpha^{-1} \int_0^{t_2} ds\, S_s^{(n)} \{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \} S_{t_2-s} \left(S_{t_1-t_2} - I\right) a \ .$$

Therefore

$$\left\| \left(B_{t_1}^{(n)} - B_{t_2}^{(n)} \right) \right\| \le 2\alpha^{-1} \left\{ \left(t_1 - t_2\right) \|a\| + \left\| \left(S_{t_1-t_2} - I\right) a \right\| \right\}$$

and the convergence is again uniform for t in any finite interval of $[0, \infty\rangle$.

Combining these conclusions we see that

$$\lim_{n\to\infty} \left\| \left(S_t^{(n)} - S_t\right) b \right\| = 0$$

for all $b \in D\left(H^2\right)$, and consequently for all $b \in \mathcal{B}_0$. Moreover the convergence is uniform for t in finite intervals of $[0, \infty\rangle$. □

Although Proposition 1.8.1 could be viewed as a criterion for strong convergence of semigroups it does have two distinct drawbacks. First it gives an indirect link between semigroup convergence and convergence of the generators, because it concerns convergence of their resolvents. Second it assumes that the limit of the resolvents is the resolvent of a generator of a C_0-semigroup, at least on a subspace. The next theorem avoids both these disadvantages and relates semigroup convergence directly to graph convergence of the generators. This latter notion is introduced as follows.

If H_n is a sequence of operators on the Banach space $\mathcal{B}$ then the *graphs* $G(H_n)$ of H_n are defined as subspaces of $\mathcal{B} \times \mathcal{B}$ by

$$G(H_n) = \left\{\{a, H_n a\} \; ; \; a \in D(H_n)\right\} .$$

Now consider all sequences $a_n \in D(H_n)$ such that

$$\lim_{n\to\infty} \|a_n - a\| = 0 \; , \quad \lim_{n\to\infty} \|H_n a_n - b\| = 0$$

for some pair $\{a, b\} \in \mathcal{B} \times \mathcal{B}$. The pairs $\{a, b\}$ obtained in this way form a subspace G of $\mathcal{B} \times \mathcal{B}$ and we introduce the notation $D(G)$ for the set of a such that $\{a, b\} \in G$ for some b . Similarly $R(G)$ is the set of b such that $\{a, b\} \in G$ for some a . Moreover we write

$$G = \lim_{n\to\infty} G(H_n) .$$

In general G is not the graph of an operator but if there exists an operator H on the Banach space $\mathcal{B}$, or on a Banach subspace $\mathcal{B}_0$, such that $G = G(H)$ then H is called the *graph limit* of the H_n. Clearly in this case $D(G) = D(H)$ and $R(G) = R(H)$.

The next result demonstrates that this kind of convergence is appropriate for the characterization of semigroup convergence.

THEOREM 1.8.2. *Let* $S_t^{(n)} = \exp\{-tH_n\}$ *be a sequence of* C_0-*semigroups of contractions on the Banach space* $\mathcal{B}$ *and define the subspaces* $G_\alpha \subseteq \mathcal{B} \times \mathcal{B}$

$$G_\alpha = \lim_{n\to\infty} G(I+\alpha H_n) .$$

The following conditions are equivalent:

1. *There exists a Banach subspace* $\mathcal{B}_0$ *of* $\mathcal{B}$ *and a* C_0-*semigroup* S *on* $\mathcal{B}_0$ *such that*

$$\lim_{n\to\infty} \left\| \left(S_t^{(n)} - S_t\right)a \right\| = 0$$

for all $a \in \mathcal{B}_0$ *and* $t > 0$, *uniformly for* t *in any finite interval of* $[0, \infty\rangle$,

2. *There exists a Banach subspace* $\mathcal{B}_0$ *of* $\mathcal{B}$ *such that*

$$\mathcal{B}_0 = \overline{\{a \; ; \; \{a, b\} \in G_\alpha \; \textit{for some} \; b \in \mathcal{B}_0\}}$$

$$\mathcal{B}_0 = \overline{\{b \; ; \; \{a, b\} \in G_\alpha \; \textit{for some} \; a \in \mathcal{B}_0\}}$$

for some $\alpha > 0$ *, where the bar denotes norm closure.*

If these conditions are satisfied then S *is a contraction semigroup on* $\mathcal{B}_0$ *,* G_α *is the graph of* $I + \alpha H$ *where* H *is the generator of* S *, and* $\mathcal{B}_0 = \overline{D(G_\alpha)} = R(G_\alpha)$.

Proof. 1 ⇒ 2. It follows from strong convergence that

$$\|S_t a\| = \lim_{n\to\infty} \|S_t^{(n)} a\| \leq \|a\|$$

for all $a \in \mathcal{B}_0$ and hence S is a contraction semigroup on $\mathcal{B}_0$. Let H denote the generator of S then

$$\lim_{n\to\infty} \left\|(I+\alpha H_n)^{-1}a - (I+\alpha H)^{-1}a\right\| = 0$$

for all $a \in \mathcal{B}_0$ by Proposition 1.8.1. Thus if $a_n = (I+\alpha H_n)^{-1}a$ one has

$$\lim_{n\to\infty} \left\|a_n - (I+\alpha H)^{-1}a\right\| = 0$$

and

$$(I+\alpha H_n)a_n = a .$$

Consequently $\{(I+\alpha H)^{-1}a, a\} \in G$. This demonstrates Condition 2 and gives the identification of G_α , $D(G_\alpha)$, and $R(G_\alpha)$.

$2 \Rightarrow 1$. Define G to be the set of pairs $\{a, b\} \in \mathcal{B}_0 \times \mathcal{B}_0$ such that there exists a sequence $a_n \in D(H_n)$ with the property that $a_n \to a$, and $H_n a_n \to b$, as $n \to \infty$. To prove that G is the graph of an operator on $\mathcal{B}_0$ we must demonstrate that $a = 0$ implies $b = 0$. But suppose $a = 0$ and for an arbitrary pair $\{a', b'\} \in G$ choose $a'_n \in D(H_n)$ such that $a'_n \to a'$, and $H_n a'_n \to b'$, as $n \to \infty$. Then

$$\|\alpha(a'+b) + \alpha^2 b'\| = \lim_{n\to\infty} \|(I+\alpha H_n)(a_n+\alpha a'_n)\|$$

$$\geq \lim_{n\to\infty} \|a_n + \alpha a'_n\| = \alpha\|a'\| .$$

Dividing by α and taking the limit $\alpha \to 0$ one obtains

$$\|b + a'\| \geq \|a'\| .$$

But this inequality is true for all $a' \in D(G)$. Moreover $D(G) = D(G_\alpha)$ and hence $D(G)$ is norm dense in $\mathcal{B}_0$, by assumption. Therefore one must have $b = 0$ and consequently G is the graph of a norm densely defined operator H on $\mathcal{B}_0$.

Now $G_\alpha = G(I+\alpha H)$ and it follows by limiting that $\|(I+\alpha H)a\| \geq \|a\|$ for all $a \in D(H) = D(G_\alpha)$. The same inequality then extends to the closure $\overline{H}$ of H and it readily follows that $R(I+\alpha\overline{H})$ is norm closed. But $R(I+\alpha\overline{H}) = \overline{R(G_\alpha)} = \mathcal{B}_0$ and hence $\overline{H}$ is the generator of a C_0-semigroup of contractions S by the Hille-Yosida theorem. Now if $a_n \to a$ and $b_n = (I+\alpha H_n)a_n \to (I+\alpha H)a = b$ then

$$\left\|\left\{(I+\alpha H_n)^{-1} - (I+\alpha\overline{H})^{-1}\right\}b\right\| = \left\|(I+\alpha H_n)^{-1}(b-b_n) + (a_n-a)\right\|$$

$$\leq \|b - b_n\| + \|a_n - a\| .$$

Since $R(I+\alpha\overline{H}) = \mathcal{B}_0$ it follows that $(I+\alpha H_n)^{-1}a \to (I+\alpha\overline{H})^{-1}a$ for all $a \in \mathcal{B}$ and $S^{(n)}$ converges to S by Proposition 1.8.1. But the resolvent convergence also implies that G_α is closed and hence $H = \overline{H}$. □

In Theorem 1.8.2 there is not necessarily any unique or natural subspace $\mathcal{B}_0$ of convergence, e.g., if $S^{(n)}$ converges to S on $\mathcal{B}_0$ and $\mathcal{B}_1 \subset \mathcal{B}_0$ is an S-invariant subspace of $\mathcal{B}_0$ then $S^{(n)}$ converges to $S|_{\mathcal{B}_1}$ on $\mathcal{B}_1$. Of course the largest possible subspace of convergence is determined by the closure of $D(G_\alpha)$. If $S^{(n)}$ converges strongly to S on $\mathcal{B}_0$ it follows from the argument used to prove $1 \Rightarrow 2$ in Theorem 1.8.2 that $G_\alpha = G(I+\alpha H)$ where H is the generator of S. Thus $D(G_\alpha) = D(H)$ and in particular one has the following.

COROLLARY 1.8.3. *Adopt the assumption of Theorem 1.8.2. The following conditions are equivalent*

1. *There exists a* C_0*-semigroup* S *on* $\mathcal{B}$ *such that* $\left\|\left(S_t^{(n)}-S_t\right)a\right\| \to 0$ *for all* $a \in \mathcal{B}$, *uniformly for* t *in any finite interval of* $[0, \infty)$.

2. $D(G_\alpha)$ *and* $R(G_\alpha)$ *are norm dense in* $\mathcal{B}$ *for some* $\alpha > 0$.

Proposition 1.8.1, Theorem 1.8.2, and Corollary 1.8.3, have a variety of uses. The latter results give a clear delineation of the infinitesimal properties which characterize semigroup convergence. But unfortunately these properties are often difficult to verify in particular examples. There is, however, one situation in which the first result is easily applicable.

PROPOSITION 1.8.4. *Let* $S_t^{(n)} = \exp\{-tH_n\}$ *be* C_0*-semigroups of contractions on the Banach space* $\mathcal{B}$ *and* $S_t = \exp\{-tH\}$ *a similar semigroup on the Banach subspace* $\mathcal{B}_0$.

If there exists a core D *of* H *such that* $D \subseteq D(H_n)$ *for all* n *or, more generally,*

$$D \subseteq \bigcup_m \left\{ \bigcap_{n \geq m} D(H_n) \right\}$$

and if

$$\lim_{n\to\infty} \|(H_n - H)a\| = 0$$

for all $a \in D$ *, then*

$$\lim_{n\to\infty} \left\| \left(S_t^{(n)} - S_t\right) a \right\| = 0$$

for all $a \in \mathcal{B}_0$ *, uniformly for* t *in finite intervals of* $[0, \infty\rangle$. *In particular* H *is the graph limit of* H_n .

Proof. If $\alpha > 0$ the set $R_\alpha = \{(I+\alpha H)a \; ; \; a \in D\}$ is norm dense in $\mathcal{B}_0$ because D is a core of H and $R(I+\alpha H) = \mathcal{B}_0$. But for $b = (I+\alpha H)a$ with $a \in D$ one has

$$\left\| \left\{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \right\} b \right\| = \left\| \left(I+\alpha H_n\right)^{-1} \left(H - H_n\right) a \right\|$$

$$\leq \|\left(H - H_n\right) a\|$$

and hence $(I+\alpha H_n)^{-1}b$ converges strongly to $(I+\alpha H)^{-1}b$ for all $b \in \mathcal{B}_0$. The convergence of $S^{(n)}b$ to Sb now follows from Proposition 1.8.1. The identification of H as the graph limit of the H_n is a consequence of Corollary 1.8.3. □

There are two general corollaries of this last result which are useful throughout semigroup theory and which we have already partly exploited in Example 1.3.6. These corollaries concern the approximation of a given semigroup by a family of semigroups with bounded generators.

First let

$$H_s = \left(1-e^{-sH}\right)/s$$

where $S_s = \exp\{-sH\}$ is assumed to be a contraction semigroup. It is evident that the H_s are bounded but they also generate contraction semigroups because

$$\left\|e^{-tH_s}\right\| = e^{-t/s}\left\|e^{tS_s/s}\right\| \leq 1 \; .$$

Moreover

$$\lim_{s\to 0}\left\|\left(H_s-H\right)a\right\| = 0$$

for all $a \in D(H)$ by definition. Hence Proposition 1.8.4 implies that

$$(*) \qquad \lim_{s\to 0}\left\|\left(e^{-tH}-e^{-t\left(1-e^{-sH}\right)/s}\right)a\right\| = 0$$

for all $a \in \mathcal{B}$ uniformly for t in compact intervals of $[0, \infty\rangle$.

Second let

$$H_s = H(I+sH)^{-1}$$

$$= \{I - (I+sH)^{-1}\}/s \; .$$

Again H_s is bounded and $\exp\{-tH_s\}$ is a contraction semigroup because

$$\|e^{-tH_s}\| = e^{-t/s}\|e^{t(I+sH)^{-1}/s}\| \leq 1$$

where the last estimate uses $\|(I+sH)^{-1}\| \leq 1$. But

$$\lim_{s\to 0} \|(H_s-H)a\| = 0$$

for all $a \in D(H)$ because $(I+sH)^{-1}$ converges strongly to the identity as s tends to zero. This was established in the proof of Theorem 1.3.1. Thus the assumptions of Proposition 1.8.4 are satisfied by $\exp\{-tH\}$ and $\exp\{-tH_s\}$ and hence

$$(**) \qquad \lim_{s\to 0} \left\|\left(e^{-tH}-e^{-tH(1+sH)^{-1}}\right)a\right\| = 0$$

for all $a \in \mathcal{B}$, uniformly for t in finite intervals of $[0, \infty\rangle$.

The algorithms (*) and (**) give two methods of approximating a given C_0-semigroup of contractions. The first of these was proposed by Hille and the second by Yosida. Consequently we refer to the semigroups $\exp\{-t(I-S_s)/s\}$ as the *Hille approximants* and $\exp\{-tH(I+sH)^{-1}/s\}$ as the *Yosida approximants* of $S_t = \exp\{-tH\}$.

The Hille and Yosida approximants have many applications. The following example describes the connection between Taylor's series expansion, the Stone-Weierstrass theorem, and the Hille approximants of the semigroup of right translations.

Example 1.8.5. Let $\mathcal{B} = C_0(0, \infty)$, the continuous functions on $[0, \infty\rangle$ which vanish at infinity, equipped with the supremum norm and let S denote the C_0-semigroup of right translations

$$(S_t f)(x) = f(x+t) .$$

If $S_t^{(h)} = \exp\{-t(I-S_h)/h\}$ denotes the Hille approximants then

$$f(x+t) = \lim_{h\to 0} \left(S_t^{(h)} f\right)(x+t)$$

$$= \lim_{h\to 0} \sum_{n\geq 0} \frac{1}{n!} (t/h)^n \left(\Delta_h^{(n)} f\right)(x)$$

where

$$\left\{\Delta_h^{(n)} f\right\}(x) = \left\{\left(S_h - I\right)^n f\right\}(x)$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \, {}^nC_k f(x+kh)$$

and the limit is uniform for $x \in [0, \infty\rangle$ and t in any finite interval of $[0, \infty\rangle$. This is a generalization of Taylor's theorem. But setting $x = 0$ one also deduces that for each $\varepsilon > 0$ one can choose N such that

$$\left| f(t) - \sum_{n=0}^{N} \frac{1}{n!} (t/h)^n \left(\Delta_h^{(n)} f\right)(0) \right| < \varepsilon$$

uniformly for t in any finite interval $[0, t_0]$. This is an explicit version of the Stone-Weierstrass theorem. □

Alternatively these approximation techniques can be applied in a variety of ways to differential operators. The following example illustrates a typical problem of statistical mechanics, the independence of the thermodynamic limit from the choice of boundary conditions. In statistical mechanics one describes systems confined to a finite region of the appropriate phase space, e.g., a bounded subset $\Lambda \subset \mathbb{R}^3$, and then attempts to calculate bulk properties, e.g., properties such as the specific heat per unit volume. For sufficiently large systems these properties should be insensitive to the size and any boundary effects.

Example 1.8.6. Let $\mathcal{B} = L^2(\mathbb{R}^\nu)$ and let H denote the usual self-adjoint Laplacian. Thus

$$D(H) = \left\{f \; ; \quad f \in L^2(\mathbb{R}^\nu) \; , \; \int d^\nu p \; p^4 |\tilde{f}(p)|^2 < +\infty\right\}$$

and

$$(Hf)(x) = -\nabla_x^2 f(x) = (2\pi)^{-\nu/2} \int d^\nu p \; p^2 \tilde{f}(p) e^{ipx}$$

where $\tilde{f}$ denotes the Fourier transform. The operator H generates the semigroup of contractions which solves the heat equation

$$\frac{\partial f(x,t)}{\partial t} = -\nabla_x^2 f(x, t)$$

on $L^2(\mathbb{R}^\nu)$. The action of this semigroup is given by

$$(S_t f)(x) = (4\pi t)^{-\nu/2} \int d^\nu y\, e^{-(x-y)^2/4t} f(y) .$$

It is well known, and easily verified, that the space of infinitely often differentiable functions with compact support forms a core D of H .

Next for each bounded open set $\Lambda \subset \mathbb{R}^\nu$ let H_Λ denote any positive self-adjoint extension of H restricted to the infinitely often differentiable functions with support in Λ . There are many such extensions corresponding to different choices of boundary conditions for the Laplace operator on the boundary $\partial\Lambda$ of Λ . Some of these will be discussed explicitly in Section 1.11. But if Λ_n is any increasing sequence such that any open bounded set Λ is contained in Λ_n for n sufficiently large then

$$D \subset \bigcup_m \left(\bigcap_{n\geq m} D\left(H_{\Lambda_m}\right)\right)$$

by definition. Hence

$$\lim_{n\to\infty} \left\| \left(e^{-tH_{\Lambda_n}} - e^{-tH} \right) f \right\| = 0$$

for all $f \in \mathcal{B}$, uniformly for t in finite intervals of $[0, \infty\rangle$, by a direct application of Proposition 1.8.4. Consequently the net of contraction semigroups $\Lambda \mapsto \exp\{-tH_\Lambda\}$ converges strongly to $S_t = \exp\{-tH\}$. □

The Hille and Yosida approximants are just particular examples of a much broader class. If $t \in \mathbb{R}_+ \mapsto F(t) \in \mathcal{L}(\mathcal{B})$ is a family of contraction operators satisfying

$$\lim_{t\to 0} \|\{(I-F(t))/t - H\}a\| = 0$$

for all a in a core D of H then $t \to \exp\{-t(I-F(s))/s\}$ is a family of contraction semigroups and

$$\lim_{s\to 0} \|(S_t-\exp\{-t(I-F(s))/s\})a\| = 0$$

for all $a \in \mathcal{B}$, uniformly for t in finite intervals of $[0, \infty\rangle$. This is again a direct corollary of Proposition 1.8.4. Next we examine an alternative set of approximations of S by powers $F(t/n)^n$. The first basic estimate which relates the power approximations to the foregoing exponential approximations is provided by the following lemma.

LEMMA 1.8.7. *Let* A *be a bounded operator on the Banach space* $\mathcal{B}$ *with* $\|A\| \leq 1$.

It follows that

$$\|(e^{-n(I-A)}-A^n)a\| \leq \sqrt{n}\|(I-A)a\|$$

for all $n = 1, 2, 3, \ldots$.

Proof. One estimates

$$\|e^{-n(I-A)}-A^n a\| \leq e^{-n} \sum_{m\geq 0} \frac{n^m}{m!} \|(A^m-A^n)a\|$$

$$\leq e^{-n} \sum_{m\geq 0} \frac{n^m}{m!} \|(A^{|n-m|}-I)a\|$$

$$\leq \|(I-A)a\| e^{-n} \sum_{m\geq 0} \frac{n^m}{m!} |n-m|$$

$$\leq \sqrt{n}\|(I-A)a\|$$

where the last estimate follows from a straightforward application of the Cauchy Schwarz inequality. □

Combination of this estimate with the previous convergence theorems then leads to the following product formula, which generalizes the construction of the Hille-Yosida theorem.

THEOREM 1.8.8. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup of contractions and* $t \in \mathbb{R}_+ \mapsto F(t) \in \mathcal{L}(\mathcal{B})$ *a family of contractions operators on the Banach space* $\mathcal{B}$ *. Further assume that*

$$\lim_{t\to 0+} \|\{(I-F(t))/t - H\}a\| = 0$$

for all a *in a core* D *of* H *.*

It follows that

$$\lim_{n\to\infty} \|\{e^{-tH} - F(t/n)^n\}a\| = 0$$

for all $a \in \mathcal{B}$ *uniformly for* t *in finite intervals of* $[0, \infty)$ *.*

Proof. First it follows from Proposition 1.8.4 that

$$\lim_{s\to 0+} \|(e^{-tH} - e^{-t(I-F(s))/s})a\| = 0$$

for all $a \in \mathcal{B}$, uniformly for t in finite intervals. Therefore

$$\lim_{n\to\infty} \|(e^{-tH} - e^{-n(I-F(t/n))})a\| = 0$$

uniformly for t in finite intervals. But Lemma 1.8.7 gives the estimate

$$\|(e^{-n(I-F(t/n))} - F(t/n)^n)a\| \leq (t/\sqrt{n})\|(I-F(t/n))a\|/(t/n)$$

and for $a \in D$, the core of H, the right hand side converges to zero uniformly for t in finite intervals. Since $D(H)$ is norm dense the desired result follows from combination of these two estimates. $\square$

Product formulae of the type described by the theorem have a wide variety of applications. As a first illustration we again consider the semigroup of right translations and the Stone-Weierstrass theorem.

Example 1.8.9. Adopt the notation and assumptions of Example 1.8.5. Next for $0 < \lambda < 1$ set

$$F(t) = (1-\lambda)I + \lambda S_{t/\lambda}$$

in Theorem 1.8.8. Clearly the hypotheses of the theorem are valid and one has

$$\lim_{n\to\infty} \|(S_t - ((1-\lambda)I + \lambda S_{t/n\lambda})^n)a\| = 0 .$$

Therefore if $f \in C_0(0, \infty)$

$$f(x+t) = \lim_{n\to\infty} \sum_{m=0}^{n} {}^mC_n (1-\lambda)^{n-m} \lambda^m f\left(x + \frac{mt}{n\lambda}\right)$$

and the limit is uniform for $x \in [0, \infty\rangle$ and t in any finite interval of $[0, \infty\rangle$. Thus for $f \in C_b(0, 1)$ one deduces that

$$f(t) = \lim_{n\to\infty} \sum_{m=0}^{n} {}^nC_m (1-t)^m t^{n-m} f(m/n)$$

uniformly for $t \in [0, 1]$. This is Bernstein's version of the Stone-Weierstrass theorem. □

As a second, completely different, application of the product formula we derive an approximation procedure for the semigroup generated by the Dirichlet Laplacian. This example is of some importance because it provides the operator theoretic structure behind the Wiener integral, i.e., the functional integration description of the heat equation.

Example 1.8.10. Let S denote the C_0-semigroup generated by the Laplacian $H = -\nabla^2$ on $L^2(\mathbb{R}^\nu)$. Furthermore identify $L^2(\Lambda)$ as the subspace of $L^2(\mathbb{R}^\nu)$ formed by the functions with support in the bounded open set $\Lambda \subset \mathbb{R}^\nu$, Now define H_Λ as the restriction of H to the twice continuously differentiable with support in the interior of Λ. Since H is norm closed in $L^2(\mathbb{R}^\nu)$ its restriction H_Λ is norm closable in $L^2(\mathbb{R}^\nu)$ and we also use H_Λ to denote the closure. One can establish that H_Λ is a positive self-adjoint operator on $L^2(\Lambda)$ and it corresponds to the Laplacian with Dirichlet boundary conditions, i.e., $f \in D(H)$ implies that $f = 0$ on the boundary of Λ at least in some distributional sense. Further details about this Laplacian and others corresponding to different boundary conditions will be given in Section 1.11.

Now consider the family of operators on $L^2(\Lambda)$ defined by

$$F(t) = \chi_\Lambda S_t$$

where χ_Λ denotes multiplication by the characteristic function of the set Λ. If $f \in L^2(\Lambda)$ is twice continuously differentiable with support in Λ one has

$$\lim_{t\to 0+} \left\| \{(I-F(t))/t - H_\Lambda\} f \right\| = 0 \ .$$

But these f form a core for H_Λ and hence Theorem 1.8.8 is applicable. Thus

$$\lim_{n\to\infty} \left\| \{e^{-tH_\Lambda} - (\chi_\Lambda e^{-tH/n})^n\} f \right\| = 0$$

for all $f \in L^2(\Lambda)$ uniformly for t in finite intervals of $[0, \infty\rangle$. Note that from the explicit form of S one has

$$\left((\chi_\Lambda e^{-tH/n})^n f\right)(x) = \left(\frac{4\pi t}{n}\right)^{-n\nu/2}$$

$$\int_\Lambda dy_1 \ \ldots \int_\Lambda dy_n \ e^{-n\{(x-y_1)^2+(y_1-y_2)^2+\ldots+(y_{n-1}-y_n)^2\}/4t} f(y_n) .$$

These results extend directly to the corresponding semigroups on on $L^p(\Lambda)$, and $L^p(\mathbb{R}^\nu)$, and they provide a proof that the Dirichlet semigroup is positive, i.e., it maps positive functions into positive functions. □

Theorem 1.8.8 can also be applied to semigroups whose generators are sums of generators.

Let $S_t = \exp\{-tH\}$ and $T_t = \exp\{-tK\}$ be two C_0-semigroups of contractions and assume that $H + K$ is a norm closable operator whose closure $\overline{H + K}$ generates a C_0-semigroup U . A slight extension of the argument preceding Proposition 1.3.4 demonstrates that $H + K$ is dissipative and then the closure $\overline{H + K}$ is also dissipative. Thus U is contractive.

Now we choose $F_t = S_tT_t$ and $D = D(H+K)$. One readily checks that the assumptions of Theorem 1.8.8 are satisfied for F and U . Consequently

$$\lim_{n\to\infty} \left(U_t-\left(S_{t/n}T_{t/n}\right)^n\right)a = 0$$

for all $a \in \mathcal{B}$. This relation is called the *Trotter product formula*. A second possible choice of F is

$$F(t) = (I+tH)^{-1}(I+tK)^{-1}$$

and this leads to the product formula

$$\lim_{n\to\infty} \left\|\left(U_t-(I+tH/n)^{-1}(I+tK/n)^{-1}\right)a\right\| = 0 .$$

Exercises.

1.8.1. Let H_n be a uniformly bounded sequence of bounded operators. Prove that the graph limit of H_n exists if, and only if, H_n converges strongly.

1.8.2. Let H_n be a sequence of operators for which the graph limit H exists and P_n a sequence of bounded operators which converges strongly to P . Prove that $H + P$ is the graph limit of $H_n + P_n$.