1.6. Analytic Vectors.

In the previous sections we examined various methods of constructing a contraction semigroup from the resolvent of its generator. Next we analyze the possibility of a direct construction based on an operator extension of the numerical algorithms

$$\exp\{-tx\} = \sum_{n\geq 0} \frac{(-t)^n}{n!} x^n$$

$$= \lim_{n\to\infty} \left(1-\frac{t}{n}x\right)^n .$$

The problem with this new construction is that it is not applicable to all C_0-semigroups, or contraction semigroups, although it is applicable to all C_0-groups. The basic new concept is that of an analytic element.

If H is an operator on a Banach space $\mathcal{B}$ an element $a \in \mathcal{B}$ is defined to be an *(entire) analytic element for* H if

$$a \in \bigcap_{n\geq 1} D(H^n)$$

and the function

$$t \geq 0 \longmapsto \sum_{n\geq 0} \frac{t^n}{n!} \|H^n a\|$$

has a non-zero (infinite) radius of convergence. It is not at all evident that an operator possesses analytic elements but this is indeed the case if H is the generator of a strongly continuous group (a C_0-group). In fact one can explicitly construct a norm dense set of entire analytic elements by the following regularization procedure.

Let $S = \{S_t\}_{t\in\mathbb{R}}$ be a C_0-group with generator H and to each $a \in \mathcal{B}$ associate the sequence a_n defined by

$$a_n = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt\; e^{-t^2} S_{t/n}a\ .$$

Since $\|S_t\| \leq M \exp\{\omega|t|\}$ for some $M \geq 1$ and $\omega \geq 0$ the integral is well defined. Moreover

$$a_n - a = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt\; e^{-t^2} \left(S_{t/n}a-a\right)$$

and it follows from strong continuity and the Lebesgue dominated convergence theorem that a_n converges uniformly to a. But since H is norm closed one may argue recursively that $a_n \in D\left(H^m\right)$ for all $m = 1, 2, \ldots$ and

$$H^m a_n = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt\left\{(-n)^m \frac{d^m}{dt^m}\, e^{-t^2}\right\} S_{t/n}a$$

$$= (-n)^m \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt\; H_m(t) e^{-t^2} S_{t/n}a$$

where H_m is the usual Hermite function. Thus

$$\|H^m a_n\|^2 \leq n^{2m} \pi^{-1} M^2 \left(\int_{-\infty}^{\infty} dt\; H_m(t) e^{-t^2} e^{\omega|t|}\right)^2 \|a\|^2$$

$$\leq n^{2m} \pi^{-1} M^2 \int_{-\infty}^{\infty} dt\; e^{2\omega|t|} e^{-t^2} \int_{-\infty}^{\infty} dt\, |H_m(t)|^2 e^{-t^2} \|a\|^2$$

where we have used the Cauchy-Schwarz inequality. Using the normalization properties of the Hermite functions,

$$\int_{-\infty}^{\infty} dt \, |H_m(t)|^2 e^{-t^2} = 2^m m! \pi^{\frac{1}{2}} \, ,$$

one finally deduces that

$$\|H^m a_n\|^2 \leq M^2 e^{\omega^2} 2^{m+1} n^{2m} m! \, .$$

Hence a_n is an entire analytic element for H and the set of such elements is norm dense.

Despite this positive result the generator of the semigroup of left translations on $C_0[0, \infty\rangle)$ has no non-zero analytic elements. The action of this semigroup is given by $(S_t f)(x) = f(x-t)$ if $x \geq t$, and 0 if $x < t$. It follows that for f to be an analytic element it must vanish with all its (right) derivatives at the origin but it must also be analytic in a strip about the right half axis. Thus $f = 0$. Nevertheless the translation group acting on $C_0(\mathbb{R})$ does have dense sets of analytic elements and a function is analytic for this group if, and only if, it is an analytic function in the usual sense.

Now we consider the construction of a semigroup through analytic elements and for simplicity we again restrict the discussion to contraction semigroups.

PROPOSITION 1.6.1. *Let* H *be a norm closed operator on a Banach space* $\mathcal{B}$. *Suppose that*

1. H *possesses a norm dense set of analytic elements,*

2. H *is norm-dissipative.*

It follows that H *is the generator of a* C_0*-semigroup of contractions.*

Proof. Let a be an analytic element for H . Thus there is a $t_a > 0$ such that

$$S_t a = \sum_{n\geq 0} \frac{(-t)^n}{n!} H^n a$$

converges uniformly for $|t| < t_a$. Moreover for t fixed in this range $S_t a$ is again an analytic element for H and one can define $S_s(S_t a)$ for suitably small s . Calculation with norm convergent power series then establishes that

$$S_t(S_t a) = S_{s+t} a$$

for all s, t satisfying $|s| + |t| < t_a$. Next we examine properties of the function $t \in \langle -t_a, t_a\rangle \mapsto \|S_t a\|$.

First one has

$$\big|\|S_t a\| - \|S_s a\|\big| \leq \|S_t a - S_s a\|$$

by the triangle inequality. But another power series estimation of the right hand side then establishes that $t \mapsto \|S_t a\|$ is continuous. Second for $0 < h < t < t_a$ one has

$$\|S_{t-h}a\| = \left\|S_{-h}\left(S_t a\right)\right\|$$

$$= \lim_{n\to\infty} \left\|\left(I+\frac{h}{n}H\right)^n \left(S_t a\right)\right\|$$

$$\geq \|S_t a\|$$

where we have used the assumed norm-dissipativity of H. But this estimate implies that $t \in \langle 0, t_a\rangle \mapsto \|S_t a\|$ is decreasing and hence

$$\|S_t a\| \leq \|a\|$$

for $0 \leq t < t_a$. This contractive estimate now allows one to extend the definition of $S_t a$ to all $t \geq 0$.

Since H is closed $S_t a \in D(H)$ and

$$HS_t a = S_t(Ha) .$$

Therefore

$$\|HS_t a\| = \|S_t(Ha)\| \leq \|Ha\|$$

for $0 < t < t_a$. Iteration of this argument establishes that if $0 < t < t_a$ then $S_t a$ is an analytic element for S with associated radius of convergence equal to t_a. Thus it is possible to iterate the definition of S_t

$$S_{t+s}a = S_t\left(S_s a\right) = \sum_{n\geq 0} \frac{(-t)^n}{n!} H^n\left(S_s a\right)$$

for $0 < s$, $t < t_a$ and consequently deduce that $\|S_t a\| \le \|a\|$ for all $0 < t < 2t_a$. Repeating this argument one defines $S_t a$ for all $t \ge 0$ by

$$S_t a = \left(S_{t/n}\right)^n a$$

where n is chosen so that $n > t/t_a$. It is then easy to establish that this definition is independent of the choice of n ,

$$S_s\left(S_t a\right) = S_{s+t} a ,$$

for all $s, t > 0$,

$$\|S_t a\| \le \|a\|$$

for all $t > 0$, and

$$\lim_{t \to 0} \|S_t a - a\| = 0 .$$

Therefore, since the analytic elements are assumed to be norm dense, S extends by continuity to a C_0-semigroup of contractions on $\mathcal{B}$. □

The foregoing result readily extends to C_0-groups of contractions. But if $S = \{S_t\}_{t \in \mathbb{R}}$ is a group of contractions with $S_0 = I$ then S is automatically isometric because

$$\|a\| = \|S_{-t} S_t a\| \le \|S_t a\| \le \|a\| .$$

Second if S is also strongly continuous then $S_\pm = \{S_{\pm t}\}_{t \ge 0}$ are both C_0-semigroups of isometries. But

$$\left\|\frac{(I-S_t)}{t}a - b\right\| = \left\|S_t\frac{(I-S_{-t})}{t}a + b\right\|$$

and hence the generator of S_+ is minus the generator of S_- . Combining these observations with Proposition 1.4.1 and the construction of analytic elements described prior to the propositon one obtains the following.

THEOREM 1.6.2. *Let* H *be an operator on the Banach space* $\mathcal{B}$. *The following conditions are equivalent:*

1. H *is the infinitesimal generator of a* C_0*-group of isometries of* $\mathcal{B}$.

2. H *is norm closed;* H *possesses a norm dense set of analytic elements* $\pm H$ *are both norm-dissipative.*

Proof. $1 \Rightarrow 2$. The entire analytic elements for H are dense by the construction preceding Proposition 1.6.1. The rest of the properties of H follow from the Hille-Yosida theorem.

$2 \Rightarrow 1$. This follows by successively applying Proposition 1.6.1 to $\pm H$ and then using the above observation that a group $S_t = \exp\{-tH\}$ of contractions is automatically isometric. □

One can also give a C_0^*-version of Proposition 1.6.1 and then deduce a weak*-version of Theorem 1.6.2. Since the second result is deduced by the same argument given above we will merely prove the analogue of Proposition 1.6.2.

PROPOSITION 1.6.3. *Let* $\mathcal{B}$ *be a Banach space with a predual* $\mathcal{B}_*$ *and* H *a weak*-weak*-closed operator on* $\mathcal{B}_*$. *Suppose*

1. *the unit ball of the set of analytic elements for* H *is weak*-dense in the unit ball of* $\mathcal{B}$,

2. H *is norm-dissipative.*

It follows that H *is the generator of a* C_0^**-semigroup of contractions.*

Proof. Let $\mathcal{B}_a \subseteq \mathcal{B}$ denote the norm closure of the subspace of all analytic elements for H and let H_a denote the restriction of H to $\mathcal{B}_a$. It follows immediately that H_a is norm closed and hence by Proposition 1.6.1 it generates a C_0-semigroup S of contractions on $\mathcal{B}_a$. In particular H_a is norm-dissipative and $R(I+\alpha H_a) = \mathcal{B}_a$ for all $\alpha > 0$.

Now by Condition 1 we may choose for each $f \in \mathcal{B}$ a family $f_\beta \in \mathcal{B}_a$ such that f_β converges to f in the weak*-sense and $\|f_\beta\| \leq \|f\|$. But it follows from the foregoing argument that there exist $g_\beta \in D(H_a) \subseteq D(H)$ such that $f_\beta = (I+\alpha H)g_\beta$ and

$$\|g_\beta\| \leq \|(I+\alpha H_a)g_\beta\| = \|f_\beta\| \leq \|f\| .$$

Thus $\{\|g_\beta\|\}$ is uniformly bounded. But the unit ball in $\mathcal{B}$ is weak*-compact, by the Alaoglu-Birkhoff theorem, and hence one may choose a weak*-convergent subfamily $g_{\beta'}$ of g_β . Let g denote its limit. Then $g_{\beta'} \to g$ and $f_{\beta'} = (I+\alpha H_a)g_{\beta'} \to f$ where both limits are in the weak*-sense.

But H is weak*-weak*-closed and so H_a is weak*-weak*-closable and its closure $\overline{H}_a$ is both norm-dissipative and satisfies the range condition $R(I+\alpha\overline{H}_a) = \mathcal{B}$ for $\alpha > 0$. Therefore $\overline{H}_a$ generates a C_0^*-semigroup S by Theorem 1.5.2. But H is a norm dissipative extension of $\overline{H}_a$ and since the latter is a generator one must have $H = \overline{H}_a$. □

We conclude this section with a Hilbert space example.

Example 1.6.4. Consider the criteria of Theorems 1.3.1 and 1.6.2 for a C_0-group of isometries on a Hilbert space $\mathcal{H}$. Norm-dissipativity of $\pm H$ is equivalent to

$$\mathrm{Re}(a, Ha) = 0$$

for all $a \in D(H)$. Setting $H = iK$ this becomes

$$(a, Ka) = (Ka, a)$$

for all $a \in D(K)$, i.e., K must be a symmetric operator. Thus Theorems 1.3.1 and 1.6.2 state that H is the generator of a C_0-group of isometries if, and only if, $H = iK$ where K is a densely defined, closed, symmetric operator satisfying

either $R(I+i\alpha K) = \mathcal{H}$, $\alpha \in \mathbb{R}\setminus\{0\}$

or K possesses a dense set of analytic elements.

The first of these conditions is the usual criterion for self-adjointness of K. Hence one can conclude from this argument that a densely defined, closed, symmetric opperator is self-adjoint if, and only if, it possesses a dense set of analytic elements.

If these conditions are satisfied then the associated operators $S_t = \exp\{-iKt\}$ form a unitary group, e.g., $S_t^* = S_{-t}$. Both the unitary group and the generator can be represented by spectral theory as direct integrals of multiplication operators. In particular there exists a family of projection valued probability measures E over $\mathbb{R}$ such that

$$(a, S_t b) = \int_{-\infty}^{\infty} d(a, E(\lambda)b)e^{-i\lambda t}$$

for all $a, b \in H$ and

$$(a, Kb) = \int_{-\infty}^{\infty} d(a, E(\lambda)b)\lambda$$

for all $a \in H$, and $b \in D(K)$, where the domain of K is defined by

$$D(K) = \left\{b \ ; \int_{-\infty}^{\infty} d(b, E(\lambda)b)\lambda^2 < +\infty\right\} . \qquad \square$$

In the Hilbert space context one can further elaborate the extension theory mentioned at the end of Section 1.3. Thus given a symmetric operator K one tries to construct self-adjoint extensions. This construction is a repetition of the procedure outlined in Section 1.3. Both $\pm iK$ must be extended to generators iK_1, $-iK_2$, of contraction semigroup, $S^{\pm}$. But these semigroups determine a C_0-group of isometries, by $S_t = S_t^+$ if $t \geq 0$ and

$S_t = S_t^-$ if $t \le 0$, if, and only if, $K_1 + K_2 = 0$. To obtain this latter relation it is imperative that the deficiency indices of $\pm K$ are identical.

Exercises.

1.6.1. An element $a \in \mathcal{B}$ is defined to be bounded for H if $a \in D(H^n)$ for all $n \ge 1$ and

$$\|H^n a\| \le r^n \|a\|$$

for some $r \ge 0$. Prove that if H is the generator of the C_0-semigroup S and a is bounded for H then

$$Ha = t^{-1} \sum_{n\ge1} (I-S_t)^n a/n$$

for $rt \le 1$.

Hint: Use $(I-S_t)a = \int_0^t ds\, S_s Ha$.