1.5. C_0^*-semigroups.

If the Banach space $\mathcal{B}$ is the dual of a Banach space $\mathcal{B}_*$, the pre-dual of $\mathcal{B}$, then it is of interest to study families of bounded operators $S = \{S_t\}_{t\geq 0}$ with the semigroup property $S_sS_t = S_{s+t}$ which are weak*-continuous in the sense that

1. $$\lim_{t\to 0+} (S_tf, a) = (f, a)$$

   for all $f \in \mathcal{B}$ and $a \in \mathcal{B}_*$,

2. $$\lim_{\alpha} (S_tf_\alpha, a) = (S_tf, a)$$

   for all $t > 0$, all $a \in \mathcal{B}_+$, and all families f_α such that

   $$\lim_{\alpha} (f_\alpha, a) = (f, a) .$$

Such families are called C_0^*-semigroups. The simplest example is translations on $L^\infty(\mathbb{R})$ which has pre-dual $L^1(\mathbb{R})$.

Our first aim is to show that if S is a C_0^*-semigroup there exists an adjoint semigroup S_* on $\mathcal{B}_*$ such that

$$(S_tf, a) = (f, S_{*t}a) .$$

The weak*-continuity of S then implies the weak, and hence strong, continuity of S_*, i.e., the C_0^*-semigroup S is the adjoint of a C_0-semigroup S_*. This explains the name C_0^*-semigroup. In the sequel we demonstrate that much of the foregoing theory of C_0-semigroups can be carried over to the C_0^*-semigroups by duality arguments.

We begin by recalling a number of standard definitions.

A family $f_\alpha \in \mathcal{B}$ is weak*-convergent if there is an $f \in \mathcal{B}$ such that

$$\lim_\alpha \left(f_\alpha, a\right) = (f, a)$$

for all $a \in \mathcal{B}_*$, and a set $\mathcal{D} \subseteq \mathcal{B}$ is weak*-closed if each weak*-convergent family $f_\alpha \in \mathcal{D}$ has a limit $f \in D$. Alternatively a set $\mathcal{D} \subseteq \mathcal{B}$ is weak*-dense if each $f \in \mathcal{B}$ can be approximated by $f_\alpha \in \mathcal{D}$ in the weak*-sense, i.e.,

$$\lim_\alpha \left(f_\alpha, a\right) = (f, a)$$

for all $a \in \mathcal{B}_*$.

Next an operator H on $\mathcal{B}$ is weak*-densely defined if its domain $D(H)$ is weak*-dense in $\mathcal{B}$ and it is weak*-weak*-closed if $f_\alpha \in D(H)$ and

$$\lim_\alpha \left(f_\alpha, a\right) = (f, a)$$

$$\lim_\alpha \left(Hf_\alpha, a\right) = (g, a) ,$$

for all $a \in \mathcal{B}_*$, imply that $f \in D(H)$ and $g = Hf$. Moreover H is weak*-weak*-closable if it has a weak*-weak*-closed extension or, equivalently, if $f_\alpha \in D(H)$ and $\left(f_\alpha, a\right) \to 0$, $\left(Hf_\alpha, a\right) \to (g, a)$, for all $a \in \mathcal{B}_*$, imply that $g = 0$.

The basic duality properties of operators rely upon two versions of the bipolar theorem. Specifically *if* A *is a weak*-closed sbuspace of* $\mathcal{B}$ *and one defines*

$$A^{\perp} = \{a \in \mathcal{B}_* \; ; \; (f, a) = 0 \text{ for all } f \in A\} \, ,$$

$$A^{\perp\perp} = \{f \in \mathcal{B} \; ; \; (f, a) = 0 \text{ for all } a \in A^{\perp}\}$$

then $A = A^{\perp\perp}$. Similarly *if* A_* *is a closed subspace of* $\mathcal{B}_*$ *and*

$$A_*^{\perp} = \{f \in \mathcal{B} \; ; \; (f, a) = 0 \text{ for all } a \in A_*\} \, ,$$

$$A_*^{\perp\perp} = \{a \in \mathcal{B} \; ; \; (f, a) = 0 \text{ for all } f \in A_*^{\perp}\}$$

then $A_* = A_*^{\perp\perp}$. Both these statements are a consequence of the Hahn-Banach theorem. Consider, for example, the second statement.

It follows by definition that $A_* \subseteq A_*^{\perp\perp}$. Next define p over $\mathcal{B}_*$ by

$$p(a) = \inf\{\|a - c\| \; ; \; c \in A_*\} \, ,$$

then $p(a) = 0$ for all $a \in A_*$ but $p(a) \neq 0$ for $a \notin A_*$. Moreover p satisfies the hypotheses of the Hahn-Banach theorem cited in Section 1.4. Hence for $a \in A_*$ and $b \notin A_*$ one has

$$p(a+\lambda b) = \pm\lambda p(b \pm a/\lambda) = |\lambda| p(b)$$

where the $+$ and $-$ signs correspond to positive and negative λ respectively. Next introduce C as the subspace spanned by A_*

and b and define a linear functional f over $\mathcal{C}$ by

$$(f, a+\lambda b) = \lambda p(b)$$

for $a \in \mathcal{A}_*$. One has $|(f, c)| = p(c)$ for $c \in \mathcal{C}$ and hence, by the Hahn-Banach theorem, there exists a linear extension F of f to $\mathcal{B}_*$ satisfying $|F(a)| \le p(a)$ for all $a \in \mathcal{B}_*$. Since $p(a) \le \|a\|$ it follows that $F \in \mathcal{B}$ and since $F(a) = 0$ for all $a \in \mathcal{A}_*$ one also concludes that $F \in \mathcal{A}_*^{\perp}$. Finally $F(b) = (f, b) = p(b) \ne 0$ and hence $b \notin \mathcal{A}_*^{\perp\perp}$. Thus $\mathcal{A}_*^{\perp\perp} \subseteq \mathcal{A}_*$. and the two sets must be identical.

LEMMA 1.5.1. *Let $\mathcal{B}$ be a Banach space with a predual $\mathcal{B}_*$ and H an operator on $\mathcal{B}$.*

The following conditions are equivalent:

1. H *is weak*-densely defined and weak*-weak*-closed,*

2. H *is the adjoint of a norm densely defined, norm closed, operator* H_* *on* $\mathcal{B}_*$.

If these conditions are fulfilled and H *is bounded then* $\|H\| = \|H_*\|$.

Proof. 1 ⇒ 2. Consider $\mathcal{B} \times \mathcal{B}$ equipped with the norm $\|(f, g)\| = \left(\|f\|^2+\|g\|^2\right)^{\frac{1}{2}}$ and $\mathcal{B}_* \times \mathcal{B}_*$ with the norm $\|(a, b)\| = \left(\|a\|^2+\|b\|^2\right)^{\frac{1}{2}}$. These two spaces are then in duality through the relation

$$\big((f, g), (a, b)\big) = (f, a) + (g, b) .$$

Next introduce the graph $G(H)$ of H in $\mathcal{B} \times \mathcal{B}$ as the subspace

$$G(H) = \{(f, Hf) \; ; \; f \in D(H)\} .$$

Thus the orthogonal complement $G(H)^{\perp}$ of $G(H)$ in $\mathcal{B}_* \times \mathcal{B}_*$ consists of the pairs (a, b) which satisfy

$$(f, a) + (Hf, b) = 0$$

for all $f \in D(H)$. Now define

$$G = \{(-b, a) \; ; \; (a, b) \in G(H)^{\perp}\} .$$

Then G is the graph of an operator H_* on $\mathcal{B}_*$. This follows because if $(0, a) \in G$ the orthogonality relation gives

$$(f, a) = 0 ,$$

for all $f \in D(H)$, and $a = 0$ because $D(H)$ is weak*-dense. But $G(H)^{\perp}$, and G, are norm closed by definition and hence H_* is norm closed. Finally, if H_* is not norm densely defined there must exist a non-zero element of $G^{\perp}$ of the form $(-f, 0)$. Thus $(0, f) \in G(H)^{\perp\perp}$. But since H is weak*-weak*-closed $G(H)$ is a weak*-closed subspace and $G(H)^{\perp\perp} = G(H)$, by the first version of the bipolar theorem cited above. Hence $(0, f) \in G(H)$. This, however, contradicts the linearity of H and consequently $D(H_*)$ must be norm dense.

$2 \Rightarrow 1$. The proof is identical but $\mathcal{B}_*$ replaces $\mathcal{B}$, H_* replaces H, etc., and one uses the second version of the bipolar theorem.

Finally the equality of the norms for bounded operators follows because

$$\|H\| = \sup\{|(Hf, a)| \ ; \ f \in \mathcal{B} \ , \ a \in \mathcal{B}_*\}$$

$$= \sup\{|(f, H_* a)| \ ; \ f \in \mathcal{B} \ , \ a \in \mathcal{B}_*\} = \|H_*\| \ . \qquad \square$$

If $S = \{S_t\}_{t \geq 0}$ is a C_0^*-semigroup on $\mathcal{B}$ then the S_t are everywhere defined and weak*-weak*-closed, by the second continuity hypothesis. Hence Lemma 1.5.1 establishes the existence of an adjoint semigroup $S_* = \{S_{*t}\}_{t \geq 0}$ on $\mathcal{B}_*$ such that

$$(S_t f, a) = (f, S_{*t} a) \ ,$$

for all $f \in \mathcal{B}$ and $a \in \mathcal{B}_*$. Moreover,

$$\|S_t\| = \|S_{*t}\| \ .$$

But weak*-continuity of S is equivalent to weak, and hence strong, continuity of S_* . Thus S_* is a C_0-semigroup and in general satisfies bounds of the form $\|S_{*t}\| \leq M \exp\{\omega t\}$. Hence the C_0^*-semigroup S satisfies similar bounds. Now by exploiting the Hille-Yosida theorem for the C_0-semigroup S_* and the duality properties of Lemma 1.5.1 one can obtain a Hille-Yosida theorem for the C_0^*-semigroup S . But first we must define the generator of S .

If S is a C_0^*-semigroup its generator H is defined as the weak*-derivative of S at the origin. Explicitly $D(H)$ consists of those $f \in \mathcal{B}$ for which there is a $g \in \mathcal{B}$ such that the limits

$$(g, a) = \lim_{t \to 0+} \big((I-S_t)f, a\big) \big/ t$$

exist for all $a \in \mathcal{B}_*$ and the action of H is then given by $Hf = g$. Note that if K is the generator of the C_0-semigroup S_* on $\mathcal{B}_*$, which is adjoint to S, then

$$(Hf, a) = \lim_{t \to 0+} \big((I-S_t)f, a\big) \big/ t$$

$$= \lim_{t \to 0+} \big(f, (I-S_{*t})a\big) \big/ t = (f, Ka)$$

for all $f \in D(H)$ and $a \in D(K)$. This demonstrates that the adjoint K^*, of K, extends H but part of the proof of the following result is to show that in fact $K^* = H$.

THEOREM 1.5.2. *Let $\mathcal{B}$ be a Banach space with a predual $\mathcal{B}_*$ and H an operator on $\mathcal{B}$. The following conditions are equivalent:*

1. H *is the infinitesimal generator of a C_0^*-semigroup of contractions,*

2. H *is weak*-densely defined, weak*-weak*-closed,*

$$R(I+\alpha H) = \mathcal{B}$$

for all $\alpha > 0$ *(or for one* $\alpha = \alpha_0 > 0$*), and*

$$\|(I+\alpha H)f\| \ge \|f\|$$

for all $f \in D(H)$ *and all* $\alpha > 0$ *(or for all* $\alpha \in \langle 0, \alpha_0]$*).*

Proof. 1 ⇒ 2. The proof of this implication follows the reasoning used to establish Proposition 1.2.1.

First for $\alpha > 0$ one can define a bounded operator $R_\alpha(H)$ on $\mathcal{B}$ by

$$\left(R_\alpha(H)f,\ a\right) = \int_0^\infty dt\ e^{-t}\left(S_{\alpha t}f,\ a\right)$$

and since S is contractive one has the bound

$$\|R_\alpha(H)\| \leq 1\ .$$

But a weak*-version of the calculation used in the proof of Proposition 1.2.1 demonstrates that

$$R_\alpha(H) = (I+\alpha H)^{-1}\ .$$

Hence

$$R(I+\alpha H) = \mathcal{B}$$

and

$$\|(I+\alpha H)f\| \geq \|f\|$$

for all $f \in D(H)$. But $R_\alpha(H)f \in D(H)$ for all $f \in \mathcal{B}$ and

$$\lim_{\alpha\to 0+}\left(R_\alpha(H)f,\ a\right) = \lim_{\alpha\to 0+}\int_0^\infty dt\ e^{-t}\left(S_{\alpha t}f,\ a\right)$$

$$= (f,\ a)$$

for all $a \in \mathcal{B}_*$ by weak*-continuity of S and the Lebesgue dominated convergence theorem. Thus $D(H)$ is weak*-dense. Finally suppose $f_\beta \in D(H)$ and

$$\lim_\beta \left(f_\beta, a\right) = (f, a)$$

$$\lim_\beta \left((I+\alpha H)f_\beta, a\right) = (g, a)$$

for all $a \in \mathcal{B}_*$. Then

$$(f, a) = \lim_\beta \left(R_\alpha(H)(I+\alpha H)f_\beta, a\right)$$

$$= \left(R_\alpha(H)g, a\right)$$

for all $a \in \mathcal{B}_*$ by another application of the Lebesgue dominated convergence theorem. Thus $(I+\alpha H)$, and hence H , is weak*-weak*-closed.

$2 \Rightarrow 1.$ It follows from Lemma 1.5.1 that H is the adjoint of a norm densely defined, norm closed, operator H_* on $\mathcal{B}_*$. But for $\alpha > 0$ and $a \in D(H)$

$$\|(I+\alpha H_*)a\| = \sup\left\{\left|(f, (I+\alpha H_*)a)\right| \; ; \; f \in D(H) \; , \; \|f\| \le 1\right\}$$

$$= \sup\left\{\left|((I+\alpha H)f, a)\right. \; ; \; f \in D(H) \; , \; \|f\| \le 1\right\} .$$

Thus since $\|(I+\alpha H)f\| \ge \|f\|$ and $R(I+\alpha H) = \mathcal{B}$ one concludes that

$$\|(I+\alpha H_*)a\| \ge \sup\left\{\left|(g, a)\right| \; ; \; \|g\| \le 1\right\}$$

$$\|a\| \; ,$$

i.e., H_* is norm-dissipative.

Next suppose there is an $f \in \mathcal{B}$ such that

$$(f,\ (I+\alpha H_*)a) = 0$$

for all $a \in D(H_*)$. But then

$$(f,\ H_*a) = -(f,\ a)/\alpha$$

is continuous in a . Hence $f \in D(H)$ and

$$((I+\alpha H)f,\ a) = 0$$

for all $a \in D(H_*)$. Since $D(H_*)$ is norm dense it follows that $(I+\alpha H)f = 0$ and then $f = 0$ because H is norm-dissipative. Hence $R(I+\alpha H_*) = \mathcal{B}$.

Finally we can apply the Hille-Yosida theorem to deduce that H_* generates a C_0-semigroup of contractions S_* on $\mathcal{B}_*$. Then the adjoint semigroup S on $\mathcal{B}$ is a C_0^*-semigroup of contractions. But if K denotes the generator of this latter semigroup then by Laplace transformation

$$\big((I+\alpha K)^{-1}f,\ a\big) = \big(f,\ (I+\alpha H_*)^{-1}a\big)$$

for all $f \in \mathcal{B}$ and $a \in \mathcal{B}_*$. Thus $(I+\alpha K)^{-1} = (I+\alpha H)^{-1}$, i.e., $K = H$ is the generator of S . □

There is also a pre-generator version of the foregoing theorem. If H is weak*-densely defined and weak*-weak*-closable then its weak*-closure $\overline{H}$ generates a C_0^*-semigroup of contractions if, and only if, H is norm-dissipative and $R(I+\alpha H)$ is weak*-dense in $\mathcal{B}$ for all sufficiently small $\alpha > 0$.

Finally we remark that a result analogous to Theorem

1.5.2 can be obtained for a general C_0^*-semigroup. The norm-dissipativity which is characteristic of contraction semigroups is replaced by a family of lower bounds of the type described in Remark 1.3.3.

Exercises.

1.5.1. Let $\mathcal{L}(H)$ denote the algebra of all bounded operators on the Hilbert space H and $\mathcal{T}(H)$ the Banach space of trace class operators, with the norm

$$T \in \mathcal{T}(H) \mapsto \|T\|_{tr} = \mathrm{Tr}\left((T^*T)^{\frac{1}{2}}\right) .$$

Prove that $\mathcal{L}(H)$ is the dual of $\mathcal{T}(H)$ with the duality

$$(T, B) \mapsto \mathrm{Tr}(TB) .$$

1.5.2. Let S be a C_0^*-semigroup on the Banach space $\mathcal{B}$ with generator H. Prove that $f \in D(H)$ if, and only if,

$$\sup_{0<t<1} \|(I-S_t)f\|/t < +\infty .$$

Hint: The unit ball of $\mathcal{B}$ is weakly*-compact by the Alaoglu-Birkhoff theorem.

1.5.3. Let S be a C_0^*-semigroup with generator H and define $\mathcal{B}_0 \subseteq \mathcal{B}$ as the norm closure of $D(H)$. Prove that $S\mathcal{B}_0 \subseteq \mathcal{B}_0$ and that the restriction of S to $\mathcal{B}_0$ is a C_0-semigroup.