1.4. Norm-dissipative Operators.

The Hille-Yosida theorem establishes that norm-dissipativity of a generator H, i.e., the condition

$$\|(I+\alpha H)a\| \geq \|a\| , \qquad a \in D(H) ,$$

for small $\alpha > 0$, is an infinitesimal reflection of contractivity of the associated semigroup. Next we discuss a reformulation of dissipativity which corresponds to a more geometric interpretation of contractivity. This reformulation is the Banach space analogue of the condition

$$\mathrm{Re}(a, Ha) \geq 0 , \qquad a \in D(H) ,$$

which characterizes dissipative operators H on Hilbert space.

The semigroup S is contractive if, and only if, it maps the unit sphere, $\{a ; \|a\| = 1\}$, into the unit ball, $\mathcal{B}_1 = \{a ; \|a\| \leq 1\}$. Thus the change $S_t a - a$ of an element a must be toward the interior of the ball of radius $\|a\|$. To describe this last geometric idea in a quantitative manner it is necessary to introduce the notion of a tangent functional.

An element $f_a \in \mathcal{B}^*$ is defined to be a norm-tangent functional at a if

$$\|b\| \geq \|a\| + \mathrm{Re}\left(f_a, b-a\right)$$

for all $b \in \mathcal{B}$. Geometrically each such functional describes a hyperplane tangent to the graph of $b \in \mathcal{B} \mapsto \|b\| \geq 0$ at the point a. The functional f_a divides the space into two sets

$E_a = \{b\ ;\ \mathrm{Re}(f_a, b) \geq 0\}$ and $I_a = \{b\ ;\ \mathrm{Re}(f_a, b) \leq 0\}$. The first set can be interpreted as the b which are directed toward the exterior of the ball $\{b\ ;\ \|b\| \leq \|a\|\}$ and the second set the b which are directed toward the interior. Hence the geometric rephrasing of contractivity of S given in the last paragraph can be quantitatively expressed as

$$\mathrm{Re}(f_a, S_t a - a) \leq 0 ,$$

i.e., the change $S_t a - a$ of a is toward the interior of the ball. Indeed this property follows directly from the definition of the tangent functional f_a ,

$$\mathrm{Re}(f_a, S_t a - a) \leq \|S_t a\| - \|a\| \leq 0 .$$

Thus if H is the generator of the C_0-contraction semigroup S one concludes that

$$\mathrm{Re}(f_a, Ha) = \lim_{t\to 0+} \mathrm{Re}(f_a, a - S_t a)/t \geq 0$$

for all $a \in D(H)$ and all norm-tangent functionals f_a at a . This is the alternative reformulation of norm-dissipativity of H ; equivalence with the original formulation is provided by the following.

THEOREM 1.4.1. *Let* H *be an operator on the Banach space* $\mathcal{B}$. *The following conditions are equivalent:*

1. (1') $\|(I+\alpha H)a\| \geq \|a\|$

for all $a \in D(H)$ *and all* $\alpha > 0$ *(for all small* $\alpha > 0$*)*,

2. $$\mathrm{Re}\left(f_a, Ha\right) \geq 0$$

for one non-zero norm-tangent functional at each $a \in D(H)$.

Moreover if H *is norm densely defined these conditions are equivalent to the following:*

3. $$\mathrm{Re}\left(f_a, Ha\right) \geq 0$$

for all norm-tangent functionals f_a *at each* $a \in D(H)$.

The proof uses an alternative characterization of norm-tangent functional which can be used to establish the existence of such functionals.

LEMMA 1.4.2. *For* $f \in \mathcal{B}^*$ *the following conditions are equivalent:*

1. f *is a norm-tangent functional at* a ,

2. $$|(f, b)| \leq \|b\| , \qquad b \in \mathcal{B} ,$$

and

$$(f, a) = \|a\| .$$

Hence for each $a \in \mathcal{B} \setminus \{0\}$ *there exists a non-zero norm-tangent functional.*

Proof. $1 \Rightarrow 2$. Condition 1 states that

$$(*) \qquad \|b\| \geq \|a\| + \mathrm{Re}(f, b-a) .$$

Thus replacing b by $\lambda e^{i\theta} b$ one finds

$$\|b\| \geq \lim_{\lambda \to \infty} \left\{ \|a\|/\lambda + \mathrm{Re}\, e^{i\theta}\left(f, b - a/\lambda\right) \right\}$$

$$= \mathrm{Re}\, e^{i\theta}(f, b) .$$

Hence $|(f, b)| \leq \|b\|$. But setting $b = 0$ in $(*)$ one also obtains $(f, a) \geq \|a\|$ and therefore $(f, a) = \|a\|$.

$2 \Rightarrow 1$. Successively applying the two relations of Condition 2 one has

$$\|b\| \geq \mathrm{Re}(f, b)$$
$$= \mathrm{Re}(f, a) + \mathrm{Re}(f, b-a)$$
$$= \|a\| + \mathrm{Re}(f, b-a) .$$

Finally the Hahn-Banach theorem states that *if* p *is a real-valued function over* $\mathcal{B}$ *satisfying*

$$p(a+b) \leq p(a) + p(b) , \qquad a, b \in \mathcal{B} ,$$

$$p(\lambda a) = \lambda p(a) \qquad , \qquad \lambda \geq 0 , \quad a \in \mathcal{B}$$

and f *is a linear functional over a subspace* $C \subseteq \mathcal{B}$ *such that* $|(f, c)| \leq p(c)$ *for* $c \in C$ *then there exists a linear extension* F *of* f *to* $\mathcal{B}$ *such that* $|F(a)| \leq p(a)$ *for all* $a \in \mathcal{B}$. Therefore choosing $p(\cdot) = \|\cdot\|$, $C = \{\lambda a \,;\ \lambda \in \mathbb{C}\}$, and setting $(f, \lambda a) = \lambda\|a\|$, one can find a linear extension F to $\mathcal{B}$ satisfying $|F(b)| \leq \|b\|$ and $F(a) = (f, a) = \|a\|$. Hence F is a non-zero norm-tangent functional at a by Condition 2 of the lemma. □

Proof of Theorem 1.4.1. $1' \Rightarrow 2$. Set $b = Ha$ and for each sufficiently small α choose a norm-tangent functional g_α at the point $a + \alpha b$. Then from Condition 1

$$\begin{aligned} \|a\| &\leq \|a + \alpha b\| \\ &= \mathrm{Re}\left(g_\alpha, a+\alpha b\right) \\ &= \mathrm{Re}\left(g_\alpha, a\right) + \alpha\, \mathrm{Re}\left(g_\alpha, b\right) \\ &\leq \mathrm{Re}\left(g_\alpha, a\right) + \alpha\|b\| \ . \end{aligned}$$

Now the unit ball of $\mathcal{B}^*$ is weakly* compact by the Alaoglu-Birkhoff theorem, i.e., for every net $f_\alpha \in \mathcal{B}^*$ with $\|f_\alpha\| \leq 1$ there is a subset $f_{\alpha'}$ which converges to an $f \in \mathcal{B}^*$ in the sense that $(f_{\alpha'}, a) \to (f, a)$ for all $a \in \mathcal{B}$. Hence one deduces from the foregoing inequality that

$$\begin{aligned} \|a\| &\leq \lim_{\alpha'\to 0} \left\{\mathrm{Re}\left(g_{\alpha'}, a\right) + \alpha'\|b\|\right\} \\ &= \mathrm{Re}(g, a) \end{aligned}$$

where g is the weak* limit of the subset $g_{\alpha'}$. Now since $\|g_\alpha\| = 1$ one has $\|g\| \leq 1$ and then

$$\mathrm{Re}(g, a) \leq \|g\|\ \|a\| \leq \|a\| \ .$$

Hence

$$(g, a) = \|a\| .$$

This proves that g is a norm-tangent functional at a . But one also has

$$\|a\| \leq \mathrm{Re}(g_\alpha, a) + \alpha\, \mathrm{Re}(g_\alpha, b)$$

$$\leq \|a\| + \alpha\, \mathrm{Re}(g_\alpha, b)$$

and hence in the limit $\alpha' \to 0$ one obtains

$$0 \leq \mathrm{Re}(g, b) = \mathrm{Re}(g, Ha) ,$$

i.e., Condition 2 is satisfied.

$2 \Rightarrow 1$. Let f be a norm-tangent functional at $a \in D(H)$ satisfying

$$\mathrm{Re}(f, Ha) \geq 0 .$$

Then

$$\|a\| = \mathrm{Re}(f, a)$$

$$\leq \mathrm{Re}(f, a+\alpha Ha)$$

$$\leq \|(I+\alpha H)a\|$$

for all $\alpha > 0$.

$1 \Rightarrow 1'$. This is evident.

Finally 3 ⇒ 2 and it remains to prove 1 ⇒ 3 under the assumption that D(H) is norm dense.

Now if a, b ∈ D(H) and f is a non-zero norm-tangent functional at a one has

$$\|(I-\alpha H)a\| \geq \|a\| - \alpha \operatorname{Re}(f, Ha) .$$

Therefore

$$\operatorname{Re}(f, Ha) \geq \limsup_{\alpha \to 0+} \left(\|a\| - \|(I-\alpha H)a\|\right) \Big/ \alpha$$

But

$$\begin{aligned} \|(I-\alpha H)a\| &\leq \|a + \alpha b\| + \alpha\|b + Ha\| \\ &\leq \|(I+\alpha H)(a+\alpha b)\| + \alpha\|b + Ha\| \\ &\leq \|a\| + 2\alpha\|b + Ha\| + \alpha^2\|Hb\| \end{aligned}$$

for all sufficiently small $\alpha > 0$ by Condition 1. Therefore by combination of these results

$$\operatorname{Re}(f, Ha) \geq -2\|b + Ha\| .$$

But since D(H) is norm dense we may choose b arbitrarily close to -Ha and deduce that

$$\operatorname{Re}(f, Ha) \geq 0 ,$$

i.e., Condition 3 is satisfied. □

Example 1.4.3. Let H be a Hilbert space and hence identifiable with its dual. If a, b ∈ H then

$$|(a,\ b)| \leq \|a\|\ \|b\|$$

with equality if, and only if, $a = \lambda b$ for some $\lambda \in \mathbb{C}$. Therefore $a/\|a\|$ is the unique norm-tangent functional at $a \in \mathcal{H}$ and Theorem 1.4.1 states that an operator H is norm-dissipative if, and only if,

$$\mathrm{Re}(a,\ Ha) \geq 0$$

for all $a \in D(H)$. This is the characterization used in Section 1.3. □

Example 1.4.4. If $\mathcal{B} = L^p(X;\ d\mu)$ with $p \in \langle 1, \infty \rangle$ then there is a unique norm-tangent functional at each $f \in \mathcal{B}$ given by $\left(|f|^{p-1} \arg f\right) \|f\|_p^{p-1}$ where $\arg f(x) = f(x)/|f(x)|$ if $|f(x)| \neq 0$ and $\arg f(x) = 0$ if $|f(x)| = 0$. If $p = 1$ this gives the tangent functional $\arg f$, but this is not unique if $f = 0$ on a set Y of non-zero measure. In this case $g + \arg f$, where g has support in Y and $|g| \leq 1$, is also a tangent functional. □

Theorem 1.4.1 allows an immediate reformulation of the Hille-Yosida theorem which is often more convenient for applications.

THEOREM 1.4.5. (Lumer and Phillips). *Let* H *be an operator on the Banach space* $\mathcal{B}$ *. The following conditions are equivalent:*

1. H *is the generator of a* C_o*-contraction semigroup* S ,

2. H *is (norm closed), norm densely defined*

$$R(I+\alpha H) = \mathcal{B}$$

for all $\alpha > 0$ *(or for an* $\alpha > 0$*) and*

$$\mathrm{Re}\left(f_a, Ha\right) \geq 0$$

for one norm-tangent functional f_a *at each* $a \in D(H)$.

The alternative characterization of norm-dissipativity provided by Theorem 1.4.1 also allows an easy proof of a version of the Hille-Yosida theorem in which the range condition $R(I+\alpha H) = \mathcal{B}$ does not occur explicitly.

THEOREM 1.4.6. *Let* H *be an operator on the Banach space* $\mathcal{B}$ *and consider the following conditions:*

1. H *is norm densely defined with norm densely defined adjoint* H^* *and both* H *and* H^* *are norm dissipative,*

2. H *is norm closable and its closure* $\overline{H}$ *generates a* C_0*-contraction semigroup.*

Then $1 \Rightarrow 2$ *and if* $\mathcal{B}$ *is reflexive* $2 \Rightarrow 1$.

Proof. $1 \Rightarrow 2$. Suppose $R(I+H)$ is not norm dense in $\mathcal{B}$. The Hahn-Banach theorem then implies the existence of a non-zero $f \in \mathcal{B}^*$ such that $(f, (I+H)a) = 0$ for all $a \in D(H)$. Therefore

$$|(f, Ha)| = |(f, a)| \leq \|f\| \, \|a\|$$

and hence $f \in D(H^*)$. Moreover since $D(H)$ is norm dense $(I+H^*)f = 0$. Thus if $b \in \mathcal{B}^{**}$ is a norm-tangent functional at $f \in \mathcal{B}^*$ one has

$$(b, H^*f) = -(b, f) = -\|f\|$$

which contradicts the norm-dissipativity of H^* . Hence $R(I+H)$ is norm dense and the desired implication follows from Theorem 1.3. .

Next assume $\mathcal{B}$ is reflexive and consider the converse.

$2 \Rightarrow 1$. If $\overline{H}$ generates the C_0-contraction semigroup S then $\overline{H}^*$ generates the C_0-contraction semigroup S^* (see Exercise 1.3.4). Hence Condition 1 follows from the Hille-Yosida theorem applied to S and S^* . □

Of course the drawback of this criterion is that one has to specifically identify the adjoint H^* before it is applicable.

Finally we illustrate the notion of norm-dissipativity with two examples of matrices acting on finite-dimensional spaces.

Example 1.4.7. (Matrix Semigroups). Let $x = (x_1, x_2, \ldots, x_n)$ denote an element of the finite-dimensional space $\mathbb{C}^n$. Further let $H = (H_{ij})$ be a complex-valued $n \times n$ matrix acting on $\mathbb{C}^n$ and $S_t = \exp\{-tH\}$, $t \geq 0$, the corresponding matrix semigroup.

The space $\mathbb{C}^n$ can be equipped with various norms which are all equivalent in the topological sense. But S can be contractive with respect to one norm without being contractive with respect to an equivalent norm. Nevertheless if a norm is given then S is contractive if, and only if, H is dissipative. Dissipativity with respect to the ℓ^∞- and ℓ^1-norms is particularly easy to describe because of the simple geometry of the corresponding balls. We will not pursue, however, the geometric aspects but proceed analytically.

Define the ℓ^∞-norm on $\mathbb{C}^n$ by

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| .$$

It follows that $S_t = \exp\{-tH\}$ *is* ℓ^∞*-contractive if, and only if,*

$$(*) \qquad \mathrm{Re}H_{ii} - \sum_{j\neq i} |H_{ij}| \geq 0$$

for all $i = 1, 2, \ldots, n$. This is established as follows. For i fixed choose x such that $x_i = 1$, $x_j = -\overline{H}_{ij}\big/|H_{ij}|$ if $j \neq i$ and $H_{ij} \neq 0$, and $x_j = 0$ if $j \neq i$ and $H_{ij} = 0$. Next choose $f = (f_1, \ldots, f_n)$ such that $f_i = 1$ and $f_j = 0$ if $j \neq i$. Then f is a norm-tangent functional at x and

$$\mathrm{Re}(f, Hx) = \mathrm{Re}\, H_{ii} - \sum_{j\neq i} |H_{ij}| .$$

Thus (*) is necessary for S to be ℓ^∞-contractive. Conversely let x be a non-zero element of $\mathbb{C}^n$ and choose i such that $|x_i| \geq |x_j|$ for all $j \neq i$. Set $f_i = x_i\big/|x_i|$ and $f_j = 0$ if $j \neq i$. It follows that f is a norm-tangent functional at x and

$$\begin{aligned}\mathrm{Re}(f, Hx) &= |x_i|\ \mathrm{Re}\, H_{ii} + \mathrm{Re} \sum_{j\neq i} H_{ij}\overline{x}_i x_j \Big/ |x_i| \\ &\geq |x_i| \left(\mathrm{Re}\, H_{ii} - \sum_{j\neq i} |H_{ij}|\right).\end{aligned}$$

Thus (*) is sufficient for H to be ℓ^∞-dissipative and S to be

ℓ^∞-contractive.

Next if

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

denotes the ℓ^1-norm it follows by duality that $S_t = \exp\{-tH\}$ is ℓ^1-contractive if, and only if, the adjoint semigroup $S_t^* = \exp\{-tH^*\}$ is ℓ^∞-contractive. Thus $S_t = \exp\{-tH\}$ *is* ℓ^1*-contractive if, and only if,*

$$\mathrm{Re}\, H_{ii} - \sum_{j\neq i} |H_{ji}| \geq 0$$

for all $i = 1, 2, \ldots, n$.

Finally one can equip $\mathbb{C}^n$ with the ℓ^p-norms

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

for $1 < p < \infty$ and consider ℓ^p-contractivity. If S is both ℓ^1- and ℓ^∞-contractive it follows by abstract interpolation that S is ℓ^p-contractive for all $p \in [1, \infty]$. This conclusion can, however, be reached by explicit estimate. For example if $p = 2$ then $x/\|x\|_2$ is the unique tangent functional at x and

$$\mathrm{Re}(x, Hx) = \sum_{i=1}^{n} \left\{|x_i|^2 \,\mathrm{Re}\, H_{ii} + \sum_{j\neq i} H_{ij}\overline{x}_i x_j\right\}$$

$$\geq \sum_{i=1}^{n} \left\{|x_i|^2 \,\mathrm{Re}\, H_{ii} - \sum_{j\neq i} |H_{ij}|\left(|x_i|^2+|x_j|^2\right)/2\right\}$$

$$= \sum_{i=1}^{n} \left(|x_i|^2 \left(\text{Re } H_{ii} - \sum_{j \neq i} \left(|H_{ij}| + |H_{ji}| \right) \Big/ 2 \right) \right)$$

where we have used the Cauchy-Schwarz inequality. Thus combination of the conditions for ℓ^1- and ℓ^∞-contractivity imply that H is ℓ^2-dissipative. A similar argument using the Minkowski inequality establishes that ℓ^1- and ℓ^∞-contractivity imply that H is ℓ^p-dissipative.

If $p \neq 1$ or ∞ the ℓ^p-dissipative conditions cannot be expressed in any particularly practical terms of the matrix element H_{ij}. Nevertheless if H is self-adjoint, i.e., if $H = H^*$, then ℓ^∞-contractivity of S implies ℓ^1-contractivity by duality and ℓ^p-contractivity, $p \in \langle 1, \infty \rangle$, by interpolation. Thus *a self-adjoint matrix semigroup is* ℓ^p*-contractive for all* $p \in [1, \infty]$ *if, and only if,* (*) *is valid.* More generally if H is normal, i.e., if $HH^* = H^*H$, then ℓ^2-dissipativity is implied by ℓ^1- or ℓ^∞-dissipativity. This will be established in the next example. □

Example 1.4.8. **(Normal matrix semigroups).** Let $S_t = \exp\{-tH\}$ denote the matrix semigroup of Exmaple 1.4.7. We first argue that if the conditions

$$(*) \qquad \text{Re } H_{ii} - \sum_{j \neq i} |H_{ij}| \geq 0 , \qquad i = 1, 2, \ldots, n$$

for ℓ^∞-contractivity are valid then $\text{Re}\,\lambda \geq 0$ for all eigenvalues λ of H. This follows by noting that if $(H-\lambda I)x = 0$ then

$$|H_{ii} - \lambda|\ |x_i| = \Big|\sum_{j\neq i} - H_{ij}x_j\Big|$$

$$\leq \sum_{j\neq i} |H_{ij}|\ |x_i|$$

where i has been chosen such that $|x_i| \geq |x_j|$ for $j \neq i$. Thus if x is non-zero the conditions (*) imply that

$$|H_{ii} - \lambda| \leq \text{Re}\ H_{ii}$$

and hence $\text{Re}\ \lambda \geq 0$. Consequently Example 1.3.6 implies that *if* S *is a normal matrix semigroup then* ℓ^∞*-contractivity implies* ℓ^2*-contractivity* and hence, by interpolation or by explicit estimation, it implies ℓ^p-contractivity for all $p \in [2, \infty]$. Similarly if S is a normal matrix semigroup then ℓ^1-contractivity implies ℓ^2-contractivity, and hence ℓ^p-contractivity for all $p \in [1, 2]$. □

Exercises.

1.4.1. Let H be the generator of a C_0-semigroup of contractions. Prove that the operators $H_\alpha = H(I+\alpha H)^{-1}$, $\alpha \geq 0$, are norm-dissipative.

1.4.2. Prove that if H is an invertible norm-dissipative operator on a Hilbert space then H^{-1} is norm-dissipative.

1.4.3. Prove that the closure of a norm densely defined, norm-dissipative, operator is norm-dissipative.