1.3. Generators and Semigroups.

Proposition 1.2.1 states necessary conditions for an operator to generate a C_0-semigroup of contractions. Next we examine sufficient conditions and also study the construction of a semigroup from its generator.

The problem of characterizing a generator H is equivalent to the problem of proving existence and uniqueness of global solutions of a differential equation

$$\frac{da_t}{dt} + Ha_t = 0 \,, \qquad a_t = a$$

for all a in a suitable Banach space $\mathcal{B}$. Formally the solution of the differential equation is

$$a_t = \exp\{-tH\}a$$

and the difficulty is to give an appropriate definition of the exponential. Various algorithms and approximation techniques are of use. For example the algorithm

$$\exp\{-tx\} = \lim_{n\to\infty} (1+tx/n)^{-n}$$

for the numerical exponential can be extended to an operator relation if the (pseudo-) resolvent $(I+\alpha H)^{-1}$ has suitable properties for small positive α.

It should perhaps be emphasized that in applications the Banach space $\mathcal{B}$ is not necessarily specified in advance. Typically one might encounter a differential equation of the above type for functions over some measure space but without specification of a particular norm. Thus the problem consists of choosing the norm and reinterpreting the operator H such that an appropriate solution can be found.

The first basic result which characterizes generators is the following:

THEOREM 1.3.1. (Hille-Yosida). *Let* H *be an operator on the Banach space* $\mathcal{B}$. *The following conditions are equivalent:*

1. H *is the infinitesimal generator of a* C_0*-semigroup of contractions* S ,

2. H *is norm closed, norm densely defined:*

$$R(I+\alpha H) = \mathcal{B}$$

for all $\alpha > 0$ *(or for one* $\alpha = \alpha_0 > 0$*) :*

$$\|(I+\alpha H)a\| \geq \|a\|$$

for all $a \in D(H)$ *and all* $\alpha > 0$ *(or for all* $\alpha \in \langle 0, \alpha_0]$*)* .

If these conditions are satisfied

$$\lim_{n\to\infty} \|S_t a - (I+tH/n)^{-n}a\| = 0$$

for all $a \in \mathcal{B}$, *uniformly for* t *in any finite interval of* $[0, \infty\rangle$.

Proof. $1 \Rightarrow 2$. This follows from Proposition 1.2.1, it suffices to set $\lambda = -1/\alpha$.

$2 \Rightarrow 1$. Assume $R(I+\alpha_0 H) = \mathcal{B}$ and $\|(I+\alpha H)a\| \geq \|a\|$ for all $a \in D(H)$ and $\alpha \in \langle 0, \alpha_0]$. Thus $(I+\alpha_0 H)^{-1}$ is a bounded operator with norm one. First we extend this conclusion to all $\alpha \in \langle \alpha_0/2, \alpha_0]$ and then by iteration to all $\alpha \in \langle 0, \alpha_0]$.

If $\alpha \in \langle \alpha_0/2, \alpha_0]$ then

$$R_N = \left(\frac{\alpha_0}{\alpha}\right) \sum_{n=0}^{N} \left(\frac{\alpha-\alpha_0}{\alpha}\right)^n (I+\alpha_0 H)^{-n-1}$$

converges in norm to a bounded operator R . But for $a \in D(H)$ one has $R_N a \in D(H)$ and a simple rearrangement argument proves that $\|(I+\alpha H)R_N a - a\| \to 0$ and $\|R_N(I+\alpha H)a - a\| \to 0$ as $N \to \infty$. Since H is norm closed it follows that $R = (I+\alpha H)^{-1}$ and then $\|R\| \leq 1$ by the bound $\|(I+\alpha H)a\| \geq \|a\|$.

The remainder of the proof consists of establishing that the strong limit of the operators

$$r_n(t) = (I+tH/n)^{-n}$$

exists as $n \to \infty$ and that it defines a C_0-semigroup of contractions with generator H . Note that for n sufficiently large $t/n < \alpha_0$ and $\|(I+\alpha H)^{-1}\| \leq 1$ for all α relevant to the remainder of the proof.

As a preliminary to studying the above limit we note that if $a \in D(H)$

$$\|(I+\alpha H)^{-1}a - a\| = \alpha\|(I+\alpha H)^{-1}Ha\|$$
$$\leq \alpha\|Ha\| \xrightarrow[\alpha\to 0+]{} 0 .$$

Since $D(H)$ is dense one concludes that $(I+\alpha H)^{-1}$ converges strongly to the identity as $\alpha \to 0+$. This has several implications.

First if for $a \in \mathcal{B}$ one defines

$$a_n = (I+H/n)^{-2}a$$

then $a_n \in D(H^2)$ and a_n is norm convergent to a . Thus $D(H^2)$ is dense. Second if $a \in D(H)$ then a_n converges to a and Ha_n also converges to Ha . Thus $D(H^2)$ is a core for H . Third $r_n(t)$ is strongly convergent to the identity as $t \to 0$.

Next one claculates that $dr_n(t)/dt$ is bounded for $t > 0$ and, more specifically,

$$\frac{dr_n(t)}{dt} = -H(I+tH/n)^{-n-1} .$$

Combining these facts one calculates that if $a \in D(H^2)$ then

$$r_n(t)a - r_m(t)a = \lim_{\varepsilon\to 0+} \int_\varepsilon^{t-\varepsilon} ds \frac{d}{ds}\left\{r_n(s)r_m(t-s)a\right\}$$
$$= \lim_{\varepsilon\to 0+} \int_\varepsilon^{t-\varepsilon} ds \left\{r_n'(s)r_m(t-s)a - r_n(s)r_m'(t-s)a\right\}$$
$$= \lim_{\varepsilon\to 0+} \int_\varepsilon^{t-\varepsilon} ds\, r_n(s)r_m(t-s) \left\{-H(I+sH/n)^{-1}a + H(I+(t-s)H/m)^{-1}a\right\}$$
$$= \int_0^t ds \left(\frac{s}{n} - \frac{t-s}{m}\right)(I+sH/n)^{-n-1}(I+(t-s)H/m)^{-m-1}H^2a .$$

This immediately yields the estimate

$$\|r_n(t)a - r_m(t)a\| \le \frac{t^2}{2}\left(\frac{1}{n}+\frac{1}{m}\right)\|H^2a\| .$$

Thus $\{r_n(t)a\}_{n\ge 1}$ is a Cauchy sequence which is norm convergent, uniformly for t in any finite interval of $[0, \infty\rangle$. But since $D(H^2)$ is norm dense, and $\|r_n(t)\| \le 1$ for all $n = 1, 2, \ldots$, it follows that $\{r_n(t)\}_{n\ge 1}$ is strongly convergent, uniformly for t in any finite interval of $[0, \infty\rangle$. If $S = \{S_t\}_{t\ge 0}$ denotes the strong limit one readily deduces that $S_0 = I$, $t \in \mathbb{R}_+ \mapsto S_t \in (B)$ is strongly continuous, and $\|S_t\| \le 1$. To establish the semigroup property we use the combinatoric identity

$$x^n - y^n = \sum_{m=1}^{n} x^{n-m}(x-y)y^{m-1} .$$

Hence for $a \in D(H^2)$ one calculates that

$$\begin{aligned} r_n(s)r_n(t)a - r_n(s+t)a &= \sum_{m=1}^{n} (I+sH/n)^{-n+m}(I+tH/n)^{-n+m}(I+(s+t)H/n)^{-m+1} \times \\ &\qquad \times \left\{(I+sH/n)^{-1}(I+tH/n)^{-1} - (I+(s+t)H/n)^{-1}\right\}a \\ &= \sum_{m=1}^{n} (I+sH/n)^{-n+m-1}(I+tH/n)^{-n+m-1}(I+(s+t)H/n)^{-m}\,\frac{st}{n^2}H^2a . \end{aligned}$$

Therefore

$$\|r_n(s)r_n(t)a - r_n(s+t)a\| \le \frac{st}{n}\|H^2a\| .$$

In the limit $n \to \infty$ one finds

$$S_s S_t a = S_{s+t} a$$

and the semigroup property follows from the density of $D(H^2)$ and the contractivity of S.

It remains to identify the generator of S.

Again one calculates for $a \in D(H^2)$

$$t^{-1}\bigl(r_n(t)-I\bigr)a + Ha = t^{-1}\sum_{m=0}^{n-1} (I+tH/n)^{-m}\bigl((I+tH/n)^{-1}-I\bigr)a + Ha$$

$$= -\frac{1}{n}\sum_{m=1}^{n} \bigl((I+tH/n)^{-m}-I\bigr)Ha$$

$$= \frac{t}{n^2}\sum_{m=1}^{n}\sum_{p=1}^{m} (I+tH/n)^{-p}H^2 a$$

Consequently

$$\|t^{-1}\bigl(r_n(t)-I\bigr)a + Ha\| \le \frac{t(n+1)}{2n}\|H^2 a\|$$

and in the limit $n \to \infty$

$$\|t^{-1}\bigl(S_t-I\bigr)a + Ha\| \le t\|H^2 a\| .$$

Thus if $\hat{H}$ denotes the generator of S then

$$\hat{H}a = Ha$$

for all $a \in D(H^2)$. But $D(H^2)$ is a core for H and hence $\hat{H}$ is an extension of H. This, however implies that $(I+\alpha\hat{H})^{-1}$

is an extension of $(I+\alpha H)^{-1}$ for all small $\alpha > 0$. Since the latter operator is everywhere defined it is not possible that $\hat{H}$ is a strict extension of H . Therefore $\hat{H} = H$. $\square$

There are a number of possible variations of the Hille-Yosida theorem. It follows from Condition 1 of the theorem that H is norm closed but for the implication $2 \Rightarrow 1$ it is not necessary to assume the closedness since it follows from the other hypotheses of Condition 2. For example if $a_n \in D(H)$, $\|a_n - a\| \to 0$, and $\|Ha_n - b\| \to 0$ then there is a c such that

$$(I+\alpha H)c = a + \alpha b ,$$

by the range condition, and consequently

$$\|c - a_n\| \leq \|(I+\alpha H)(c-a_n)\| \to 0 ,$$

by the lower bound. Hence $c = a$, $b = Ha$, and H is norm closed. This redundancy will reoccur, without comment, in several of the subsequent statements.

The Hille-Yosida theorem can also be rephrased as a criterion for an operator to be a *pre-generator*, i.e., a closable operator whose closure is a generator.

THEOREM 1.3.2. *Let* H *be a norm densely defined operator on the Banach space* $\mathcal{B}$ *and assume that*

$$\|(I+\alpha H)a\| \geq \|a\|$$

for all $a \in D(H)$ *and all* $\alpha \in \langle 0, \alpha_0]$ *, for some* $\alpha_0 > 0$.

It follows that H *is norm closable and the following conditions are equivalent:*

1. *The closure* $\overline{H}$ *of* H *is the generator of a* C_0*-semigroup of contractions,*

2. $$\overline{R(I+\alpha H)} = \mathcal{B}$$

   for one $\alpha \in \langle 0, \alpha_0]$, *where the bar denotes norm closure.*

Proof. If $a_n \in D(H)$, $\|a_n\| \to 0$, and $\|Ha_n - b\| \to 0$, then H is norm closable if, and only if, $b = 0$. Now suppose $a' \in D(H)$ and $b' = Ha'$ then

$$\|(I+\alpha H)(a_n+\alpha a')\| \geq \|a_n + \alpha a'\|$$

for $\alpha \in \langle 0, \alpha_0]$. Therefore taking the limit over n and subsequently dividing by α one finds

$$\|b + a' + \alpha b'\| \geq \|a'\| .$$

Hence

$$\|a' + b\| \geq \|a'\| .$$

But $D(H)$ is norm dense and so for each $\varepsilon > 0$ one can choose a' such that $\|b + a'\| < \varepsilon$ and $\|a'\| \geq \|b\|$. Therefore $\|b\| < \varepsilon$ and $b = 0$.

Next suppose $a_n \in D(H)$, $\|a_n - a\| \to 0$, and $\|Ha_n - \overline{H}a\| \to 0$ then

$$\|(I+\alpha\overline{H})a\| = \lim_{n\to\infty} \|(I+\alpha H)a_n\|$$

$$\geq \lim_{n\to\infty} \|a_n\| = \|a\| .$$

Moreover if $c \in \mathcal{B}$ and one chooses $c_n \in R(I+\alpha H)$ such that $\|c_n - c\| \to 0$ then $c_n = (I+\alpha H)a_n$ for some $a_n \in D(H)$ and

$$\|c_n - c_m\| = \|(I+\alpha H)(a_n-a_m)\|$$

$$\geq \|a_n - a_m\| .$$

Therefore a_n must be a convergent sequence. But

$$\|H(a_n-a_m)\| \leq \alpha^{-1}\left\{\|(I+\alpha H)(a_n-a_m)\| + \|a_n - a_m\|\right\}$$

$$= \alpha^{-1}\left\{\|c_n - c_m\| + \|a_n - a_m\|\right\}$$

and consequently Ha_n is also convergent. Hence if $\|a_n - a\| \to 0$ then $a \in D(\overline{H})$ and $\|Ha_n - \overline{H}a\| \to 0$ because H is norm closable. Thus

$$c = (I+\alpha\overline{H})a$$

and this establishes that

$$R(I+\alpha\overline{H}) = \overline{R(I+\alpha H)} .$$

Therefore Conditions 1 and 2 are equivalent by the Hille-Yosida theorem. $\square$

Remark 1.3.3. Results analogous to Theorem 1.3.1 and 1.3.2 are valid for general C_0-semigroups. For example if one replaces the lower bound in Condition 2 of Theorem 1.31 by the set of lower bounds

$$(*)\quad \|(I+\alpha H)^n a\| \geq M^{-1}(1-\alpha\omega)^n\|a\| \;, \qquad a \in D(H^n) \;, \quad n = 1, 2, 3, \ldots$$

for all $\alpha \in \langle 0, \omega]$ and repeats the proof of $2 \Rightarrow 1$ then the new bounds give the estimates

$$\|r_n(t)\| \leq M(1-\alpha\omega)^{-n}$$

and one readily concludes that H generates a C_0-semigroup S satisfying

$$\|S_t\| \leq Me^{\omega t} \;.$$

Conversely if S satisfies these bounds then the lower bounds $(*)$ follow from the Laplace transforms

$$(I+\alpha H)^{-n}a = \frac{1}{n!}\int_0^\infty dt\; t^n e^{-t} S_{\alpha t} a \;.$$

Remark 1.3.4. If S is a C_0-semigroup with generator H it is customary to write

$$S_t = e^{-tH} \;.$$

This is justified by the definition of the generator and also by the construction of Theorem 1.3.1. Moreover if H is bounded S_t coincides with $\exp\{-tH\}$ defined as a uniformly convergent power series.

The Hille-Yosida theorem can be reformulated in a much neater manner: H *is the generator of a* C_0*-semigroup of contractions if, and only if,* $(I+\alpha H)^{-1}$ *is a bounded contraction operator for all sufficiently small positive* α . Nevertheless it is useful to identify explicitly the two pieces of information which are contained in the statement that $(I+\alpha H)^{-1}$ is a bounded contraction operator, the range condition

$$R(I+\alpha H) = \mathcal{B} ,$$

and the lower bounds

$$\|(I+\alpha H)a\| \geq \|a\| , \qquad a \in D(H) .$$

These latter lower bounds can often be re-expressed in quite different terms. They are related to the maximum principle when applied to differential operators and to a spectral property for operators on Hilbert space. In the next section we discuss the interpretation of these bounds as a criterion of dissipation. But for the present we adopt the terminology that *the operator* H *is norm-dissipative if*

$$\|(I+\alpha H)a\| \geq \|a\|$$

for all $a \in D(H)$ *and all small* $\alpha > 0$.

The following example illustrates this concept for elliptic differential operators.

Example 1.3.5. (The Laplace Operator). Let $\mathcal{B} = C_0(\mathbb{R}^\nu)$, the space of continuous functions over $\mathbb{R}^\nu$ which vanish at infinity, equipped with the usual supremum norm. The Laplace operator $-\nabla^2$ is defined on $C_0^2(\mathbb{R}^\nu)$, the twice continuously differentiable functions in $C_0(\mathbb{R}^\nu)$, by

$$-\nabla^2 a = -\sum_{i=1}^{\nu} \frac{\partial^2 a}{\partial x_i^2} ,$$

and one has the obvious identity

$$2|\underline{\nabla}a|^2 - \nabla^2|a|^2 = (-\nabla^2\bar{a})a + \bar{a}(-\nabla^2 a) .$$

Therefore if $\alpha > 0$

$$\begin{aligned} |(1-\alpha\nabla^2)a|^2 &= |a|^2 + \alpha^2|\nabla^2 a|^2 + \alpha(-\nabla^2\bar{a})a + \alpha\bar{a}(-\nabla^2 a) \\ &= |a|^2 + \alpha^2|\nabla^2 a|^2 + 2\alpha|\underline{\nabla}a|^2 - \alpha\nabla^2|a|^2 \\ &\geq |a|^2 - \alpha\nabla^2|a|^2 . \end{aligned}$$

Now if $|a|$ has a maximum at $x = x_0$ then the maximum principle states that $x \mapsto -\nabla^2|a(x)|^2$ is non-negative at $x = x_0$. Therefore the preceding estimate establishes that

$$\begin{aligned} \|(I-\alpha\nabla^2)a\|_\infty^2 &\geq |(1-\alpha\nabla^2)a(x_0)|^2 \\ &\geq |a(x_0)|^2 = \|a\|_\infty^2 , \end{aligned}$$

i.e., the Laplace operator is $\|\cdot\|_\infty$-dissipative. A similar conclusion is true for more general elliptic operators by the same calculation. $\square$

Now let us examine operators on Hilbert space. In this case one has

$$\|(I+\alpha H)a\|^2 = \|a\|^2 + \alpha^2\|Ha\|^2 + \alpha(Ha, a) + \alpha(a, Ha)$$
$$= \|a\|^2 + \alpha^2\|Ha\|^2 + 2\alpha \operatorname{Re}(a, Ha) .$$

Therefore H is norm-dissipative if, and only if,

$$\operatorname{Re}(a, Ha) \geq 0$$

for all $a \in D(H)$. But under certain quite general circumstances these latter conditions are equivalent to a spectral property of H . For example if H is bounded and normal, i.e., if H commutes with its adjoint H^* , these conditions are equivalent to

$$\operatorname{Re} \sigma(H) \geq 0 .$$

This follows by a numerical range argument. Define the numerical range $W(H)$ of H by

$$W(H) = \{(a, Ha) ; \quad a \in D(H)\} .$$

If H is bounded then the Hausdorff-Stone theorem establishes that $W(H)$ is convex. If, moreover, H is normal then the closure $\overline{W(H)}$ of $W(H)$ coincides with the convex closure of $\sigma(H)$. Therefore in this latter case $\operatorname{Re} W(H) \geq 0$ if, and only if, $\operatorname{Re} \sigma(H) \geq 0$. This conclusion can be extended to unbounded generators of normal semigroups.

Example 1.3.6. (Normal Semigroups). Let $S = \{S_t\}_{t\geq 0}$ be a C_0-semigroup acting on a Hilbert space H . The adjoints

$S^* = \{S_t^*\}_{t\geq 0}$ form a weakly, hence strongly (Exercise 1.2.3), continuous semigroup called the *adjoint semigroup*. The semigroup S is defined to be *normal* if S_s and S_t^* commute for all $s, t > 0$ and *self-adjoint* if $S_t = S_t^*$ for all $t > 0$. Note that $\|S_t\| = \|S_t^*\|$ and hence S and S^* are simultaneously contractive. Moreover if H generates S then the adjoint H^* of H generates S^* (Exercise 1.3.4).

If S *is contractive then* $\mathrm{Re}\,\sigma(H) \geq 0$ *and if* S *is normal it is contractive if, and only if,* $\mathrm{Re}\,\sigma(H) \geq 0$.

The first statement was established in Proposition 1.2.1. Moreover if H is bounded it is normal if, and only if, S is normal and the second statement follows from the discussion preceding the example. The case of unbounded H can now be deduced by an approximation technique based on the functional analysis of generators.

If $\mathrm{Re}\,\sigma(H) \geq 0$ then $(I+\alpha H)^{-1}$ is a well-defined bounded operator for all $\alpha \geq 0$. Consequently the operators

$$H_\alpha = H(I+\alpha H)^{-1} = \alpha^{-1}\left(I-(I+\alpha H)^{-1}\right)$$

are bounded. But if S is normal it follows that H_α is normal and the uniformly continuous semigroups $S_t^\alpha = \exp\{-tH_\alpha\}$ are also normal. Moreover it follows from the identity

$$\lambda(I+\alpha\lambda)^{-1}I - H_\alpha = (I+\alpha\lambda)^{-1}(\lambda I-H)(I+\alpha H)^{-1}$$

that if $\lambda \in r(H)$ then $\lambda(I+\alpha\lambda)^{-1} \in r\left(H_\alpha\right)$, unless $\lambda = -\alpha^{-1}$.

Therefore $r(H_\alpha)$ contains the open left hand plane, $\mathrm{Re}\,\sigma(H_\alpha) \geq 0$, and S^α is contractive by the preceding argument for bounded generators. Finally the formula

$$S_t^\alpha a - S_t a = \int_0^1 d\lambda \frac{d}{d\lambda} S_{\lambda t}^\alpha \, S_{(1-\lambda)t} a$$

$$= t \int_0^1 d\lambda \, S_{\lambda t}^\alpha \, S_{(1-\lambda)t} \left(I - (I+\alpha H)^{-1}\right) Ha$$

with $a \in D(H)$, and the fact that $(I+\alpha H)^{-1} \mapsto I$ as $\alpha \to 0$ (see the proof of Theorem 1.3.1), establish that S_t^α converges strongly to S_t. Hence S is contractive.

Note that if $S_t = \exp\{-tH\}$ is contractive then $\mathrm{Re}\,\sigma(H) \geq 0$ but the converse is not necessarily true if S is not normal. For example if

$$H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then $\sigma(H) = 0$ but

$$S_t = e^{-tH} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$$

and $\|S_t\| > 1$ for all $t \neq 0$. $\square$

Throughout this section we have examined criteria for an operator H on a Banach space $\mathcal{B}$ to be the generator of a C_0-semigroup of contractions. More generally one can ask whether a given operator H has extensions which are generators, and then try to classify all such extensions. Unfortunately the theory of extensions is poorly developed except for specific examples, or for the special case of norm-dissipative operators on Hilbert space. This Hilbert space theory can be briefly described as follows.

Assume H is a norm closed, norm densely defined, norm-dissipative, operator on the Hilbert space $\mathcal{H}$ and for small $\alpha > 0$ define

$$\mathcal{H}_\alpha = R(I+\alpha H) .$$

It follows by a simple variation of the argument used in the proof of Theorem 1.3.2 that $\mathcal{H}_\alpha$ is a closed subspace of $\mathcal{H}$. Now if $\mathcal{H}_\alpha = \mathcal{H}$ then H is the generator of a C_0-contraction semigroup by the Hille-Yosida theorem. Therefore we consider the situation $\mathcal{H}_\alpha \neq \mathcal{H}$ and try to construct extensions of H which generate contraction semigroups.

It is useful to introduce the spaces $D_\alpha = D_\alpha(H) = \mathcal{H}_\alpha^\perp$ which measure the extent to which the range spaces $R(I+\alpha H)$ fail to equal $\mathcal{H}$. The D_α are called *deficiency spaces* and the first key observation is that *the dimension of* D_α *is independent of* α. This dimension is called the *deficiency index* of H. To prove the independence statement one first remarks that

$$\|(\lambda I+H)a\| \geq \lambda\|a\|$$

for all $a \in D(H)$ and all sufficiently large $\lambda > 0$ because H is norm-dissipative. Next define E_α as the orthogonal projection onto D_α and note that

$$\|(I-E_\alpha)b\| = \sup_{a \in D(H)} |(b, (I+\alpha H)a)| \big/ \|(I+\alpha H)a\| .$$

Therefore if $b \in D_{1/\lambda}$ then

$$\begin{aligned} \|(I-E_{1/\mu})b\| &= \sup_{a \in D(H)} |(b, (\mu I+H)a)| \big/ \|(\mu I+H)a\| \\ &\le \sup_{a \in D(H)} \{|(b, (\lambda I+H)a)| + |\mu-\lambda| \, |(b, a)|\} \big/ \|(\mu I+H)a\| \\ &\le (|\xi-\lambda| \big/ \mu) \|b\| \end{aligned}$$

where the last estimate uses the norm dissipativity of H. Consequently

$$\|(I-E_\alpha)E_\beta\| \le |\alpha-\beta| \big/ \beta .$$

But this implies that

$$\begin{aligned} \|E_\alpha - E_\beta\| &= \|(I-E_\alpha)E_\beta - E_\beta(I-E_\alpha)\| \\ &\le |\alpha-\beta| \big/ \beta + |\alpha-\beta| \big/ \alpha . \end{aligned}$$

Thus if $\alpha > 0$ is in a sufficiently small open interval around β one has $\|E_\alpha - E_\beta\| < 1$ which is equivalent to $E_\alpha = E_\beta$. Therefore D_α and D_β have the same dimension. But since $\beta > 0$ was arbitrary the general independence statement follows immediately.

The second crucial observation is that $D(H) \cap D_\alpha = \{0\}$

for each $\alpha > 0$. This is established by noting that if $a \in D(H) \cap D_\alpha$ then $(a, (I+\alpha H)a) = 0$ and

$$(a, Ha) = -\|a\|^2/\alpha \ .$$

But H is norm-dissipative, hence $\mathrm{Re}(a, Ha) \geq 0$, and $a = 0$.

Now one can construct generator extensions of H by iteration of the following procedure for the simplest case that the deficiency index is one.

Assume D_α is one-dimensional. Then define H_α by $D(H_\alpha) = D(H) \oplus D_\alpha$ and

$$H_\alpha(a+b) = Ha + b/\alpha$$

for $\alpha \in D(H)$ and $b \in D_\alpha$. If $a + b = 0$ one has $a = 0 = b$ because $D(H) \cap D_\alpha = \{0\}$. Therefore $Ha = 0 = b/\alpha$ and $H_\alpha(a+b) = 0$, i.e., the operator H_α is linear. But

$$\begin{aligned}\mathrm{Re}\big((a+b), H_\alpha(a+b)\big) &= \mathrm{Re}((a+b), (Ha+b/\alpha)) \\ &= \mathrm{Re}(a, Ha) + \|b\|^2/\alpha + \mathrm{Re}(b, (I+\alpha H)a)/\alpha \\ &\geq \mathrm{Re}(a, Ha) \geq 0\end{aligned}$$

where we have used $b \in D_\alpha$. Thus H_α is norm-dissipative. Finally if $c \in R\big(I+\alpha H_\alpha\big)^\perp$ then

$$\big(c, \big(I+\alpha H_\alpha\big)(a+b)\big) = (c, (I+\alpha H)a) + 2(c, b) = 0$$

for $a \in D(H)$ and $b \in D_\alpha$. But $R(I+\alpha H) = \mathcal{H}_\alpha$ and $b \in \mathcal{H}_\alpha^\perp$. Therefore $c = 0$. Thus to summarize $\alpha > 0 \longmapsto H_\alpha$ is a one-parameter family of norm densely defined, norm-dissipative, operators with $R(I+\alpha H_\alpha) = \mathcal{H}$. Hence the H_α are norm closed and each H_α generates a C_0-semigroup of contractions by the Hille-Yosida theorem.

The above construction generalizes quite easily. If D_α has dimension $n > 1$ one first chooses a one-dimensional subspace $D_\alpha^{(1)} \subset D_\alpha$ and defines H_α by $D(H_\alpha) = D(H) \oplus D_\alpha^{(1)}$ and

$$H_\alpha(a+b) = a + b/\alpha$$

for all $a \in D(H)$ and $b \in D_\alpha^{(1)}$. It then follows as above that H_α is norm-dissipative and the corresponding deficiency space is given by $D_\alpha(H_\alpha) = D_\alpha(H) \setminus D_\alpha^{(1)}$. Thus the deficiency index is reduced by one. Iteration of this procedure then produces a family of extensions of H which generate contraction semigroups. If n is finite, or countably infinite, this iterative procedure is straightforward. In the general case it is necessary to appeal to complete induction.

Although the foregoing method allows the construction of some generator extensions, in the Hilbert space context, it does not give all possible extensions. A complete classification of such extensions is only known for the even more special cases of symmetric operators,

$$\mathrm{Im}(a, Ha) = 0 , \qquad a \in D(H) ,$$

or anti-symmetric operators,

$$\mathrm{Re}(a, Ha) = 0 , \qquad a \in D(H) .$$

These particular cases will be discussed in greater detail in Chapter 2.

Example 1.3.7. Define $H = d^2/dx^2$ on the twice continuously differentiable functions with compact support in $\langle 0, \infty\rangle$. Then H is a symmetric norm-dissipative operator on $L^2(0, \infty)$ because

$$(f, Hf) = \int_0^\infty dx \, |\tfrac{df}{dx}(x)|^2$$

for all $f \in D(H)$. But the deficiency index of H is one because $R(I+\alpha^2 H)^{\perp}$ consists of multiples of the function f_α where

$$f_\alpha(x) = \exp\{-x/\alpha\} .$$

Hence the above construction gives a one-parameter family of norm-dissipative extensions H_α of H satisfying the range condition $\overline{R(I+\alpha^2 H_\alpha)} = L^2(0, \infty)$. But

$$\left(\partial f_\alpha(x)-\alpha^{-1}f_\alpha(x)\right)\Big|_{x=0} = 0 ,$$

where ∂ denotes the right derivative. Therefore the family of extensions of $-d^2/dx^2$ to the twice differentiable functions $f \in L^2(0, \infty)$ with

$$\left(\partial f(x)-\alpha^{-1}f(x)\right)\Big|_{x=0} = 0$$

must also satisfy the range condition. But these extensions, which we also denote by H_α , are also symmetric and norm dissipative because

$$\left(f,\ H_\alpha f\right) = \alpha^{-1}|f(0)|^2 + \int_0^\infty dx\ \left|\frac{df}{dx}(x)\right|^2$$

for all $f \in D\left(H_\alpha\right)$. Hence the H_α are pre-generators of contraction semigroups S^α . This construction omits, however, two extensions which formally correspond to the values $\alpha = 0$ and $\alpha = \infty$; the first is related to *Dirichlet boundary conditions* $f(0) = 0$ and the second to *Neumann boundary conditions* $\partial f(0) = 0$.

Exercises.

1.3.1. If H generates the C_0-semigroup S prove that

$$\lim_{\alpha\to 0+} \|S_t a - \exp\{-tH(I+\alpha H)^{-1}\}a\| = 0$$

and

$$\lim_{s\to 0+} \|S_t a - \exp\{-t\left(I-S_s\right)/s\}a\| = 0 .$$

Hint: See Example 1.3.6.

1.3.2. Complete the proof of Remark 1.3.3 that a norm closed, norm densely defined, operator generates a C_0-semigroup S satisfying $\|S_t\| \le M \exp\{\omega t\}$ if, and only if,

$$R(I+\alpha H) = \mathcal{B}$$

and

$$\|(I+\alpha H)^n a\| \geq M^{-1}(1-\alpha\omega)^n \|a\| \ , \qquad a \in D(H^n) \ ,$$

for all small $\alpha > 0$.

1.3.3. Let Λ be a bounded open subset of $\mathbb{R}^\nu$. Define the Laplace operator $H = -\nabla^2$ on the twice continuously differentiable functions with compact support in Λ . Prove that H is norm-dissipative on $L^p(\Lambda)$ for all $p \in [1, \infty\rangle$.

1.3.4. Let $S = \exp\{-tH\}$ be a C_0-semigroup on a reflexive Banach space, i.e., $\mathcal{B} = (\mathcal{B}^*)^*$. Prove that the adjoints $S^* = \{S_t^*\}_{t\geq 0}$ define a C_0-semigroup, the adjoint semigroup, with generator H^* , the adjoint of H .

Hint: Use Exercises 1.2.4 and 1.2.5 together with the definition

$$D(H^*) = \{f \ ; \quad f \in \mathcal{B}^* \ , \quad |(f, Ha)| \leq c_f\|a\| \ , \quad a \in D(H)\}$$

$$(H^*f, a) = (f, Ha) \quad \text{for} \quad f \in D(H^*) \ , \qquad a \in D(H) \ .$$

1.3.5. Consider the Laplacian $H = -\nabla^2$ defined on the infinitely often differentiable functions in $L^2(\mathbb{R}^\nu)$ which vanish in a neighbourhood of the origin. Prove that the deficiency index $d(H)$ of H satisfies

$$d(H) = 2 \qquad \text{if} \quad \nu = 1$$

$$d(H) = 1 \qquad \text{if} \quad \nu = 2 \ , \ \ 3$$

$$d(H) = 0 \qquad \text{if} \quad \nu \geq 4 \ .$$