NUMERICAL ANALYSIS OF THE BOUNDARY INTEGRAL METHOD

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One of the important recent developments in numerical engineering has been the boundary integral method (bim)([5], [6], [10], [18]). This technique is useful because the most common practical problems can sensibly be expressed as linear equations in a homogeneous medium, and the full generality of the finite element method is not needed. In these "simple" situations the bim is a valuable means of using the special structure of a problem to save computational effort. Additionally the scope of the bim is currently being extended to include some non-linear, time dependent and non-homogeneous problems ([6], [18]).

A parallel effort has also been made to give a theoretical basis for the bim. Two of the important early papers were Nedelec and Planchard [14] and Hsiao and Wendland [9]. The difficulty is the appearance of first kind integral equations. The classical boundary integral equations (bie's) described in [12] or [13] reformulate potential problems as second kind integral equations. The numerical solution of these equations is understood ([1], [3], [4], [11], [16]), and is well described by the most elementary functional analysis. To deal with first kind equations more theory is required, and fractional order Sobolev spaces are introduced to describe the boundary values of the solution to the underlying differential equation.

Because of their importance the fractional order spaces have been developed in great generality ([15]). However for the analysis of numerical solutions to bie's most of this is unnecessary, and references to the literature can give a false impression of the difficulty of the prerequisites. The aim here is to give a very gentle self-contained introduction to the ideas required to deal with first kind equations in the simplest situations. More precisely, we use nothing that is not commonly used in the theory of second kind equations. A better review of the range and generality of the problems that can be covered is contained in Wendland [20].

1. BOUNDARY INTEGRAL EQUATIONS

The simplest problem to which the bim can be applied is the interior Dirichlet problem. Here $\Omega^+ \subset \mathbb{R}^2$ is a bounded region with a smooth boundary Γ, and $g: \Gamma \to \mathbb{R}$ is the given boundary data. We seek the unknown function $U: \Omega^+ \to \mathbb{R}$ satisfying

$$\Delta U(x) = 0 \ , \quad x \in \Omega^+ \ , \tag{1}$$

and

$$U|_\Gamma = g \ ; \tag{2}$$

where $\Delta U = \nabla . \nabla U$ (∇ is the gradient operator). For any $x \in \Gamma$, $\nu(x)$ denotes the outward pointing unit normal, and $\Omega^- = \mathbb{R} \backslash \overline{\Omega}^+$ is the unbounded exterior. The bim is only possible because the *fundamental solution* (or Green's function) is known to be

$$G(x,\xi) = -\frac{1}{2\pi} \ln|x-\xi| , \quad x,\xi \in \mathbb{R}^2 \quad .$$

For $x \neq \xi$ let

$$G_\nu(x,\xi) = \frac{\partial}{\partial \nu(\xi)} G(x,\xi)$$

Then for any $u: \Gamma \to \mathbb{R}$ define the *double layer potentials* V_D^+, V_D and V_D^- by

$$(3) \qquad (V_D^{\pm}u)(x) := \int_\Gamma G_\nu(x,\cdot)u \;, \quad x \in \Omega^{\pm} \;,$$

$$(4) \qquad (V_D u)(x) := \int_\Gamma G_\nu(x,\cdot)u \;, \quad x \in \Gamma \;.$$

Define the *single layer potentials* V_S^+, V_S, V_S^-, by

$$(5) \qquad (V_S^{\pm}u)(x) := \int_\Gamma G(x,\cdot)u \;, \quad x \in \Omega^{\pm} \;,$$

$$(6) \qquad (V_S u)(x) := \int_\Gamma G(x,\cdot)u \;, \quad x \in \Gamma \;.$$

Direct differentiation shows that V_S^+u and V_D^+u are harmonic. Thus if u or v can be chosen so that V_D^+u or V_S^+u satisfy the boundary conditions (2), the original differential equation has been solved. The classical or *indirect* bim chooses to represent the solution as $U = V_D^+u$; for then the boundary conditions are satisfied provided u satisfies a second kind equation. However once the difficulties of the first kind formulation are overcome, the alternative of the direct bim is generally preferred in practice.

All bim's depend on the behaviour of the potentials as the boundary is crossed. This is best motivated by remembering the physical interpretations. $V_S^{\pm}v$ is the potential created when positive charges are distributed around Γ with a density of $v(x)$ per unit length at $x \in \Gamma$. As $x^{\pm} \to x$ $(x^+ \in \Omega^+ , x^- \in \Omega^-)$ physics suggests both $V_S^+v(x^+)$ and $V_S^-v(x^-)$ increase to a common limit $(V_S v)(x)$ (see Fig. 1). That is

$$(7) \qquad V_S^+v|_\Gamma = V_S v = V_S^-v|_\Gamma \;.$$

However Fig. 1 also suggests that the normal derivatives $(V_S^+v)_\nu$ and $(V_S^-v)_\nu$ have oppositive signs. Thus denoting $(V_S^{\pm}v)_\nu(x) = \lim_{x^{\pm}\to x} (V_S^{\pm}v)_\nu(x)$, we have $(V_S^+v)_\nu|_\Gamma \neq (V_S^-v)_\nu|_\Gamma$. More precisely, if V_D^* is the adjoint of V_D defined by

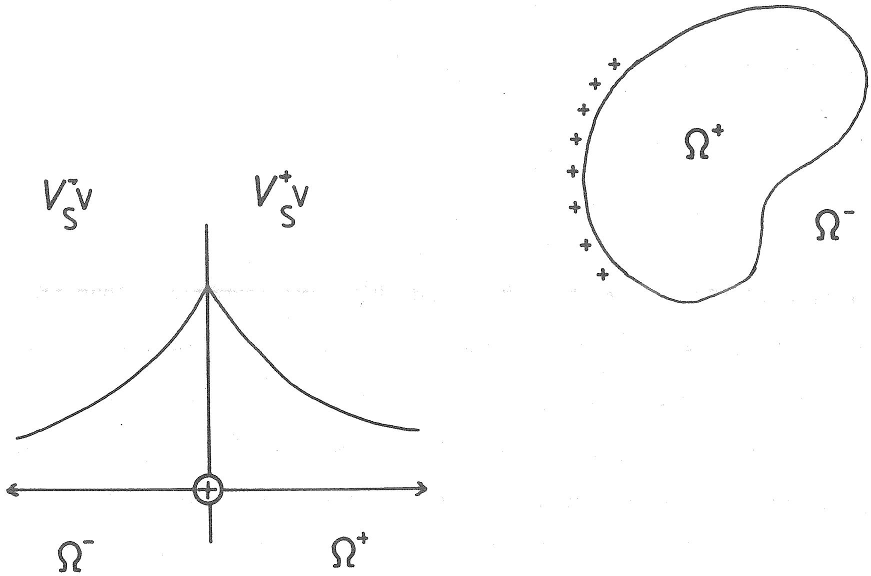

FIG. 1

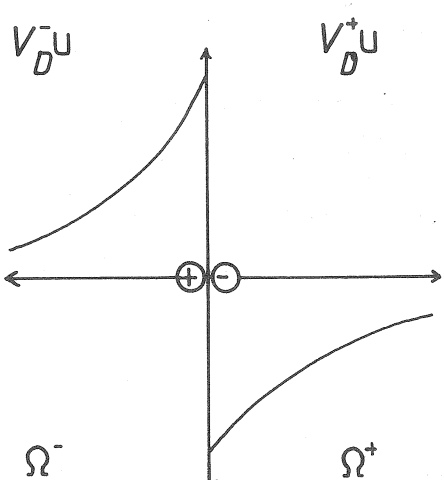

FIG. 2

$$(V_D^* v)(x) := \int_\Gamma G_\nu(\xi,x)v(\xi)d\xi ,$$

then

(8) $$(V_S^{\pm} v)_\nu|_\Gamma = \pm\tfrac{1}{2}v + V_D^* v .$$

$V_D^+ u$ is the potential of dipoles distributed around Γ with density u (positive charges pointing out). Intuitively as $x^+ \to x$, $(V_D^+ u)(x^+)$ becomes more negative and as $x^- \to x$, $(V_D u)(x^-)$ becomes more positive (because of the dipoles' allignment. See Fig. 2.) In fact

(9) $$(V_D^{\pm} u)|_\Gamma = \mp\tfrac{1}{2}u + V_D u .$$

Note the difference between $V_D^+ v|_\Gamma$ which is defined by a careful limiting process, and $V_D u$ which is the direct application of (4).

The "jumps" (7)-(9) are not too difficult to prove when u and v are smooth ([12], p.160 ff.) If only the minimum regularity of u and v is assumed (to ensure that $V_D^+ u$ and $V_D^- v$ are weak solutions to (1)) matters become more complex. This is avoided, here.

The direct bim combines (7)-(9) by examining the potential V^+ defined by

(10) $$V^+ := V_S^+ v - V_D^+ u .$$

But this is familiar from Green's third identity, which states for a harmonic function $U: \Omega^+ \to \mathbb{R}$

(11) $$\begin{aligned} \int_\Gamma G(x,\cdot)U_\nu - G_\nu(x,\cdot)U &= U(x) \quad x\in\Omega^+ \\ &= \tfrac{1}{2}U(x) \quad x\in\Gamma \\ &= 0 \quad x\in\Omega^- . \end{aligned}$$

The "converse" is also true.

PROPOSITION 1. *If smooth functions* u,v *on* Γ *, with* $\int_\Gamma v = 0$ *, satisfy the bie*

$$V_S v - V_D u = \tfrac{1}{2}u \,, \tag{12}$$

then v^+ *defined by* (10) *satisfies*

$$v^+|_\Gamma = u \,, \tag{13}$$

and

$$v^+_\nu|_\Gamma = v \;. \tag{14}$$

Proof. Equation (13) is a consequence of (7) and (9). To prove (14) let $\omega = v^+_\nu - v$. Applying (11) to v^+ and then using (13) shows $V_S\omega = 0$. Hence Green's first identity and (8) give

$$\begin{aligned} 0 = (V_S\omega,\omega) &= (V_S\omega,(V^+_S\omega)_\nu - (V^-_S\omega)_\nu) \\ &= \int_{\Omega^+}|\nabla V^+_S\omega|^2 + \int_{\Omega^-}|\nabla V^-_S\omega|^2 \;; \end{aligned} \tag{15}$$

where (u,v) denotes $\int_\Gamma uv$. Therefore $\nabla V^\pm_S\omega = 0$, and so

$\omega = (V^+_S\omega)_\nu|_\Gamma - (V^-_S\omega)_\nu|_\Gamma = 0$. ////.

To arrange u and v so that v^+ defined by (10) solves the interior Dirichlet problem, first choose $u=g$. Then (13) shows the boundary conditions are satisfied if v satisfies the bie

$$V_S v = \tfrac{1}{2}g + V_D g \;. \tag{16}$$

Once v has been found approximately, U can be approximated at any interior point by (10). But (14) shows that v is the unknown normal derivative of U , and in practice this is often the quantity of interest. Thus the further (expensive) computations can be avoided. With indirect methods U_ν must be calculated after the bie is solved. (For an analogous description of the direct and indirect methods in elastostatics see Hartmann, Ch. 4 in [5].)

2. NUMERICAL SOLUTIONS

The analysis of the numerical solution of the bie(16) requires some manipulation of V_S and the introduction of some new function spaces. Then (16) becomes a second kind integral equation, to which the standard theory applies.

Suppose Γ is parameterized as the smooth curve $\gamma: [-\pi,\pi] \to \mathbb{R}^2$, and for convenience suppose $|\dot{\gamma}|$ is constant (i.e. the arc-length around Γ to $\gamma(s)$ is proportional to s). Any function $v: \Gamma \to \mathbb{R}$ may be identified with the periodic function $v \circ \gamma: [-\pi,\pi] \to \mathbb{R}$. We write $v(s)$ for $v(\gamma(s))$ when appropriate. Then for $v \in C_0^\infty$, the set of infinitely differentiable functions with $(v,1) = 0$,

$$v(s) = \frac{1}{\sqrt{2\pi}} \sum_{m \neq 0} \hat{v}(m) \exp ims$$

with

$$\hat{v}(m) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} v(s) \exp -ims \, ds \quad .$$

Define the norm $|\cdot|_t$ for any $t \in \mathbb{R}$ by

$$(17) \qquad |v|_t^2 = \sum |m^t \hat{v}(m)|^2$$

(the range of summation, $m \neq 0$, is omitted for convenience). If $t = 0$, $|v|_0$ is the L^2 norm of v. For a positive integer k, $|v|_k = |D^k v|_0$, the L^2 norm of the k^{th} derivative. For negative integers, define the anti-derivative D^{-1} by

$$D^{-1} D v = v \,, \qquad (D^{-1} v, 1) = 0 \quad .$$

Then if k is positive $|v|_{-k} = |D^{-k}v|_0$.

Define the fractional order Sobolev spaces $H_0^t(\Gamma)$ (or just H_0^t) to be the completion of C_0^∞ under $|\cdot|_t$; or equivalently for $t \geq 0$,

$$H_0^t = \{u \in L^2 : |u|_t < \infty \ \& \ (u,1) = 0\}$$

When $t < 0$, H_0^t includes functions which do not belong to L^2 . Thus the delta "function" δ_σ defined by $(u,\delta_\sigma) = u(\sigma)$, is the limit of the smooth functions

$$\phi_n(s) := \frac{1}{\sqrt{2\pi}} \sum_{0<|m|<n} \exp{-im\sigma} \exp ims$$

when $t < -\frac{1}{2}$, and hence $\delta_\sigma \in H_0^t$ if $t < -\frac{1}{2}$.

These spaces have a number of properties which hold in more general circumstances. Here however the proofs involve the simplest properties of Fourier series and some standard limit arguments. Thus they are omitted.

THEOREM 2. (i) *H_0^t is a Hilbert space with inner product*

$$(u,v)_t = \sum m^{2t}\hat{u}(m)\overline{\hat{v}(m)}$$

(ii) *If* $s < t$ *then* $H_0^s \supset H_0^t$ *and the inclusion is a compact map.*

(iii) *The dual of* H_0^t *is* H_0^{-t} . *That is the functional* $u \mapsto (u,v)$ *is bounded on* H_0^t *if and only if* $v \in H_0^{-t}$, *and*

(18) $$|v|_{-t} = \sup\{(u,v)/|u|_t : u \in H_0^t\} .$$

(iv) *If* $t = \theta t_1 + (1-\theta)t_2$, $t_1 < t_2$, $0 \leq \theta \leq 1$ *we have the interpolation inequality*

(19) $$|u|_t \leq |u|_{t_1}^{\theta} |u|_{t_2}^{1-\theta} .$$ ////

The relevance of these spaces can be seen when Γ is the unit circle C. Writing $(\cos s, \sin s) = \operatorname{cis} s$, and using V_{SC}, V_{DC} etc for the potentials in this special case

$$(20) \qquad (V_{SC}v)(s) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|2\sin\frac{s-\sigma}{2}\right| v(\sigma)\,d\sigma = \frac{1}{2\sqrt{2\pi}}\sum \frac{\hat{v}(m)}{|m|}\exp ims$$

and

$$(V_{SC}^{\pm}v)(r \operatorname{cis} s) = \frac{1}{2\sqrt{2\pi}}\sum \frac{\hat{v}(m)}{|m|}\, r^{\pm|m|} \exp ims \quad .$$

(These formulae may be obtained by contour integration, or by using the uniqueness of the exterior and interior Dirichlet problems and checking (7) holds. See [13]). Clearly

$$(21) \qquad (V_{SC}v, v) = (v,v)_{-\frac{1}{2}} \quad \text{and} \quad |v|_{-\frac{1}{2}} = |V_{SC}v|_{\frac{1}{2}} \quad .$$

Thus V_{SC} is an isometry between $H_0^{-\frac{1}{2}}$ and $H_0^{+\frac{1}{2}}$.

For general regions the simple formula (20) is no longer true. However we have the decomposition

$$(V_S v)(\gamma(s)) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|\gamma(s)-\gamma(\sigma)|\,v(\sigma)\,d\sigma$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\ln|\operatorname{cis} s - \operatorname{cis}\sigma| + \ln\frac{|\gamma(s)-\gamma(\sigma)|}{|\operatorname{cis} s - \operatorname{cis}\sigma|}\right]v(\sigma)\,d\sigma \; .$$

That is

$$(22) \qquad (V_S v)(\gamma(s)) = (V_{SC}v)(s) + (K_S v)(s) \quad ,$$

where

$$(K_S v)(s) = \int_{-\pi}^{\pi} k(s,\cdot)v \;, \quad k(s,\sigma) := \ln\frac{|\gamma(s)-\gamma(\sigma)|}{|\operatorname{cis} s - \operatorname{cis}\sigma|} \quad .$$

Note that the kernel function k is smooth. Therefore for any $v \in H_0^t$, $K_S v$ has derivatives of all orders with

(23) $$\frac{\partial^\ell}{\partial s^\ell}(K_S v)(s) = \left(\frac{\partial^\ell}{\partial s^\ell} k(s,\cdot), v\right) .$$

(Since $v \in H_0^t$ the inner product on the rhs of (23) is defined. A simple argument shows that it is the derivative of $K_S v$.) Hence $K_S: H_0^{-\frac{1}{2}} \to H_0^{\ell+1}$ is bounded for any ℓ , and by properly (ii) $K_S: H_0^{-\frac{1}{2}} \to H_0^{\ell}$ is compact. Therefore $K := (V_{SC})^{-1} K_S$ is a compact operator on $H_0^{-\frac{1}{2}}$, and the decomposition (22) becomes

(24) $$V_S = V_{SC}(I+K)$$

(I is the identity operator). Therefore the bie may be rewritten as the operator equation on $H_0^{-\frac{1}{2}}$:

(25) $$(I+K)v = f , \quad f = V_{SC}^{-1}(\tfrac{1}{2}g+V_D g) .$$

(It is necessary to check that $f \in H_0^{-\frac{1}{2}}$. Since we assume g is smooth $\frac{1}{2}g + V_D g$ is also smooth, and it suffices to show that $(\frac{1}{2}g+V_D g,1) = 0$. But by the definition of V_D^* and using (8)

$$(\tfrac{1}{2}g+V_D g,1) = (g,\tfrac{1}{2}+V_D^* 1) = (g,(V_S^- 1)_\nu|_\Gamma) .$$

A direct calculation shows $\nabla V_S^- 1 = 0$, and hence $(V_S^- 1)_\nu|_\Gamma = 0$ as required.)

The introduction of the spaces $H_0^{-\frac{1}{2}}$ has reduced the first kind bie to a second kind equation (24), and the standard Fredholm theory may be applied. If $v \in H_0^{-\frac{1}{2}}$ solves $(I+K)v = 0$, then $v = -Kv$ and the smoothness properties of K_S show v is smooth. (i.e. $v \in H_0^\ell$ for all integers ℓ) Then the arguments used in the proof of Proposition 1 show $v = 0$. Hence the Fredholm alternative ([13]) shows (24) (or equivalently (16)) has a unique solution.

Most numerical solutions to the bie's are use piecewise polynomial basis functions: - the boundary element method. Thus let

$-\pi = s_0 < s_1 < \ldots < s_n = \pi$ be a subdivision of $[-\pi,\pi]$ with $\max\{|s_{i+1}-s_i|\} = h$. Define S^h to be the set of piecewise polynomials of order r (degree $r-1$), which have ν continuous derivatives at the knot points $\{s_i\}$ (if $\nu = -1$ the functions are allowed to be discontinuous.) Let

$$S_0^h = \{\phi \in S^h:\ (\phi,1) = 0\} \quad .$$

We seek to approximate the solution $v\circ\gamma$ to the bie by an element of S_0^h. The element may be selected in a variety of ways, but the easiest to analyse theoretically is the Galerkin method. Here $v_h \in S_0^h$ is defined by the *Galerkin equations*

$$\forall\phi \in S_0^h \quad (V_S v_h,\phi) = (f,\phi) \quad . \tag{26}$$

If a basis $\{\phi_i\}$ is chosen for S_0^h, then $v_h = \sum\alpha_i\phi_i$ and the Galerkin equations reduce to a system of linear equations for the vector $[\alpha_i]$. These may in turn be solved by standard or non-standard methods. However the procedure breaks down if the linear equations become singular, and this possibility must be eliminated in a theoretical justification.

Applying the decomposition (24) and using (21) shows that (26) is equivalent to

$$(V_S v-\tfrac{1}{2}g-V_D g,\phi) = (V_S(v_h-v),\phi) = ((I+K)(v_h-v),\phi)_{-\frac{1}{2}} = 0 \quad .$$

Thus the Galerkin method for the first kind bie(16) is just the Galerkin method for the second kind equation (25) in the space $H_0^{-\frac{1}{2}}$. But then the standard theory ([3], [11]) shows that the Galerkin equations are non-singular for h sufficiently small and

$$|v_h-v|_{-\frac{1}{2}} \le C\ \inf\{|\phi-v|_{-\frac{1}{2}}:\ \phi \in S_0^h\} \quad , \tag{27}$$

where here and elsewhere the constant C is independent of v and h .

This bound is the basis for the qualitative convergence theorem.

THEOREM 3. (i) *The Galerkin approximate solution to* (16) *satisfies*

$$|v_h - v|_{-\frac{1}{2}} \le Ch^{r+\frac{1}{2}}|v|_r \quad . \tag{28}$$

(ii) *If in addition the mesh satisfies the quasi-uniformity condition*

$$\max\{|s_{i+1}-s_i|\}/\min\{|s_{j+1}-s_j|\} \le \sigma \tag{29}$$

for some constant σ *independent of* h *; then*

$$|v_h - v|_0 \le Ch^r|v|_r \quad . \tag{30}$$

Proof. The proof is the derivation of the required approximation theory. For any $u \in H_0^0$ let $Pu \in S_0^h$ be the orthogonal projection defined by

$$\forall\phi \in S_0^h , \quad (Pu-u,\phi) = 0 \quad . \tag{31}$$

Then it is well known ([7] for example) that

$$|Pu-u|_0 \le Ch^r|u|_r \quad . \tag{32}$$

(If ν is small, the proof of this result is fairly straight forward, depending only on a scaling argument.) For any $\Psi \in H_0^r$, (31) shows

$$(Pu-u,\Psi) = (Pu-u,\Psi-P\Psi) \le C|Pu-u|_0|\Psi-P\Psi|_0 ,$$

and hence (18) and (32) give

$$|Pu-u|_{-r} = \sup\{(Pu-u,\Psi)/|\Psi|_r\} \le Ch^{2r}|u|_r .$$

Finally (19) with $\theta = 1/2r$ gives

(33) $$|Pu-u|_{-\frac{1}{2}} = |Pu-u|_{-r}^{\theta}|Pu-u|_0^{1-\theta} \le Ch^{r+\frac{1}{2}}|u|_r ,$$

which combined with (27) gives (28).

Because of the quasi-uniformity condition, any continuous piecewise polynomial satisfies

(34) $$|\Psi|_1 \le Ch^{-1}|\Psi|_0 .$$

For any $\phi \in S_0^h$, $D^{-1}\phi$ is a continuous piecewise polynomial, and hence (34) shows

$$|\phi|_0 = |D^{-1}\phi|_1 \le Ch^{-1}|D^{-1}\phi|_0 = Ch^{-1}|\phi|_{-1} .$$

Thus the interpolation inequality gives

$$|\phi|_{-\frac{1}{2}} \le |\phi|_{-1}^{\frac{1}{2}} |\phi|_0^{\frac{1}{2}} \le Ch^{-\frac{1}{2}}|\phi|_{-1} .$$

Combined with (30) and (33) we have

$$\begin{aligned} |v_h-v|_0 &\le |v_h-Pv|_0 + |Pv-v|_0 \\ &\le Ch^{-\frac{1}{2}}|v_h-Pv|_{-\frac{1}{2}} + |Pv-v|_0 \\ &\le Ch^{-\frac{1}{2}}(|v_h-v|_{-\frac{1}{2}} + |v-Pv|_{-\frac{1}{2}}) + |Pv-v|_0 \\ &\le Ch^r|v|_r \end{aligned}$$ .////

The L^2 estimate (30) is a more deep and meaningful estimate of the error in v_h . But (28) is also useful. Assuming Green's identity (16) for piecewise polynomials,

$$\int_{\Omega^+} |\nabla V_S(v_h - v_0)|^2 + \int_{\Omega^-} |\nabla V_S(v_h - v_0)|^2 = (V_S(v_h - v), v_h - v_0)$$

$$= ((I+K)(v_h - v_0), v_h - v_0)_{-\frac{1}{2}}$$

$$\leq C|v_h - v_0|^2_{-\frac{1}{2}} \quad .$$

Thus (28) measures (in the "energy" norm) the difference between the solution U of the interior Dirichlet problem and its approximation by the potential $v^+ = V_S^+ v_h - V_D g$.

The Galerkin method is of limited usefulness because of the integrations needed to set up the Galerkin equations. Collocation methods are preferred in practice, and some of these can be treated in the above framework. For instance let S_0^h be the space of Hermite cubics. Then as V_{SC} is an isometry $H_0^{3/2} \to H_0^{5/2}$, the Galerkin equations

$$\forall \phi \in S_0^h: \quad (V_S(v_h - v), \phi)_2 = ((I+K)(v_h - v), \phi)_{3/2} = 0$$

are equivalent to the collocation equations

(35) $$\forall i: V_S(v_h - v)(s_i) = DV_S(v_h - v)(s_i) = 0$$

([2] or [20]). Thus the convergence of the collocation method (35) follows from the Galerkin theory. Indeed this particualr collocation method is used in the elastostatic software [19]. However not all collocation methods can be analysed in this way. The results and techniques of [8] cover some of the remaining cases; but the Fourier analysis there is restricted to uniform meshes. The techniques described here are more flexible, and can be more easily extended to more difficult problems. (When the boundary contains corners for example. See [17].)

4. A MIXED PROBLEM

The second advantage of the direct bim is that it can be readily applied to mixed boundary conditions. Here the boundary Γ is divided into two segments Γ_1 and Γ_2 and we need to solve

(36) $$\Delta U(x) = 0 \ , \quad x \in \Omega^+ \ ,$$

(37) $$U|_{\Gamma_1} = g \ , \quad U_\nu|_{\Gamma_2} = h \ ;$$

where we assume the boundary data g and h are given as smooth functions defined over the entire boundary with $(h,1) = 0$. The direct bim again seeks u and v such that $V^+ = V_S^+ v - V_D^+ u$ satisfies the boundary conditions (37). That is we must satisfy

$$V_S v - V_D u - \tfrac{1}{2}u = 0$$

so that $V^+|_\Gamma = u$ and $V_\nu^+|_\Gamma = v$, together with

$$(u - g)|_{\Gamma_1} = 0 \ , \quad (v - h)|_{\Gamma_2} = 0 \ .$$

Equivalently writing $u = u_0 + g$, $v = v_0 + h$ we need

(38) $$V_S v_0 - V_D u_0 - \tfrac{1}{2}u_0 = f \ , \quad f = -(V_S h - V_D g - \tfrac{1}{2}g)$$

and

(39) $$u_0|_{\Gamma_1} = 0 \ , \quad v_0|_{\Gamma_2} = 0 \ .$$

For $i = 1,2$ define $C^\infty(\Gamma_i)$ to be the smooth functions on the boundary Γ which are zero outside Γ_i. $C_0^\infty(\Gamma_i) = C^\infty(\Gamma_i) \cap C_0^\infty(\Gamma)$. Then (38) is satisfied iff

$$\forall\phi \in C^\infty(\Gamma_2) \ , \quad \forall\psi \in C_0^\infty(\Gamma_1)$$

(40) $$(\tfrac{1}{2}u_0 + V_D u_0 - V_S v_0 - f, \phi) = 0 = (V_S v_0 - V_D u_0 - f, \psi) \ .$$

Let $H_0^{-\frac{1}{2}}(\Gamma_1)$ be the completion of $C_0^\infty(\Gamma_1)$ in the $|\cdot|_{-\frac{1}{2}}$ norm. In fact

$$H_0^{-\frac{1}{2}}(\Gamma_1) = \{v|_{\Gamma_1} : v \in H_0^{-\frac{1}{2}}(\Gamma) \ \& \int_{\Gamma_1} v = 0\}$$

but this useful characterization is not necessary. $H^0(\Gamma_2) = L^2(\Gamma_2)$ is the space of square integrable functions defined on Γ_2 with norm $\|u\|^2 = (u,u)$. Then equations (39) are satisfied provided $u_0 \in H^0(\Gamma_2)$, $v_0 \in H_0^{-\frac{1}{2}}(\Gamma_1)$. For $u_0 \in H^0(\Gamma_2)$, $v_0 \in H_0^{-\frac{1}{2}}(\Gamma_1)$, $\psi \in C_0^\infty(\Gamma_1)$

$$(V_S v_0 - V_D u_0, \psi) = ((I+K)v_0 - V_{SC}^{-1} V_D u_0, \psi)_{-\frac{1}{2}}$$

and thus

$$|(V_S v_0 - V_D u_0, \psi)| \le C(|v_0|_{-\frac{1}{2}} + \|u_0\|)|\psi|_{-\frac{1}{2}} \ .$$

Hence $\psi \mapsto (V_S v_0 - V_D u_0, \psi)$ is a bounded functional on $H_0^{-\frac{1}{2}}(\Gamma_1)$. Using the compactness of K and V_D , there is a compact operator $K_1 : H^0(\Gamma_2) \times H_0^{-\frac{1}{2}}(\Gamma_1) \to H_0^{-\frac{1}{2}}(\Gamma_1)$ such that

$$(V_S v_0 - V_D u_0, \psi) = (u_0 + K_1(u_0, v_0), \psi)_{-\frac{1}{2}} \ .$$

Similarly, there is a compact operator $K_2 : H^0(\Gamma_2) \times H_0^{-\frac{1}{2}}(\Gamma_1) \to H^0(\Gamma_2)$ such that for all $\phi \in C^\infty(\Gamma_2)$

$$(u_0 + 2V_D u_0 - 2V_S v_0, \phi) = (u_0 + K_2(u_0, v_0), \phi) \ .$$

Hence the equations (40) can be expressed as the operator equation on $H^0(\Gamma_2) \times H_0^{-\frac{1}{2}}(\Gamma_1)$,

(41) $$\begin{bmatrix} u_0 \\ u_0 \end{bmatrix} + \begin{bmatrix} K_1(u_0, v_0) \\ K_2(u_0, v_0) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \ ;$$

where $f_1 \in H^0(\Gamma_2)$, $f_2 \in H_0^{-\frac{1}{2}}(\Gamma_1)$ are defined by

$$\forall\phi \in C^{\infty}(\Gamma_2): (f_1,\phi)_{-\frac{1}{2}} = (f,\phi)_{-\frac{1}{2}} \quad ,$$
$$\forall\phi \in C_0^{\infty}(\Gamma_1): (f_2,\psi) \quad = (f,\psi) \quad .$$

Thus again a first kind bie has been expressed as a second kind equation on unfamiliar spaces.

To solve the bie numerically introduce the spaces

$$S^h(\Gamma_i) = \{\phi \in S^h: \phi(x) = 0 \quad x \notin \bar{\Gamma}_i\} \quad ,$$

$$S_0^h(\Gamma_i) = S^h(\Gamma_i) \cap S_0^h \quad .$$

Note $S_0^h(\Gamma_1) \subset H_0^{-\frac{1}{2}}(\Gamma_1)$ because $S_0^h(\Gamma_1) \subset H_0^0(\Gamma_1)$ and $H_0^0(\Gamma_1) \subset H_0^{-\frac{1}{2}}(\Gamma_1)$. The Galerkin solutions to the bie (38)-(39) are the piecewise polynomials $u_h \in S^h(\Gamma_2)$, $v_h \in S_0^h(\Gamma_1)$ defined by $\forall\phi \in S^h(\Gamma_2)$, $\psi \in S_0^h(\Gamma_1)$

(42) $$(V_S u_h - V_D u_h - \tfrac{1}{2}u_h - f,\phi) = (V_S v_h - V_D u_h - f,\psi) = 0 \quad .$$

These are clearly equivalent to Galerkin's method for the second kind equation (41). We again conclude that, provided the bie has a unique solution, (42) uniquely determines u_h and v_h for h sufficiently small, and

$$\|u_h-u_0\| + |v_h-v_0|_{-\frac{1}{2}}$$
$$\leq C \inf\{\|\phi-u_0\| + |\psi-v_0|_{-\frac{1}{2}}: \phi \in S^h(\Gamma_2) \ , \quad \psi \in S_0^h(\Gamma_1)\} \ .$$

But the proof that the bie has a unique solution depends on uniqueness results for mixed problems. This in turn is complicated by the fact that u,v are not smooth even though g and h are. That is $u \in H^t(\Gamma)$, $v \in H_0^{t-1}(\Gamma)$ for $t < 1$, but not for higher t . Thus the jump formulae and Green's identities need to be proved for non-smooth functions. There is also the practical consequence that the piecewise polynomial bases must be modified to produce higher order approximations to u and v ([19], [21]). Nevertheless the above theory assures the convergence of the Galerkin method if it is applied to well posed problems.

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