LATTICES FOR MULTIPLE INTEGRATION

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1. INTRODUCTION

The subject of this paper is the numerical integration of smooth, periodic functions over the unit cube in s dimensions. In one dimension the problem reduces to

$$\int_0^1 f(x)dx \ , \tag{1}$$

where f is a smooth, periodic function with period 1. In this situation the recommended method (see for example [3, p. 106]) is usually the trapezoidal rule, or equivalently the rectangle rule

$$\frac{1}{n}\sum_{j=0}^{n-1} f\left(\frac{j}{n}\right) \ . \tag{2}$$

The s-dimensional analogue of (1) is

$$\begin{aligned} I(f) &= \int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_s)dx_1 \cdots dx_s \\ &= \int_{U^s} f(\underset{\sim}{x})d\underset{\sim}{x} \ , \end{aligned} \tag{3}$$

where U^s is the s-dimensional unit cube,

$$U^s = \{\underset{\sim}{x} \in \mathbb{R}^s : \ 0 \le x_i < 1 \ , \ 1 \le i \le s\} \ .$$

We shall always assume that f is periodic with period 1 with respect

to each coordinate separately, i.e.

$$f(\underset{\sim}{x} + \underset{\sim}{u}) = f(\underset{\sim}{x}) \ , \quad \underset{\sim}{u} \in \mathbb{Z}^s \ , \ \underset{\sim}{x} \in \mathbb{R}^s \ . \tag{4}$$

What, then, is the s-dimensional analogue of (2)? There is no unique answer to this question. Perhaps the most obvious generalization of (2) is the product rectangle rule

$$\frac{1}{n^s} \sum_{j_1=0}^{n-1} \cdots \sum_{j_s=0}^{n-1} f\left(\frac{j_1}{n}, \ \ldots, \frac{j_s}{n}\right) \ , \tag{5}$$

which we shall refer to hereafter simply as the rectangle rule. The total number of quadrature points in this rule is $N = n^s$, which for constant n rises very rapidly with s. Thus the rectangle rule suffers badly from the 'curse of dimensionality'.

To see the extent of the problem, consider, for example, the function

$$f(\underset{\sim}{x}) = F(x_1) \ \ldots \ F(x_s) \ , \tag{6}$$

where

$$F(x) = \begin{cases} x^2(1-x)^2 \ , & 0 \le x \le 1 \ , \\ F(x+1) \ , & x \in \mathbb{R} \ . \end{cases} \tag{7}$$

Because f is a product, the rectangle rule reduces in this case to a product of one-dimensional rules for the function F. Now the error in the one-dimensional rule for F is known to be of order $O(n^{-4})$, and from this it follows that the error in the s-dimensional rule for f is also of order $O(n^{-4})$. The trouble is that in terms of N the order is merely $O(N^{-4/s})$, so the convergence is very slow indeed if s is large.

A second and quite different way of generalizing the rule (2) is the number-theoretic 'good-lattice' method of Korobov [9] and others [2, 4, 6, 7, 11, 14]. (For a lucid recent summary, see [5].) In this method one first chooses the total number of points N (with N often a prime, or a product of two primes). Then an integer vector $\underset{\sim}{p} \in \mathbb{Z}^s$ is determined, and the integral $I(f)$ is approximated by

$$I_N(f) = \frac{1}{N} \sum_{j=0}^{N-1} f\left(\left\{j \frac{\underset{\sim}{p}}{N}\right\}\right) , \tag{8}$$

where the braces $\{\ \}$ about a vector indicates that the fractional part of each component is to be taken. (The braces can be omitted because of the periodicity property (4), but it is convenient to think of all quadrature points as lying in the unit cube.) Without loss of generality, we may assume that the components of $\underset{\sim}{p}$ satisfy $0 \le p_i < N$. We shall also assume that at least one component of $\underset{\sim}{p}$ is relatively prime with N — in practice this is often achieved by setting $p_1 = 1$.

If $s = 1$ and N is set equal to n then the rule (8) reduces to the one-dimensional rectangle rule (2). In higher dimensions, however, the rule (8) is different from the rectangle rule (5). As a simple illustration, we set $s = 2$, $N = 5$ and $\underset{\sim}{p} = (1,2)$. Then (8) yields a five-point rule in two dimensions,

$$I_5(f) = \frac{1}{5}\left[f(0,0) + f\left(\frac{1}{5},\frac{2}{5}\right) + f\left(\frac{2}{5},\frac{4}{5}\right) + f\left(\frac{3}{5},\frac{1}{5}\right) + f\left(\frac{4}{5},\frac{3}{5}\right)\right] .$$

Much of the literature on the number-theoretic good-lattice method is concerned with proving that in a certain precise sense there exist 'good lattices', corresponding to good choices of N and $\underset{\sim}{p}$. In Figure 1 we show the quadrature points corresponding to one such good choice in two dimensions. (The numbers 55 and 89 used in this construction are con-

secutive Fibonacci numbers. For the theory of the Fibonacci construction in two dimensions see [14].) In Figure 2 we show the quadrature points for the same value of N but a different (and rather arbitrary) choice of $\underset{\sim}{p}$. In both cases the points are seen to form slanting lattices, but the distribution appears to be more uniform in the first case than the second. (Later we shall give a numerical comparison of the corresponding quadrature rules, which will confirm that the second quadrature rule is indeed less good than the first.)

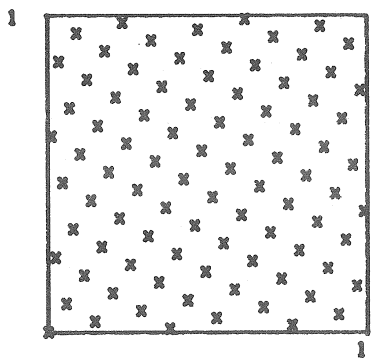

Fig. 1. Quadrature points for $N = 89$, $\underset{\sim}{p} = (1,55)$.

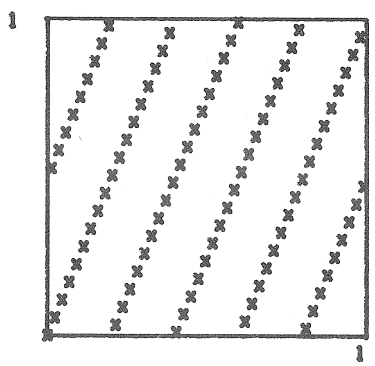

Fig. 2. Quadrature points for $N = 89$, $\underset{\sim}{p} = (1,47)$.

Korobov and Hlawka have shown that as N runs through the primes it is possible to choose a corresponding sequence of vectors $\underset{\sim}{p}$ such that the quadrature error for the function f defined by (6) and (7) satisfies

$$|I_N(f) - I(f)| \le c \frac{(\log N)^{\beta}}{N^4},$$

where c and β depend only on s. Asymptotically that convergence is much faster than the order $O(N^{-4/s})$ obtained for the rectangle rule. However, Haber [5] points out that β increases very rapidly with s, so

that the curse of dimensionality is not entirely dispelled.

There are perhaps two main problems with the number-theoretic method (apart, that is, from the practical problem of finding good lattices). The first is that the convergence as $N \to \infty$ is irregular, because there is no relation between the $\underset{\sim}{p}$ vectors for one value of N and another. Thus there is no possibility of using Richardson extrapolation to improve the estimate, nor any valid way of estimating the error by comparing the values for different values of N. The other problem is that the set of quadrature points has very little symmetry: the only symmetry is inversion about the centre of the cube. It often happens that f has at least some of the symmetries of the cube — for example, a standard way of producing periodic functions from non-periodic functions makes f symmetric about each of the mid-planes perpendicular to an axis. With the number-theoretic method there is no way of taking advantage of this symmetry to reduce the number of quadrature points (except for an approximate halving through the inversion symmetry). In contrast, the rectangle rule has all of the symmetries (a vast number in high dimensions) of the s-dimensional cube.

We shall demonstrate in this paper that both the rectangle rule (5) and the number-theoretic method (8) are special cases of a much more general family of 'lattice methods', all of which can be thought of as generalizations of the one-dimensional rectangle rule, and all of which can be analysed within the one simple framework. It seems certain that some other methods within this family will be of practical interest. Indeed, one class of lattice methods introduced below (the class W_{nn}) may already be a contender. Only a brief sketch of the lattice methods can be given here. A full exposition will appear elsewhere [12].

2. LATTICE METHODS

Both the rectangle rule (5) and the number-theoretic rule (8) may be

written in the form

$$(9) \qquad I_N(f) = \frac{1}{N} \sum_{i=0}^{N-1} f(\underset{\sim}{x}_i) \ ,$$

where $\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_{N-1}$ are points in the unit cube U^s . The point sets in the two cases are different, but they have in common that they are points of a 'lattice'.

A lattice, in our definition, is an infinite set of points in $\mathbb{R}^s$ that form a group under addition, and that are such that some neighbourhood of $\underset{\sim}{0}$ contains no point of the lattice other than $\underset{\sim}{0}$ itself. The lattices that we want to use in (9) also have the additional property that they contain $\mathbb{Z}^s$ as a sub-lattice. (This ensures that the lattice has the same periodicity property as f .) We shall call a lattice with that additional property a 'multiple-integration lattice'.

To each multiple-integration lattice there corresponds a 'lattice method': it is the rule of the form (9) for which $\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_{N-1}$ are the points of the lattice that lie in U^s . Note that $\underset{\sim}{0}$ is one of the quadrature points; so we write $\underset{\sim}{x}_0 = \underset{\sim}{0}$.

The rectangle rule (5) is, of course, the lattice method corresponding to the rectangular lattice

$$(10) \qquad \left\{ \left(\frac{j_1}{n}, \ldots, \frac{j_s}{n} \right) : j_i \in \mathbb{Z}, \ 1 \le i \le s \right\} ;$$

and the number-theoretic rule (8) corresponds to the lattice

$$(11) \qquad \left\{ j \frac{\underset{\sim}{p}}{N} + \underset{\sim}{u} : j \in \mathbb{Z}, \ \underset{\sim}{u} \in \mathbb{Z}^s \right\} .$$

Of the vast number of other multiple-integration lattices, we shall consider for purposes of illustration just one family: for positive

integers n and r, the lattice W_{nr} is defined by

$$\left\{\left(\frac{j_1}{n}, \ldots, \frac{j_s}{n}\right) + k\left(\frac{1}{rn}, \ldots, \frac{1}{rn}\right) : j_1, \ldots, j_s, k \in \mathbb{Z}\right\}. \tag{12}$$

We may think of W_{nr} as consisting of cubes of side $\frac{1}{n}$, within each of which there are $r+1$ equally spaced points on the main diagonal. The lattice W_{n1} is just the rectangular lattice. The lattice W_{n2} has an additional point at the centre of each of the cubes of side $\frac{1}{n}$, and so may reasonably be called the 'body-centred cubic' lattice. It has all of the symmetries of the rectangular lattice itself. The lattices W_{nr} with $r > 2$ have somewhat less symmetry. (No doubt the definition of W_{nr} will seem rather arbitrary at this stage. We shall return to the motivation later.)

In Figures 3, 4 and 5 we show the lattices W_{41}, W_{42} and W_{44} (or more precisely, the part of each lattice that lies in U^s), for the case $s = 2$. Later we shall give some numerical results for the corresponding integration rules.

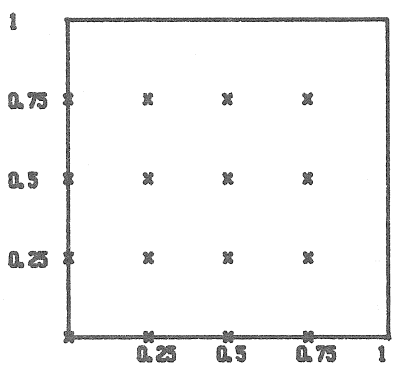

Fig. 3. Quadrature points for W_{41}, the rectangle rule with $n = 4$.

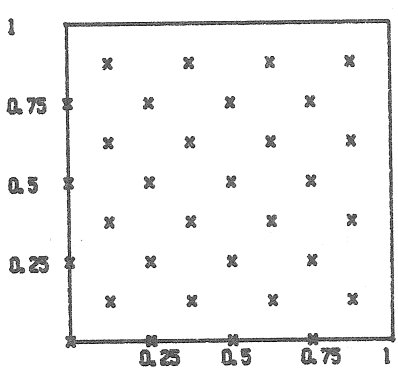

Fig. 4. Quadrature points for W_{42}.

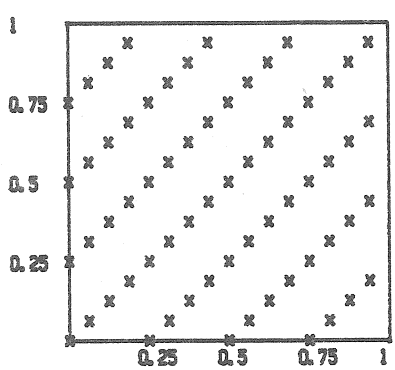

Fig. 5. Quadrature points for W_{44}.

To assess the accuracy of the competing lattice methods, we need to know the error for the integration rule (9). That is the next step.

3. ERROR ANALYSIS

We may begin, as in all treatments of number-theoretic multiple integration, by assuming that f has an absolutely and uniformly convergent Fourier series representation

$$f(\underset{\sim}{x}) = \sum_{\underset{\sim}{m} \in \mathbb{Z}^s} a(\underset{\sim}{m}) e^{2\pi i \underset{\sim}{m} \cdot \underset{\sim}{x}}, \tag{13}$$

where

$$\underset{\sim}{m} \cdot \underset{\sim}{x} = m_1 x_1 + \ldots + m_s x_s .$$

Given this representation of f, it is a natural first step to ask: what does the rule (9) give for the function $e^{2\pi i \underset{\sim}{m} \cdot \underset{\sim}{x}}$? Now in the special case of the number-theoretic method (8), this question has a very simple answer: for this case, writing $\phi_{\underset{\sim}{m}}(\underset{\sim}{x}) = e^{2\pi i \underset{\sim}{m} \cdot \underset{\sim}{x}}$, we have

$$\begin{aligned} I_N(\phi_{\underset{\sim}{m}}) &= \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi i \underset{\sim}{m} \cdot (j\underset{\sim}{p}/N)} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \left(e^{2\pi i \underset{\sim}{m} \cdot \underset{\sim}{p}/N} \right)^j \\ &= \begin{cases} 1 & \text{if } \underset{\sim}{m} \cdot \underset{\sim}{p}/N \in \mathbb{Z}, \\ 0 & \text{otherwise} . \end{cases} \end{aligned}$$

Since $\underset{\sim}{m} \cdot \underset{\sim}{p}/N$ is an integer if and only if $\underset{\sim}{m} \cdot \underset{\sim}{p} \equiv 0 \pmod N$, we obtain from (13),

$$I_N(f) = \sum_{\underset{\sim}{m} \cdot \underset{\sim}{p} \equiv 0 \pmod N} a(\underset{\sim}{m}) . \tag{14}$$

Since

$$I(f) = a(\underset{\sim}{0}) ,$$

it follows that the error in applying the number-theoretic rule (8) to f is

$$(15) \qquad I_N(f) - I(f) = \sum_{\underset{\sim}{m}\cdot\underset{\sim}{p}\equiv 0 \,(\mathrm{mod}\ N)}{}' a(\underset{\sim}{m}) ,$$

where the prime indicates that the term $\underset{\sim}{m} = \underset{\sim}{0}$ is to be omitted from the sum. Equation (15) is a standard result in the number-theoretic literature. Note that the error expression has contributions only from the integer vectors $\underset{\sim}{m}$ (other than $\underset{\sim}{0}$) that satisfy the congruence $\underset{\sim}{m}\cdot\underset{\sim}{p} \equiv 0 \pmod N$, the reason being that for all other values of $\underset{\sim}{m}$ the rule integrates $e^{2\pi i \underset{\sim}{m}\cdot\underset{\sim}{x}}$ exactly.

We now seek to analyse the general lattice method introduced in the previous section: thus we let S be a multiple-integration lattice, and let $\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_{N-1}$ be the points of S that lie in U^S. Again we ask what the rule gives for the function $e^{2\pi i \underset{\sim}{m}\cdot\underset{\sim}{x}}$. The answer is given by the following theorem, proved in [12] by a group-theoretical argument.

THEOREM [12]. *If* $I_N(\cdot)$ *denotes the lattice method defined above, and if* $\phi_{\underset{\sim}{m}}(\underset{\sim}{x}) = e^{2\pi i \underset{\sim}{m}\cdot\underset{\sim}{x}}$ *with* $\underset{\sim}{m} \in \mathbb{Z}^S$, *then*

$$I_N(\phi_{\underset{\sim}{m}}) = \begin{cases} 1 & \text{if } \underset{\sim}{m} \in S^{\perp} , \\ 0 & \text{otherwise} . \end{cases}$$

The set $S^{\perp}$ appearing in the theorem is the 'dual' or 'reciprocal' of S, defined by

$$S^{\perp} = \{\underset{\sim}{m} \in \mathbb{R}^S : \underset{\sim}{m}\cdot\underset{\sim}{x} \in \mathbb{Z} \quad \forall \underset{\sim}{x} \in S\} ,$$

and is itself a lattice. The dual lattice appears in coding theory [13] and geometric number theory [1], and in X-ray diffraction and solid-state physics [8], but as far as we know this is its first explicit appearance in multiple-integration theory. Of course the dual of the number-theoretic lattice (11) is just the set of solutions of the congruence $\underset{\sim}{m} \cdot \underset{\sim}{p} \equiv 0 \pmod{N}$.

From the theorem we have the following corollary for the error in $I_N(f)$, which generalizes the result (15) for the number-theoretic case.

COROLLARY. *If f has the absolutely and uniformly convergent Fourier series representation (13), the error in the lattice method $I_N(f)$ is*

$$(16) \qquad I_N(f) - I(f) = \sum_{\underset{\sim}{m} \in S^{\perp}}{}' a(\underset{\sim}{m}) .$$

In Figures 6 to 10 we show the duals of the multiple-integration lattices in Figures 1 to 5. (We leave aside technical questions relating to how to find the dual of a lattice — see [12].)

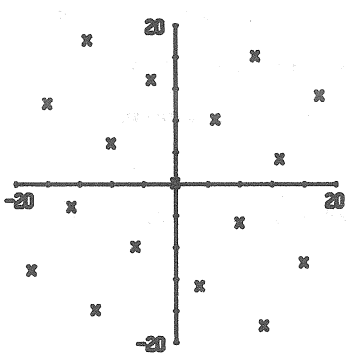

Fig. 6. Dual of the number-theoretic lattice for $N = 89$, $\underset{\sim}{p} = (1,55)$.

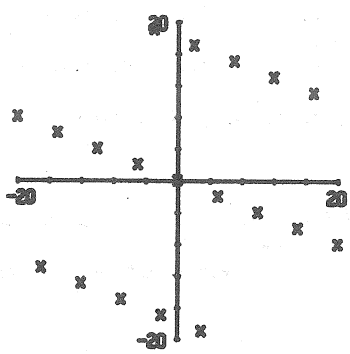

Fig. 7. Dual of the number-theoretic lattice for $N = 89$, $\underset{\sim}{p} = (1,47)$.

Figure 6 is the dual of the 'good' number-theoretic lattice in Figure 1. The goodness of that lattice can now be understood by observing that there are no non-zero elements of the dual close to the origin. Figure 7, on the other

hand, which is the dual of the less good number-theoretic lattice, has points relatively close to the origin, namely at $\pm (5,-2)$. These may give a relatively large contribution to the error expression (16), if the Fourier coefficients $a(\underset{\sim}{m})$ behave in an ordinary way (see below).

The dual of the rectangular lattice (10), illustrated for $s = 2$ and $n = 4$ in Figure 8, is itself a rectangular lattice, the edge length in this case being n . The most dangerous points in that dual, from the point of view of the error expression (16), are the innermost points on the axes, for example at $(\pm n,0, \ldots,0)$.

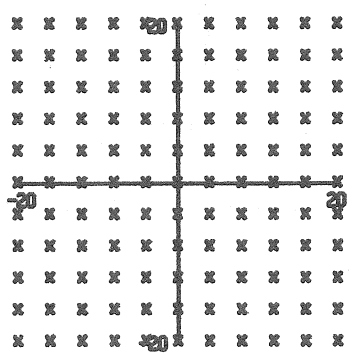

Fig. 8. Dual of the rectangular lattice W_{41} .

The body-centred cubic lattice in Figure 4, when compared with the rectangular lattice in Figure 3, has additional points, whose effect is to remove the most dangerous points from the rectangular dual — compare Figure 9 with Figure 8. This removal of the most dangerous points is a general property of the body-centred cubic lattice, and may be considered a first example of the 'dual engineering' approach to lattice design. The lattice W_{nr} goes further in this direction, in that it removes from the rectangular dual the $r - 1$ most dangerous points on each semi-axis. The motivation for this is that the points of a dual that lie on the axes turn out to be particularly significant (see (21) below). The dual of W_{44} in two dimensions is shown in Figure 10. Note that the nearest dual-lattice points on the axes are now at a distance of $n^2 = 16$ from the origin.

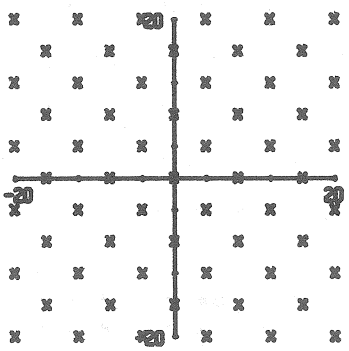

Fig. 9. Dual of W_{42} .

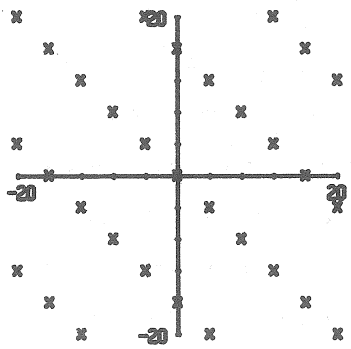

Fig. 10. Dual of W_{44} .

To proceed further it is necessary to make some assumptions about the function f . A test function that has been much used in the literature of number-theoretic multiple integration is the function f_2 defined by

$$f_2(\underset{\sim}{x}) = F_2(x_1) \ \ldots \ F_2(x_s) \ ,$$

where

$$F_2(x) = \begin{cases} 1+2\pi^2(x^2-x+\frac{1}{6}), & 0 \le x \le 1 \ , \\ F_2(x+1), & x \in \mathbb{R} \ . \end{cases}$$

It is easily shown that the Fourier series expansion of F_2 is

$$F_2(x) = \sum_{m=-\infty}^{\infty} \frac{1}{\overline{m}^2} e^{2\pi i m x} \ ,$$

where

$$\overline{m} = \begin{cases} |m| & \text{if} \quad |m| \ge 1 \ , \\ 1 & \text{if} \quad m = 0 \ , \end{cases}$$

from which it follows that

$$f_2(\underset{\sim}{x}) = \sum_{\underset{\sim}{m} \in \mathbb{Z}^s} \frac{1}{(\bar{m}_1 \dots \bar{m}_s)^2} e^{2\pi i \underset{\sim}{m} \cdot \underset{\sim}{x}} .$$

The Corollary above then tells us that the error from applying the lattice method (9) to the function f_2 is

$$I_N(f_2) - I(f_2) = \sum_{m \in S^\perp}{}' \frac{1}{(\bar{m}_1 \dots \bar{m}_s)^2} . \tag{17}$$

The particular virtue of the function f_2 is that it is in a sense the 'worst' function in a certain class. Let $E_s^\alpha(c)$ denote the class of functions f for which

$$|a(\underset{\sim}{m})| \le \frac{c}{(\bar{m}_1 \dots \bar{m}_s)^\alpha} .$$

Then for $f \in E_s^\alpha(c)$ we have, from the Corollary,

$$|I_N(f) - I(f)| \le c \sum_{m \in S^\perp}{}' \frac{1}{(\bar{m}_1 \dots \bar{m}_s)^\alpha} . \tag{18}$$

In particular, for $f \in E_s^2(c)$ we have,

$$\begin{aligned} |I_N(f) - I(f)| &\le c[I_N(f_2) - I(f_2)] \\ &= c[I_N(f_2) - 1] . \end{aligned} \tag{19}$$

In a similar way we may define

$$f_4(\underset{\sim}{x}) = F_4(x_1) \dots F_4(x_s) ,$$

where

$$F_4(x) = \begin{cases} 1 + \frac{\pi^4}{45} - \frac{2\pi^4}{3} x^2(1-x)^2 , & 0 \le x \le 1 , \\ F_4(x+1) , & x \in \mathbb{R} \end{cases}$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{\bar{m}^4} e^{2\pi i m x} ,$$

from which it follows that

$$f_4(\underset{\sim}{x}) = \sum_{\underset{\sim}{m} \in \mathbb{Z}^s} \frac{1}{(\bar{m}_1 \ldots \bar{m}_s)^4} e^{2\pi i \underset{\sim}{m} \cdot \underset{\sim}{x}} .$$

Thus for $f \in E_s^4(c)$ we have the rigorous bound

$$|I_N(f) - I(f)| \le c \sum_{\underset{\sim}{m} \in S^\perp}{}' \frac{1}{(\bar{m}_1 \ldots \bar{m}_s)^4} \tag{20}$$

$$= c[I_N(f_4) - 1] .$$

Note that the largest terms in the error bound (18) are equal to c/ρ^α , where

$$\rho = \min_{\substack{\underset{\sim}{m} \in S^\perp \\ \underset{\sim}{m} \neq \underset{\sim}{0}}} (\bar{m}_1 \ldots \bar{m}_s) , \tag{21}$$

suggesting that to minimise the error one should try to make ρ as large as possible. (A Theorem giving some theoretical support to this idea is proved in [12].) In the case of the rectangular lattice (10), the dual lattice points at $(\pm n, 0, \ldots, 0)$ ensure that $\rho = n$. For the lattice W_{nr} the 'worst' points in the dual are at $(\pm nr, 0, \ldots, 0)$ if $1 \le r \le n$, from which it follows that $\rho = nr$. For $r = n$ the points in the dual at $(n, -n, 0, \ldots, 0)$ etc. are equally 'bad'. This balancing act is the essential feature in the design of the lattice W_{nn} , and leads to a

relatively favourable value for ρ , namely $\rho = n^2$.

4. NUMERICAL RESULTS

In Table 1 we give results for the five two-dimensional lattices shown in Figures 1 to 5. The first lattice is the 'good' number-theoretic lattice of Figure 1. In this case $\rho = 34$. (Note that there are dual lattice points at $\pm(34,1)$, since $34 \times 1 + 1 \times 55 \equiv 0 \pmod{89}$.) The estimated value for $I(f_2)$ is reasonable, and that for $I(f_4)$ is excellent. (Remember that in all cases the exact integrals have the value 1 .)

Table 1. Results for $s = 2$.

N	Lattice	ρ	$I_N(f_2)$	$I_N(f_4)$
89	$\underset{\sim}{p} = (1,55)$	34	1.016	1.000 008
89	$\underset{\sim}{p} = (1,47)$	10	1.032	1.000 2
16	W_{41}	4	1.45	1.017
32	W_{42}	8	1.13	1.001 1
64	W_{44}	16	1.04	1.000 10
64	W_{81}	8	1.11	1.001

The less good number-theoretic lattice of Figure 2 has $\rho = 10$ (corresponding to the dual lattice points at $\pm(5,-2)$), and so not surprisingly gives worse values for the two integrals.

The rectangular lattice W_{41} has $\rho = 4$, as noted in the previous section, and yields relatively poor values for $I(f_2)$ and $I(f_4)$. The body-centred cubic lattice performs better in every respect, and the lattice W_{44} better still. For comparison we show also the result for the rectangular lattice W_{81} , which has the same number of points as W_{44} .

The calculations in Table 1 are really only toy calculations, to establish the ideas. We turn now to some more significant calculations

in six dimensions. In Table 2 we compare, for $s = 6$, the lattices W_{33}, W_{44} and W_{55} with three number-theoretic good-lattices with similar numbers of points. (The number-theoretic lattices and the corresponding values of ρ are taken from the tables of Maisonneuve [10].) For N near 2000 the results for W_{33} seem to be better than the number-theoretic results, whereas for N near 75000 the situation is largely reversed — which is not surprising, since the number-theoretic methods certainly have a better asymptotic rate of convergence. That is not the only consideration, however. An advantage of the sequence $\{W_{nn}\}$ is that it permits Richardson extrapolation: from (17) and (20) and an examination of the dual of W_{nn} it can be shown that

$$I_N(f_2) - 1 = c_1 n^{-4} + c_2 n^{-6} + \ldots ,$$

$$I_N(f_4) - 1 = d_1 n^{-8} + d_2 n^{-12} + \ldots .$$

In the last line of Table 2 we show the values of the integrals deduced by

Table 2. Results for $s = 6$.

N	Lattice	ρ	$I_N(f_2)$	$I_N(f_4)$
2129	$\underset{\sim}{p}=(1,41, \ldots)$	4	3.0	1.019
2187	W_{33}	9	2.5	1.007
15 019	$\underset{\sim}{p}=(1,8743, \ldots)$	8	1.2	1.000 7
16 384	W_{44}	16	1.3	1.000 7
71 053	$\underset{\sim}{p}=(1,18010, \ldots)$	18	1.033	1.000 03
78 125	W_{55}	25	1.11	1.000 1
Richardson extrapolation from W_{33}, W_{44}, W_{55}:			1.011	1.000 001

applying Richardson extrapolation to W_{33}, W_{44} and W_{55}. It is clear that for these integrals Richardson extrapolation is working well.

Finally, in Table 3 we show some ten-dimensional results. Interestingly, the lattice W_{33} predicts a much better value of $I(f_4)$ than does a reasonably comparable number-theoretic lattice from [10], the error being about 2% for W_{33}, compared with about 7% for the number-theoretic lattice. Of course it will still be the case that for large enough N the number-theoretic lattices will win, but in practice 'asymptotia' may never be reached, because we are perhaps already close to the practical limit of N.

Table 3. Results for $s = 10$.

N	Lattice	ρ	$I_N(f_4)$
155 093	$\underset{\sim}{p}$=(1,90485, ...)	4	1.069
177 147	W_{33}	9	1.020

CONCLUSION

We have shown that the dual lattice is the key to the understanding of lattice methods for multiple integration. With this knowledge it should be possible to design lattices with desired properties (e.g. good extrapolation properties, appropriate symmetries, good accuracy). The lattices $\{W_{nn}\}$ introduced above appear to be a useful first step in this direction.

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