COLLOCATION METHODS FOR SECOND KIND FREDHOLM INTEGRAL EQUATIONS

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1. INTRODUCTION

In this paper we consider the application of the collocation method and its iterated variant to the numerical solution of the Fredholm integral equation

$$(1.1) \qquad y(t) = f(t) + \lambda \int_0^1 k(t,s)y(s)ds , \quad t \in [0,1] ,$$

where f and k are known, λ is a given scalar and y is the solution to be determined. The equation can be written in operator notation as

$$y = f + \lambda Ky .$$

Taking C to be the Banach space of continuous functions on $[0,1]$ equipped with the uniform norm, we shall make the following assumptions on (1.1):

A1: $f \in C$;

A2: K is a compact operator from C to C ;

A3: the homogeneous equation $y = Ky$ has only the trivial solution.

It then follows from standard Fredholm theory that there exists a unique solution $y \in C$.

In the collocation method, y is approximated by a function belonging to some finite-dimensional subspace taken here to be a space of discontinuous piecewise polynomials. It is well-known that under suitable conditions the collocation solution converges with the same asymptotic order as the best approximation to y out of the chosen subspace. Here, however, we are mainly concerned with the iterated variant of the collocation method obtained by substituting the collocation solution into the right-hand side of (1.1). With a suitable choice of collocation points, we shall see that the iterated collocation solution exhibits (global) superconvergence, that is, it converges faster than the actual collocation solution itself. The superconvergence results for the iterated collocation method which are given here come from recently completed work in [5].

In the actual practical implementation of the collocation scheme, certain integrals need to be evaluated, but it is not always possible (or perhaps even desirable) to evaluate them analytically. Hence numerical quadrature needs to be employed and this results in the discrete collocation and discrete iterated collocation methods. For these two methods, we indicate the precision of quadrature rule that should be used so as to be best possible, in the sense that the predicted theoretical order of convergence is not improved if a quadrature rule of higher precision is used.

2. THE COLLOCATION SCHEME

We first of all define our approximating subspace within which the collocation solution will be sought. For any positive integer n, let $\Delta_n : 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$ be a mesh on $[0,1]$, and with $h = h(n) = \max_{1\le i\le n} h_i$ where $h_i = x_i - x_{i-1}$, we make the natural assumption that $h \to 0$ as $n \to \infty$. Our approximating subspace, denoted by $S_{r,n}$, is taken to be the space of piecewise polynomials of order $\le r$ (that is, degree $\le r-1$) with no continuity requirement at the knots x_i, $1 \le i \le n-1$. We arbitrarily take any member of $S_{r,n}$ to be left continuous at every non-zero knot and to be right continuous at 0. It is clear that $N = \dim(S_{r,n}) = nr$.

Now let $\{u_1, \ldots, u_N\}$ be a basis for $S_{r,n}$. The collocation solution y_n is then defined by

$$(2.1) \qquad y_n = \sum_{\ell=1}^{N} a_\ell u_\ell ,$$

where the coefficients $\{a_\ell\}_{\ell=1}^N$ are found by 'collocating' at N distinct points $\{\tau_j\}_{j=1}^N$ in $[0,1]$, known as the collocation points. Thus we require

$$(2.2) \qquad y_n(\tau_j) = f(\tau_j) + \lambda \int_0^1 k(\tau_j, s) y_n(s) ds , \quad 1 \le j \le N .$$

Substitution of (2.1) into (2.2) leads immediately to the following system of N linear equations

$$(2.3) \qquad \sum_{\ell=1}^{N} [u_\ell(\tau_j) - \lambda K u_\ell(\tau_j)] a_\ell = f(\tau_j) , \quad 1 \le j \le N .$$

We now specify the choice of collocation points that shall be used: for $1 \le i \le n$ and $1 \le j \le r$, we take $\tau_{(i-1)r+j}$ to be the j th zero of the rth-degree Legendre polynomial shifted to $J_i = (x_{i-1}, x_i)$. In other words, on each J_i , $1 \le i \le n$, there are r collocation points which come from the shifting of the zeros of the rth-degree Legendre polynomial to J_i .

Assuming that y_n exists, the iterated collocation solution y_n' is defined by

$$(2.4) \qquad \begin{aligned} y_n' &= f + \lambda K y_n \\ &= f + \lambda \sum_{\ell=1}^{N} a_\ell K u_\ell . \end{aligned}$$

We note that if y_n has been calculated, so that the $\{a_\ell\}_{\ell=1}^N$ are known, then most of the extra work required in calculating $y_n'(t)$ for general $t \in [0,1]$ is that required to calculate the integrals $\{Ku_\ell(t)\}_{\ell=1}^N$. Usually these can be done by making use of the code already present to calculate the $\{Ku_\ell(\tau_j)\}_{\ell=1}^N$, $1 \le j \le N$, in (2.3) for the collocation method.

To do an error analysis of the collocation method and its iterated variant, it is convenient to cast them into a projection method framework. Examination of (2.2) shows that y_n can be alternatively defined by

$$(2.5) \qquad y_n = P_n f + \lambda P_n K y_n ,$$

where P_n is the interpolatory projection from $C + S_{r,n}$ to $S_{r,n}$ satisfying, for $g \in C$ and $\phi_n \in S_{r,n}$,

$$P_n g(\tau_j) = g(\tau_j) , \quad 1 \le j \le N ; \quad P_n \phi_n = \phi_n .$$

Comparison of (2.4) and (2.5) shows $P_n y_n' = y_n$ and it then follows from (2.4) that an alternative definition of y_n' is

$$(2.6) \qquad y_n' = f + \lambda K P_n y_n' .$$

Taking $L_\infty(0,1)$ to be the space of all essentially bounded and measurable functions on $(0,1)$ equipped with the usual L_∞ norm, we see that P_n is a uniformly bounded operator from $C + S_{r,n}$ to $L_\infty(0,1)$, since $\|P_n\|$ is simply the norm of the Lagrange interpolation operator for polynomial interpolation at the r Gauss-Legendre points. Hence

$$(2.7) \qquad \|P_n\| \le c ,$$

where c is a constant independent of n. (In this paper c, c_1, c_2

denote generic constants which may take different values at their different occurrences but will be independent of n .)

Standard arguments, based on (2.5) and (2.6) (e.g. [1]), can then be used to obtain the following theorem.

THEOREM 1. *Suppose A1-A3 hold. Then for* n *sufficiently large*

i) y_n *exists uniquely in* $S_{r,n}$, y'_n *exists uniquely in* C ;

ii) $\|y - y_n\|_\infty \le c_1 \inf_{\phi_n \in S_{r,n}} \|y - \phi_n\|_\infty$;

iii) $\|y - y'_n\|_\infty \le c_2 \|K(y-P_n y)\|_\infty$.

3. SUPERCONVERGENCE

In this section we give our superconvergence result for the iterated collocation method, which is an extension of the results of [3, 4]. To do so, it is convenient to introduce some function spaces. For any open interval $\Omega \subset \mathbb{R}$, let $L_p(\Omega)$, $1 \le p \le \infty$, denote the space of functions with integrable pth power which we equip with the usual L_p-norm $\|\cdot\|_{p,\Omega}$. Also for m a positive integer, we define $W_p^m(\Omega)$ to be the usual Sobolev space of functions satisfying

$$W_p^m(\Omega) = \{g : g^{(m-1)} \text{ is absolutely continuous and } g^{(m)} \in L_p(\Omega)\} ,$$

where $g^{(m)}$ is the mth (distributional) derivative of g . We equip this space with the norm $\|g\|_{m,p,\Omega} = \sum_{k=0}^{m} \|g^{(k)}\|_{p,\Omega}$.

Writing $J = (0,1)$ and $k_t(s) = k(t,s)$, we have the following theorem, taken from [5].

THEOREM 2. *Suppose A1-A3 hold,* $y \in W_1^\ell(J)$ *(*$0 < \ell \le 2r$*) and* $k_t \in W_1^m(J)$ *(*$0 < m \le r$*), with* $\|k_t\|_{m,1,J}$ *bounded independently of* t . *Then*

$$\|y - y'_n\|_\infty = O(h^\beta) \quad \textit{where} \quad \beta = \min(\ell, r+m) .$$

PROOF. Only the essence of the proof is given. Full details may be found in [5].

It follows from Theorem 1.iii that the order of convergence of y_n' is given by $\|K(y - P_n y)\|_\infty$. Denoting the usual L_2 inner product on J by $(\cdot,\cdot)_J$ we have, for $t \in [0,1]$,

$$
\begin{aligned}
(3.1) \qquad K(y-P_n y)(t) &= (k_t, y-P_n y)_J \\
&= (k_t-\phi_{n,t}, y-P_n y)_J + (\phi_{n,t}, (I-P_n)(y-\psi_n))_J \\
&\quad + (\phi_{n,t}, (I-P_n)\psi_n)_J ,
\end{aligned}
$$

where $\phi_{n,t} \in S_{m,n}$ is an approximation to k_t and $\psi_n \in S_{\ell,n}$ is an approximation to y. To prove our result we need to show that each of the three terms in (3.1) is bounded uniformly in t by ch^β.

Bounds on the first two terms in (3.1) can be made independently of the choice of collocation points. Using (2.7) and the fact that $P_n\xi_n = \xi_n$ for any $\xi_n \in S_{r,n}$, one can make appropriate choices of ξ_n, $\phi_{n,t}$ and ψ_n to show that

$$
\begin{aligned}
(3.2) \qquad |(k_t-\phi_{n,t}, y-P_n y)_J| &= |(k_t-\phi_{n,t}, (I-P_n)(y-\xi_n))_J| \\
&\le ch^{m+\min(\ell-1,r)} \|k_t\|_{m,1,J} \|y\|_{\ell,1,J} \le ch^\beta ;
\end{aligned}
$$

$$
(3.3) \qquad |(\phi_{n,t}, (I-P_n)(y-\psi_n))_J| \le ch^\ell \|y\|_{\ell,1,J} \le ch^\beta .
$$

The particular choice of collocation points taken becomes important when we consider the final term $(\phi_{n,t}, (I-P_n)\psi_n)_J$ which we shall look at in a bit more detail. If $0 < \ell \le r$ then $(I-P_n)\psi_n = 0$, so we consider only the case $r < \ell \le 2r$. Clearly we can write

$$(3.4) \qquad (\phi_{n,t}, (I-P_n)\psi_n)_J = \sum_{i=1}^{n} (\phi_{n,t}, (I-P_n)\psi_n)_{J_i}$$

where $(\cdot,\cdot)_{J_i}$ is the usual L_2-inner product on J_i.

As $\phi_{n,t}$ is a polynomial of degree $\leq m-1$ on J_i we can write

$$\phi_{n,t}(s) = \sum_{k=0}^{m-1} \phi_{n,t}^{(k)}(t_i)(s-t_i)^k / k! , \qquad s \in J_i ,$$

where $t_i = (x_{i-1} + x_i) / 2$. We then have

$$(3.5) \qquad (\phi_{n,t}, (I-P_n)\psi_n)_{J_i} = \sum_{k=0}^{m-1} [\phi_{n,t}^{(k)}(t_i) \int_{J_i} (s-t_i)^k (I-P_n)\psi_n(s)ds]/k! .$$

With our specific choice of collocation points it will be shown below that for any polynomial of degree $\leq 2r-k-1$, $v_{2r-k-1}(s)$, we have

$$(3.6) \qquad \int_{J_i} (s-t_i)^k (I-P_n) v_{2r-k-1}(s)ds = 0 .$$

Then after applying the Bramble-Hilbert Lemma [2] in (3.5), we can use (3.4) as well as the properties of appropriate $\phi_{n,t}$ and ψ_n to show that

$$(\phi_{n,t}, (I-P_n)\psi_n)_J \leq ch^{2r} \leq ch^{\beta} .$$

This together with (3.2) and (3.3) completes the proof once we prove (3.6) which we now do. For convenience let the r collocation points on J_i be relabelled as $\{\tau_{ij}\}_{j=1}^{r}$. Since $(I-P_n)v_{2r-k-1}(\tau_{ij}) = 0$, $1 \leq j \leq r$, we can clearly write

$$(I-P_n)v_{2r-k-1}(s) = \prod_{j=1}^{r} (s-\tau_{ij}) . w_{r-k-1}(s) = P_{ir}(s) w_{r-k-1}(s) ,$$

where $w_{r-k-1}(s)$ is a polynomial of degree $\leq r-k-1$. From our choice of collocation points, $P_{ir}(s)$ is orthogonal to polynomials of degree $\leq r-1$ and since $(s-t_i)^k w_{r-k-1}(s)$ is a polynomial of degree $\leq k+r-k-1 = r-1$, (3.6) then follows.

REMARK. The theorem suggests that we require y to be very smooth for useful superconvergence to take place. For instance, to achieve full superconvergence of $O(h^{2r})$, the theorem requires $y \in W_1^{2r}(J)$ and $k_t \in W_1^r(J)$. This raises the question of whether the smoothness condition on y is in any sense necessary, and this has been looked at in detail in [5]. Some numerical results in Section 5 will indicate that the smoothness condition on y is essentially necessary in the sense that if we relax the condition on y in Theorem 2 to $y \in W_1^{\ell-1}(J)$, then the order of convergence given is not necessarily achieved.

4. THE DISCRETE COLLOCATION SCHEME

To calculate y_n, we recall from (2.3) that we need to calculate the integrals

$$(4.1) \qquad Ku_\ell(\tau_j) = \int_0^1 k(\tau_j,s)u_\ell(s)ds, \quad 1 \leq \ell,j \leq N,$$

while calculation of the iterated collocation solution, y_n', from y_n involves the evaluation of $Ku_\ell(t)$, $1 \leq \ell \leq N$, for general t. In practice it is not always possible to calculate these integrals analytically and it is common to use numerical integration. This gives rise to the discrete collocation and discrete iterated collocation methods. For simplicity, we shall assume that the approximation of the integrals in (4.1) and (2.4) required for these two methods respectively are done by using the same quadrature rule.

To be more specific, suppose for $g \in C$ we have points $p_1, \ldots, p_q \in \bar{J}$ and weights $\omega_1, \ldots, \omega_q$ such that the approximation $\int_0^1 g(s)ds \simeq \sum_{j=1}^q \omega_j g(p_j)$

is exact if g is a polynomial of degree γ. We assume that $\sum_{j=1}^{q} |\omega_j| \leq c$ for all q. Then for any $\phi_n \in S_{r,n}$, $K\phi_n$ is approximated by

$$(K_n\phi_n)(t) = \sum_{i=1}^{n} \sum_{j=1}^{q} \omega_j h_i k(t,s_{ij})\phi_n(s_{ij}) ,$$

where $s_{ij} = x_{i-1} + h_i p_j$. It then follows that the discrete collocation solution $\tilde{y}_n$ satisfies

$$\tilde{y}_n = P_n f + \lambda P_n K_n \tilde{y}_n , \tag{4.2}$$

while the discrete iterated collocation solution, $\tilde{y}'_n$, satisfies

$$\tilde{y}'_n = f + \lambda K_n \tilde{y}_n . \tag{4.3}$$

Now we would like to choose a suitable precision γ. Clearly using a highly accurate rule may be expensive computationally, while a rule of insufficient accuracy may not allow the full potential of the method to be realised. In the next two theorems, we indicate the precisions of the quadrature rule that should be used for the discrete collocation and discrete iterated collocation methods. The precisions indicated are best possible in the sense that increasing the precision of qudrature rule will not, in general, improve the predicted orders of convergence given in the theorems.

The proofs of the two theorems are not given but they may be found in [6] and are based on (4.2) and (4.3). The results given here generalise those of [3].

<u>THEOREM 3</u>. *Suppose A1-A3 hold,* $y \in W^{\ell}_{\infty}(J)$, $k_t \in W^{m}_{1}(J)$ *with* $\|k_t\|_{m,1,J}$ *bounded independently of* t, *and take* $\gamma = \min(\ell-1,m-1,r-1)$. *Then*

$$\|y-\tilde{y}_n\|_{\infty} = O(h^{\alpha^*}) , \quad \textit{where} \quad \alpha^* = \min(\ell,m,r) .$$

REMARK. If the integrals had been done analytically then it can be shown, using Theorem 1.ii, that $\|y-y_n\|_\infty = O(h^\alpha)$ where $\alpha = \min(\ell,r)$.

THEOREM 4. *Suppose A1-A3 hold,* $y \in W_1^\ell(J)$, $k_t \in W_1^m(J)$ *with* $\|k_t\|_{m,1,J}$ *bounded independently of* t *, and take* $\gamma = \min(\ell-1,m-1,2r-1)$. *Then*

$$\|y-\tilde{y}_n'\|_\infty = O(h^{\beta^*}) \quad \textit{where} \quad \beta^* = \min(\ell,m,2r) .$$

REMARK. We see from the two theorems that the predicted order of convergence will depend on the smoothness of the kernel, so that the order of convergence of the discrete methods may be less than that of the corresponding methods in which the integrals are done analytically. This results from the fact that we have opted for generality so that our approach to approximating the integrals is rather naive. For instance, if the kernel had singularities one would not usually use a simple quadrature rule but would use techniques such as a change of variable, singularity subtraction or product integration.

5. NUMERICAL RESULTS

In this section we give some numerical results which come from use of the iterated collocation method to solve the integral equation which has $\lambda = 2$, $k(t,s) \equiv 1$ and $y(t) = t^\delta$ where δ is a positive non-integer $< 2r$. The inhomogeneous term f is chosen to satisfy (1.1). Application of Theorem 2, with $m = r$, shows that for $y \in W_1^\ell(J)$, one would expect the iterated collocation solution to converge with order h^ℓ . It is easy to show that for $\ell-1 < \delta < \ell$ $(0 < \ell \le 2r)$, $y \in W_1^\ell(J)$ but $y \notin W_1^{\ell+1}(J)$.

Results (with $r = 3$) for three different values of δ are given in Tables 1, 2, 3. The order of convergence is estimated by using the ratio of two consecutive errors. We have taken $h = 1/n$.

TABLE 1

$\delta = 3.8$ Predicted Convergence Order = h^4

n	$\|y-y_n'\|_\infty$	Order of Convergence
4	3.144 E -8	4.781
8	1.143 E -9	4.790
12	1.639 E -10	4.795
16	4.128 E -11	

TABLE 2

$\delta = 4.8$ Predicted Convergence Order = h^5

n	$\|y-y_n'\|_\infty$	Order of Convergence
4	1.743 E -8	5.621
8	3.541 E -10	5.654
12	3.577 E -11	5.670
16	7.000 E -12	

TABLE 3

$\delta = 5.8$ Predicted Convergence Order = h^6

n	$\|y-y_n'\|_\infty$	Order of Convergence
4	1.258 E -7	5.984
8	1.988 E -9	5.990
12	1.752 E -10	5.992
16	3.126 E -11	

The results given indicate that if $y \in W_1^{\ell-1}(J)$ then $O(h^{\ell})$ convergence will not be achieved. The results also suggest that for $0 < \delta < 2r-1$, Theorem 2 gives only the integer part of the power of h correctly and in fact it is proved in [5] that the exact order of convergence is $h^{\delta+1}$. Since we are dealing with Sobolev spaces of only integral order, this result is the best possible. To get the exact order would require the introduction of 'fractional' derivative spaces. A few numerical tests have also been done using the discrete collocation and discrete iterated collocation methods and the results obtained show agreement with Theorems 3 and 4.

6. CONCLUSION

It has been shown that the iterated collocation method does have the potential for superconvergence provided the approximating subspace and collocation points have been chosen appropriately. However, because of the smoothness requirements on the kernel and especially on the solution (see Theorem 2), it is not clear whether useful superconvergence is obtained for integral equations that arise in practice, as typically the exact solution to these problems has only a limited number of derivatives. Thus there is a need for the version of the iterated collocation method described in this paper to be tested out in the "real world".

In the practical implementation of the collocation scheme, it is sometimes necessary to use numerical quadrature to calculate the required integrals and we have indicated the appropriate precision of quadrature rule that should be used for these discrete methods. Moreover, if the kernel is sufficiently smooth, it can be seen from Theorems 3 and 4 that it is possible to choose a precision of quadrature rule which is consistent, that is, the predicted order of convergence for the discrete method is the same as that for the method in which the integrals are calculated analytically. Even if the kernel is not sufficiently smooth the form of the kernel may suggest the use of a more appropriate numerical integration technique which would give a better order of convergence than that predicted by the theorems.

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