EXISTENCE VIA INTERIOR ESTIMATES FOR SECOND ORDER PARABOLIC EQUATIONS

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In memory of a former student of J.H. Michael, the late
Robin Wittwer (17th February 1954 - 26th May 1984)

1. PRELIMINARIES

Our problems will be solved on subsets of $\mathbb{R}^{n+1}$ with $n \geq 1$. We label points X in $\mathbb{R}^{n+1}$ by (x,t), $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, the $(n+1)$-th component being often associated with time in physical problems. For $X = (x,t)$, we call $|X| = (\|x\|^2 + |t|)^{\frac{1}{2}}$, the parabolic length of X, $\|x\|^2 = \sum_{i=1}^{n} x_i^2$ if $x = (x_1,\ldots,x_n)$. For $X,Y \in \mathbb{R}^{n+1}$, $d(X,Y) = |X-Y|$ denotes the parabolic distance between X and Y. Let Ω be a domain in $\mathbb{R}^{n+1}$. A point X in the topological boundary $\partial\Omega$ of Ω belongs to the parabolic boundary $P\Omega$ of Ω if for some $Y \in \Omega$, there exists a continuous path connecting X and Y, along which the "time" coordinate is non-decreasing. If $X \in \Omega$, then $d_\Omega(X)$ denotes $\inf\{d(X,Y);\ Y = (y,\tau) \in P\Omega,\ \tau \leq t\}$ if X is the point (x,t).

2. LINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

Linear parabolic partial differential operators will be defined on functions u defined on domains Ω to have the following form:

$$Lu(X) \equiv \sum_{i,j=1}^{n} a_{ij}(X) \frac{\partial^2 u}{\partial x_i \partial x_j}(X) + \sum_{i=1}^{n} b_i(X) \frac{\partial u}{\partial x_i}(X) + c(X)u(X) - \frac{\partial u}{\partial t}(X)$$

for $X \in \Omega$, a_{ij}, b_i, c, being real valued, locally Hölder continuous on Ω with exponent $\alpha \in (0,1]$, the matrix $[a_{ij}(X)]$ being symmetric positive definite for all $X \in \Omega$. We shall here be concerned with a certain class of such operators.

In this class, denoted by L , we shall assume the following: there exist $0 < \nu \le \Lambda$, $0 < \varepsilon < 1$ so that for all $X \in \Omega$, $1 \le i,j \le n$ the following are true

$$\sum_{i,j=1}^{n} a_{ij}(X)\, \xi_i \xi_j \ge \nu \sum_{i=1}^{n} \xi_i^{\,2} \quad \text{for all} \quad \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$$

$$|a_{ij}(X)| \le \Lambda , \quad |b_i(X)| \le \Lambda d(X)^{\varepsilon-1} , \quad -\Lambda d(P)^{\varepsilon-2} \le c(P) \le \Lambda .$$

3. SPECIAL DOMAINS

A non-empty domain Ω in $\mathbb{R}^{n+1}$ belongs to class A (admissible domains) if for each $0 < \nu \le \Lambda$, there exists a function $\psi \in C(\bar{\Omega}) \cap C^{2,1}(\Omega)$, $\bar{\Omega}$ denoting the closure of Ω , so that

(i) $\psi(X) = 0$ if $X \in P\Omega$, $\psi(X) > 0$ if $X \in \Omega$;

and, in addition, for each bounded open subset G of Ω , there exists $\rho > 0$, $M > 0$, $0 < \gamma < 1$, $A > 0$, $\lambda > 0$ so that

(ii) $\psi(X) \ge \rho d^2(X)$ for all $X \in G$, $d \equiv d_\Omega$;

(iii) $\psi(Y) \le M\psi(X)$ wherever $X,Y \in G$, $d(X,Y) < \gamma d(X)$;

(iv) $\left[\sum_{i,j=1}^{n} \left(\frac{\partial\psi}{\partial x_i}(X)\right)^2\right]^{\frac{1}{2}} \le A\psi(X)\, d(X)^{-1}$ for all $X \in G$;

(v) for any symmetric constant matrix $[a_{ij}]$ which satisfies

$$\sum_{i,j=1}^{n} a_{ij}\, \xi_i \xi_j \ge \nu \sum_{i=1}^{n} \xi_i^{\,2} \quad \text{for all} \quad \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$$

and $|a_{ij}| \le \Lambda$, $1 \le i,j \le n$ then for all $X \in G$,

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2\psi}{\partial x_i \partial x_j}(X) - \frac{\partial\psi}{\partial t}(X) \le -\lambda\psi(X)\, d(X)^{-2} .$$

For $\Omega \in A$, $0 < \nu \le \Lambda$, the corresponding function whose existence has just been asserted will be called a (ν,Λ) barrier for Ω . Some properties of $\mathcal{A}$ are given in Section 7.

4. BOUNDARY VALUE PROBLEMS

We wish to solve Dirichlet problem for operators in $\mathcal{L}$ and domains in $\mathcal{A}$.

4.1 THEOREM. *Given a bounded* $\Omega \in \mathcal{A}$, $L \in \mathcal{L}$, f *locally Hölder continuous on* Ω *with exponent* α *(as in Section 2), satisfying for some* $B > 0$, $|f(X)| \le Bd(X)^{\varepsilon-2}$ *for all* $X \in \Omega$, ε *being the same as that in the definition of* $L \in \mathcal{L}$, *and* ϕ *continuous on* $P\Omega$, *there exists* $u \in C^{2,1}(\Omega) \cap C(\bar{\Omega})$ *so that*

$$Lu(X) = f(X) \quad \text{for} \quad X \in \Omega$$

$$u(X) = \phi(X) \quad \text{for} \quad X \in P\Omega \ . \qquad \square$$

Further interior regularity of u can be deduced from the interior Schauder theory. The theory underlying this theorem and its generalizations to operators L with unbounded and degenerating coefficients generalizes work done by J.H. Michael for elliptic equations [1], [2], [3] and is given in [4]. The theory is based on using only interior Schander-type estimates.

5. OUTLINE OF THE PROOF OF THEOREM 4.1

One first notes that it is sufficient to consider the case when $\phi \equiv 0$ ([4], 84-88). With $0 < \alpha \le 1$, $\Omega \in \mathcal{A}$, ψ a (ν,Λ) barrier for Ω , we define the following Banach spaces $X_{2+\alpha}(\Omega,\psi)$ and $Y_\alpha(\Omega,\psi)$. For sufficiently smooth functions u defined on Ω , put

$$N_\beta(u) = \sup\{\psi(X)^{-1} d(X)^\beta |u(X)|;\ X \in \Omega\}$$

$$H_{\gamma}^{\alpha}(u) = \sup\{\min\{\psi(X)^{-1}\, d(X)^{\gamma+\alpha},\ \psi(Y)\, d(Y)^{\gamma+\alpha}\} \cdot$$

$$|u(X) - u(Y)|\, d(X,Y)^{-\alpha};\ X,Y \in \Omega,\ X \neq Y\} .$$

We say that $u \in X_{2+\alpha}(\Omega,\psi)$ if and only if

$$N_0(u) + \sum_{i=1}^{n} \left[N_1\left(\frac{\partial u}{\partial x_i}\right) + H_1^{\alpha}\left(\frac{\partial u}{\partial x_i}\right)\right]$$

$$+ \sum_{i,j=1}^{n} \left[N_2\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) + H_2^{\alpha}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\right]$$

(1) $$+ N_2\left(\frac{\partial u}{\partial t}\right) + H_2^{\alpha}\left(\frac{\partial u}{\partial t}\right) < \infty$$

and $u \in Y_{\alpha}(\Omega,\psi)$ if and only if

(2) $$N_2(u) + H_2^{\alpha}(u) < \infty .$$

The expressions in (1), (2) define norms for these spaces. Let $0 < \nu \leq 1 \leq \Lambda$, $\varepsilon > 0$ be the parameters associated with L. Then there exists a (ν,Λ) barrier for Ω which satisfies for some $\lambda > 0$, $B > 0$

(3) $$L\psi(X) \leqq -\lambda\psi(X)\, d(X)^{-2} \quad \text{if} \quad X \in \Omega$$

(4) $$\psi(X) \geqq Bd(P)^{\varepsilon} .$$

We shall use this (ν,Λ) barrier in our definition of $X_{2+\alpha}$, Y_{α}. It is now possible to show that if $u \in X_{2+\alpha}$ then $Lu \in Y_{\alpha}$ and that there exists a constant $C > 0$, independent of u and α so that

(5) $$\|Lu\|_{Y_{\alpha}} \leq C\, \|u\|_{X_{2+\alpha}}$$

and conversely if $Lu \in Y_{\alpha}$, $u \in C^{2,1}(\Omega) \cap C(\bar{\Omega})$ with $u(X) = 0$ for $X \in P\Omega$, then $u \in X_{2+\alpha}$ and

(6) $$\|u\|_{X_{2+\alpha}} \leq C\, \|Lu\|_{Y_{\alpha}} .$$

This last inequality is our interior Schander-type estimate. We note that under the conditions of Theorem 4.1 and our choice of ψ (see (4)), $f \in Y_\alpha$.

The Perron procedure is used to show that if $f \in Y_\alpha$, then there exists $u \in X_{2+\alpha}$ so that

$$Hu(X) \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(X) - \frac{\partial u}{\partial t}(X) = f(X)$$

for all $X \in \Omega$ (and $u = 0$ on $P\Omega$). As the details of this procedure (in the form we require) does not seem to be given in full in the literature we present this in Section 6.

The "method of continuation" is used to obtain the existence for L . That is, we show that

$$\mathcal{T} = \{t \in [0,1] : L_t(X_{2+\alpha}) = Y_\alpha\}$$

is both open and closed in $[0,1]$ using (5) and (6). Here $L_t \equiv tL + (1-t)H$. As $0 \in \mathcal{T}$, $\mathcal{T} = [0,1]$.

6. PERRON PROCEDURE

The author thanks Professor Gary Lieberman for discussions at the Centre for Mathematical Analysis on the version of the Perron procedure described below. In [4] a weaker version was given, where it seemed necessary to consider only "expanding domains". Akô [5] gives another method which also avoids this restriction.

6.1 LEMMA: *Let* $T > 0$, $\delta > 0$ *and*

$$\Omega = \{(x,t) \in \mathbb{R}^{n+1} ; \|x\| < \delta, 0 < t < T\} .$$

If ϕ *is uniformly continuous on* $P\Omega = \{(x,t) \in \Omega :$ *either* $t = 0$ *or* $0 < t < T$ *and* $\|x\| = \delta\}$, f *is locally Hölder continuous on* Ω , *there* *exists a function* $u \in C^{2,1}(\Omega) \cap C(\bar{\Omega})$ *equal to* ϕ *on* $P\Omega$ *and* $Hu = f$ *in* Ω *. Write* $u = H_\phi^\Omega$.

Proof: This result can be deduced from [6]. □

6.2 LEMMA: *Let* G *be an open domain in* $\mathbb{R}^n$, $T > 0$ *and let* u *be an upper semicontinuous function on* $\bar{\Omega}$ *,* $\Omega = G \times (0,T)$ *with* $u \le 0$ *on* $P\Omega$ *,* $Hu \ge 0$ *on* Ω *, then* $u \le 0$ *on* Ω *.*

Proof: Suppose to the contrary that $u(X) > 0$ at some point $X = (\xi,\tau) \in \Omega$, then u attains a positive maximum on $\bar{\Omega}_\tau$, $\Omega_\tau = \{(x,t);\ t \le \tau\} \cap \Omega$ at some point $\hat{X}$. Clearly $\hat{X} \notin \overline{P\Omega_\tau}$ by assumption, and by the strong maximum principle ([6], Theorem 1, page 34), $\hat{X} \notin \bar{\Omega}_\tau \sim P\Omega$ as $\limsup\{U(Y);\ Y \in \Omega_\tau,\ Y \to Z\} \le 0$ for each $Z \in P\Omega_\tau$. □

We will assume henceforth that $\Omega \in A, \psi$ is as in Section 5. Let $C(\Omega)$ denote the set of all cylinders of the form

$$U = \{(x,t) \in \mathbb{R}^{n+1};\ \|x - x_0\| < \delta,\ \ t_0 < t < t_0 + \eta\}$$

for some $(x_0,t_0) \in \mathbb{R}^{n+1}$, $\delta > 0$, $\eta > 0$ so that $U \cap \Omega \ne \emptyset$, but $\bar{U} \cap P\Omega = \emptyset$.

We call a *sub-temperature* in Ω any function u satisfying the following conditions:

(i) $-\infty \le u < \infty$, $u > -\infty$ on a dense subset of Ω ;

(ii) u is upper semicontinuous on $\bar{\Omega}$;

(iii) $u = 0$ on $P\Omega$;

(iv) if $V \in C(\Omega)$, ϕ is uniformly continuous on $\bar{\Omega} \cap PV$, $u \le \phi$ on $\bar{\Omega} \cap PV$, then $H_\phi^V \ge u$ on $\bar{\Omega} \cap V$.

Let S denote the set of all sub-temperatures.

6.3 LEMMA:

(a) $\mathcal{S} \neq \emptyset$;

(b) *If* $u,v \in \mathcal{S}$ *then* $\max\{u,v\} \in \mathcal{S}$.

Proof: From Section 5, (3), $\psi \in C^{2,1}(\Omega) \cap C(\bar{\Omega})$ satisfies $\psi = 0$ on $\mathcal{P}\Omega$ and $H\psi(X) \leq -\lambda\psi(X)\, d(X)^{-2}$ for $X \in \Omega$. Put

(7) $$v_0 = \lambda^{-1}(1 + \|f\|_{Y_\alpha})\psi ,$$

then $-v_0 \in \mathcal{S}$. Let $u,v \in \mathcal{S}$, then clearly $\max\{u,v\}$ satisfies (i)-(iii) for the definition of a subtemperature. Let $V \in \mathcal{C}(\Omega)$, ϕ uniformly continuous on $\bar{\Omega} \cap \mathcal{P}V$, $\max\{u,v\} \leq \phi$ on $\mathcal{P}V \cap \bar{\Omega}$, then $u \leq \phi$, $v \leq \phi$ on $\bar{\Omega} \cap \mathcal{P}V$ so $u,v \leq H_\phi^V$ on $\bar{\Omega} \cap V$. □

6.4 LEMMA: *Let* $u \in \mathcal{S}$, $V \in \mathcal{C}(\Omega)$. *Let* $\{u_m\}$ *be a decreasing sequence in* $C(\bar{\Omega})$ *converging pointwise in* $\bar{\Omega}$ *to* u , *and* ϕ_m *the restriction of* u_m *to* $\bar{\Omega} \cap \mathcal{P}V$. *Put* $v_m = H_{\phi_m}^V$, *then* $\{v_m\}$ *is a decreasing sequence in* $C(\overline{\Omega \cap V})$ *converging to a function* v *which is*

(a) *upper semicontinuous on* $\overline{\Omega \cap V}$,

(b) $Hv = f$ *in* $\Omega \cap V$,

(c) $v = u$ *on* $\Omega \cap \mathcal{P}V$.

Furthermore, v *on the set* $\Omega \cap V$ *does not depend on the particular sequence* $\{u_m\}$.

Proof: As $u \in \mathcal{S}$, $v_m \geq u$ on $\bar{\Omega} \cap V$. The sequence $\{v_m\}$ is decreasing by virtue of the maximum principle. Let v be defined on $\overline{\Omega \cap V}$ by

$$v(X) = \lim_{m\to\infty} v_m(X)$$

for all $X \in \overline{\Omega \cap V}$. Then v is upper semicontinuous on $\overline{\Omega \cap V}$, $v > -\infty$ on a dense subset of $\overline{\Omega \cap V}$ as $v \geq u$ on $V \cap \bar{\Omega}$, $v < +\infty$ as $v \leq v_1$.

By applying a parabolic version of Harnack's second theorem ([6], page 89) to the sequence $\{v_m - v_1\}$, we conclude that $Hv = f$ on $\Omega \cap V$. Part (c) is immediate. Now let $\{\tilde{u}_m\}$ be another decreasing sequence in $C(\bar{\Omega})$ converging pointwise on $\bar{\Omega}$ to u, and ψ_m the restriction of $\tilde{u}_m$ to $\bar{\Omega} \cap PV$. Put $w_m = H^V_{\psi_m}$; then again $\{w_m\}$ is a decreasing sequence in $C(\overline{\Omega \cap V})$. We claim that $w_m \geq v$. Note that $H(v - w_m) = 0$ in $\Omega \cap V$, $v - w_m = u - \psi_m \leq 0$ on $\bar{\Omega} \cap PV$, so by Lemma 6.2, $v - w_m \leq 0$ on $\Omega \cap V$. If $w = \lim_{m\to\infty} w_m$, then w satisfies (a), (b), (c) and $w \geq v$ on $\Omega \cap V$. Likewise $w \leq v$ on $\Omega \cap V$. □

6.5 DEFINITION: Let $u \in S$, $V \in C(\Omega)$. We define the *parabolic lift* of u on V (by analogy with [7], page 24), written L^V_u by

$$L^V_u(X) = \begin{cases} u(X) & \text{if } X \in \bar{\Omega}\backslash\bar{V} \\ v(X) & \text{if } X \in \Omega \cap V \\ \max\{u(X), \tilde{v}(X)\} & \text{if } X \in \partial(\Omega \cap V) \end{cases}$$

where v is given in Lemma 6.4 and

$$\tilde{v}(X) = \lim \sup\{v(Y);\ Y \in \Omega \cap V,\ Y \to X\}$$

for each $X \in \partial(\Omega \cap V)$. Note that $L^V_u(X) = u(X)$ for $X \in \Omega \cap PV$.

6.6 REMARK: With the notation in 6.5, let $w \in C(\overline{\Omega \cap V})$ and suppose that $Hw = f$ in $\Omega \cap V$, $w \geq u$ in $\overline{\Omega \cap V}$ then $w \geq L^V_u$ on $\overline{\Omega \cap V}$. Apply Lemma 6.2 to $L^V_u - w$.

6.7 LEMMA *(Properties of parabolic lift). Let* $u, v \in S$, $V \in C(\Omega)$, *then*

(a) $L^V_u \geq u$ *in* $\bar{\Omega}$;

(b) $L^V_u \in S$;

(c) *if* $u \geq v$ *on* $\bar{\Omega}$, *then* $L^V_u \geq L^V_v$ *on* $\bar{\Omega}$.

Proof: Property (a) is immediate from the construction of L^V_u. For (b), put $w = L_u^V$, then w satisfies (i) - (iii) of the definition of sub-temperature. Now let $W \in \mathcal{C}(\Omega)$, ϕ be uniformly continuous on $\bar{\Omega} \cap \mathcal{P}W$, $w \le \phi$ on $\bar{\Omega} \cap \mathcal{P}W$. We show that $H_\phi^W \ge w$ on $\bar{\Omega} \cap W$. Consider four cases:

(1) $V \subset W$. By definition of w , $w = u$ on $\bar{\Omega} \cap (\bar{W}\backslash V)$, and hence $u \le \phi$ on $\bar{\Omega} \cap \mathcal{P}W$, so $H_\phi^W \ge u$ on $\bar{\Omega} \cap W$. Applying Lemma 6.2 in V , $H_\phi^W \ge L_u^V = w$ on $\bar{\Omega} \cap W$.

(2) $W \subset V$. This case follows from the maximum principle applied in $\bar{\Omega} \cap W$.

(3) $W \cap \partial V \ne \emptyset$. We may assume $W \cap \mathcal{P}V \ne \emptyset$, as other situations are handled as above. Clearly $w \ge u$, so $H_\phi^W \ge u$ on $\bar{\Omega} \cap W$ and hence $H_\phi^W \ge w$ on $(\bar{\Omega} \cap W)\backslash(\bar{V}\backslash\mathcal{P}V)$. On $\overline{\Omega \cap V \cap W}$, $w - H_\phi^W$ satisfies the assumptions of Lemma 6.2, so $w \le H_\phi^W$ on $\Omega \cap V \cap W$. Now $H_\phi^W \ge u$ on $\overline{\Omega \cap V \cap W}$ so by the Remark 6.6, $H_\phi^W \ge L_u^V$ on $\overline{\Omega \cap V \cap W}$.

(4) $\bar{W} \cap \bar{V} = \emptyset$. Then $w = u$ on $\bar{\Omega} \cap \bar{V}$.

For (c), let $\{u_m\}$, $\{v_m\}$ be decreasing sequences in $C(\bar{\Omega})$ converging pointwise in $\bar{\Omega}$ to u,v respectively. Let $w_m = \min\{u_m, v_m\}$, then $\{w_m\}$ is a decreasing sequence in $C(\bar{\Omega})$ converging pointwise on $\bar{\Omega}$ to v. Let ϕ_m,ψ_m denote respectively the restrictions of u_m , w_m to $\bar{\Omega} \cap \mathcal{P}V$. Let $\bar{u} = \lim_{m\to\infty} H_{\phi_m}^V$, $\bar{v} = \lim_{m\to\infty} H_{\psi_m}^V$ on $\bar{\Omega} \cap \bar{V}$, then $\bar{u} \ge \bar{v}$ follows from $H_{\phi_m}^V \ge H_{\psi_m}^V$ on $\bar{\Omega} \cap \bar{V}$. We conclude the proof by considering the three cases in the definition of the parabolic lift. □

6.8 LEMMA: *For each* $u \in \mathcal{S}$, $u \le v_0$ *on* $\Omega \cup \mathcal{P}\Omega$ *(* v_0 *given in (7)).*

Proof: Let $u \in \mathcal{S}$ then $u - v_0$ is upper semicontinuous on $\bar{\Omega}$, equal to 0 on $\mathcal{P}\Omega$. Suppose that there exists $X = (\xi,\tau) \in \Omega$ so that $(u - v_0)(X) > 0$. Then $u - v_0$ attains a positive maximum on $\overline{\Omega_\tau}$ (see Lemma 6.2, proof) at $\hat{X}$ say. There exists $V \in \mathcal{C}(\Omega)$, $V \subset \Omega$ so that $\hat{X} \in V$ and

$$(u - v_0)(\hat{X}) = \sup\{(u - v_0)(X) ; X \in \bar{V}\} .$$

In V,

$$H(L_u^V - v_0) = f - Hv_0 \geq \psi d^{-2} \geq \rho > 0$$

where ρ depends on V,ψ as in Section 3 (ii). Hence by ([6], Lemma 1, page 34)

$$\begin{aligned}(u - v_0)(\hat{X}) &\leq (L_u^V - v_0)(\hat{X}) \\ &< \sup\{L_u^V(X) - v_0(X);\ X \in \mathcal{P}V\} \\ &= \sup\{u(X) - v_0(X);\ X \in \mathcal{P}V\} \\ &\leq \sup\{(u - v_0)(X);\ X \in \bar{V}\}\end{aligned}$$

a contradiction. □

6.9 THEOREM: *Let* u *be defined by*

$$u(X) = \sup\{v(X);\ v \in \mathcal{S}\}$$

for each $X \in \Omega \cup \mathcal{P}\Omega$. *Then* $u \in C(\Omega \cup \mathcal{P}\Omega) \cap C^{2,1}(\Omega)$ *and* $Hu = f$ *in* Ω *with* $u = 0$ *on* $\mathcal{P}\Omega$.

Proof: By Lemma 6.2 (the proof) and Lemma 6.8, $|u| \leq v_0$ on $\Omega \cup \mathcal{P}\Omega$. To prove the rest of the theorem it suffices to show that $Hu = f$ in Ω . Let $V \in \mathcal{C}(\Omega)$, $\bar{V} \subset \Omega$, $X = (\xi,\tau)$ be the centre point of V . By Lemma 6.3 (b), there exists an increasing sequence $\{u_n\}$ in $\mathcal{S}$ so that $u(X) = \lim_{n\to\infty} u_n(X)$. Let $\tilde{u}_n = L_{u_n}^V$, then $\{\tilde{u}_n\}$ is an increasing sequence in $\mathcal{S}$ and bounded above by v_0 and so by [6], page 89 the function $\tilde{u} = \lim_{n\to\infty} \tilde{u}_n$ (pointwise), satisfies $H\tilde{u} = f$ in V . Since $u_n \leq \tilde{u}_n$, $u(X) \leq \tilde{u}(X)$. But $\tilde{u}_n(X) \leq u(X)$ [Lemma 6.7 (n)], so $\tilde{u}(X) = u(X)$. Let $Y = (x,t)$ be an arbitrary point in V with $t \leq \tau$ and suppose that $\{v_n\}$ is an increasing sequence in $\mathcal{S}$ so that $u(Y) = \lim_{n\to\infty} v_n(Y)$. Let $\tilde{v}_n, \tilde{v}$ be defined as above. Then $H\tilde{v} = f$ in V and $\tilde{v}(Y) = u(Y)$. Let $w_n = \max\{v_n, u_n\}$, then $\{w_n\}$ is an increasing sequence in $\mathcal{S}$. Let $\tilde{w}_n, \tilde{w}$ be as above. Since $w_n \geq u_n$,

$\tilde{w}_n \geq \tilde{u}_n$ (Lemma 6.7 (c)) and hence $\tilde{w} \geq \tilde{u}$. But as $\tilde{w}_n \leq u$, $\tilde{w}(X) \leq u(X) = \tilde{u}(X)$, so $\tilde{w}(X) = \tilde{u}(X)$. We have $\tilde{w} \geq \tilde{u}$ on V, $H(\tilde{w}-\tilde{u}) = 0$ in V and $(\tilde{w}-\tilde{u})(X) = 0$, whence $\tilde{w} = \tilde{u}$ in $V_\tau = V \cap \{(x,t);\ t \leq \tau\}$, by the strong maximum principle ([6], page 34). As it can also be shown that $\tilde{w}(Y) = \tilde{v}(Y)$ we conclude that $u(Y) = \tilde{u}(Y)$. But $Y \in V_\tau$ was arbitrary, so $Hu = f$ in V_τ. As V was arbitrary, $Hu = f$ in Ω. □

7. SOME PROPERTIES OF CLASS A

EXAMPLE

(1) The set

$$\Omega = \{(x,t) \in \mathbb{R}^{n+1};\ t > 0,\ \|x - x_0\| < \delta\}$$

is in A, for any $\delta > 0$, $x_0 \in \mathbb{R}^n$. For $0 < \nu \leq \Lambda$, ψ defined by

$$\psi(X) = \psi(x,t) = t^{\frac{1}{2}}[1 - \exp(-\kappa(\|x - x_0\|^2 - \delta^2))]^{\beta/2}$$

where $\kappa = (n\Lambda)/(2\nu\delta^2)$ is a (ν,Λ) barrier for Ω for any $0 < \beta < 1$.

(2) A is closed under non-empty finite intersections.

(3) If G is a domain in $\mathbb{R}^n$ $(n \geq 1)$ which is "elliptic" admissible in the sense of J.H. Michael ([1], page 4), then $G \times (0,T) \in A$ for any $T > 0$. We can use examples in [1] to generate examples in A. This includes the case where G is a bounded domain in $\mathbb{R}^n$ with a C^2 boundary.

(4) A is closed under $C^{2,1}$ diffeomorphisms of $\mathbb{R}^{n+1}$.

8. CONCLUDING REMARKS

The above theory has been used to study the Dirichlet problem for operators with unbounded coefficients on special domains extending the results of J.H. Michael to parabolic equations (see [4]). These results will be published elsewhere.

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