ON AN ELLIPTIC BOUNDARY VALUE PROBLEM WITH MIXED NON-LINEAR BOUNDARY CONDITIONS

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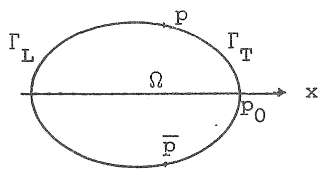
1. INTRODUCTION

E. Tuck has made a study of airflows (assumed to be irrotational of an inviscid incompressible fluid) under a thin body at a non-uniform small clearance from a plane ground surface. (See [6]). The problem is relevant to vehicle aerodynamics, especially for racing cars.

J. van der Hoek and the present author have begun an investigation whose immediate aim is to establish existence, uniqueness and regularity properties for the model used by Tuck. This paper is a report of some of that work.

We take the body to be fixed and the flow to have a uniform velocity at infinity of 1 in the positive x-direction. The plan form of the body is assumed to be a bounded convex domain Ω in $\mathbb{R}^2$ which is symmetric with respect to the x-axis and has a smooth boundary $\partial\Omega$. The height of the body above the ground surface is given by $z = a(x,y)$ where a is a positive smooth function on $\overline{\Omega}$ satisfying $a(x,y) = a(x,-y)$. Let $\varphi(x,y,z)$ be the velocity potential of the flow so that $\varphi(x,y,z) = x$ at infinity. For points $q \in \partial\Omega$ let $\nu(q)$ be the outward pointing unit normal to $\partial\Omega$ and $\tau(q)$ the clockwise pointing unit tangent. The boundary decomposes in the form $\partial\Omega = \Gamma_L \cup \Gamma_T \cup \{p,\overline{p}\}$ where Γ_L (the leading edge) and Γ_T (the trailing edge) are connected relatively open subsets of $\partial\Omega$ separated by transition points $p,\bar{p}$ which are symmetrically positioned with respect to the x-axis. See the diagram.



Tuck shows that with $u(x,y) = \varphi(x,y,0)$ the following boundary value problem arises

(1.1) $$\begin{cases} \operatorname{div} a \operatorname{grad} u = 0 & \text{in } \Omega \\ u = x & \text{on } \Gamma_L \\ |\nabla u| = 1 & \text{on } \Gamma_T \end{cases}$$

together with the supplementary condition

(1.2) $$\frac{\partial u}{\partial \nu} = -\frac{\partial x}{\partial \nu} \quad \text{at } p,\bar{p} .$$

The problem is to determine the transition points $p,\bar{p}$ as well as the funtion $u \in C^1(\bar{\Omega})$. We work in Sobolev spaces $H^\sigma(\Omega)$, σ real, defined as in Lions and Magenes [2] for example. Recall the Sobolev embedding theorem which gives $H^\sigma(\Omega) \subset C^1(\bar{\Omega})$ for $\sigma > 2$.

Remarks

(1.3) If $u \in C^1(\bar{\Omega})$ satisfies the two boundary conditions of (1.1) then at $p,\bar{p}$ we have

$$\left(\frac{\partial u}{\partial \nu}\right)^2 = 1 - \left(\frac{\partial u}{\partial \tau}\right)^2$$

$$= 1 - \left(\frac{\partial x}{\partial \tau}\right)^2$$

$$= \left(\frac{\partial x}{\partial \nu}\right)^2$$

so that $\frac{\partial u}{\partial \nu} = \pm \frac{\partial x}{\partial \nu}$ and the supplementary condition simply picks out the second alternative.

(1.4) If $a(x,y) = a(y)$ on $\overline{\Omega}$ then $u(x,y) = x$ defines a solution of (1.1), (1.2) whenever $p,\overline{p}$ are on the wing tips of Ω (that is, the set of points $q \in \partial\Omega$ of maximal distance from the x-axis). The interesting case is $a(x,y) \neq a(y)$, whereupon the velocity under the body may increase, the pressure therefore decreasing and adding to the stability of the body.

(1.5) Some supplementary condition is needed. Without it we could take $\Gamma_T = \emptyset$, $\Gamma_L = \partial\Omega$, $p = \overline{p} = p_0$ the point $(x,0) \in \partial\Omega$ with maximal x, and obtain a unique solution $u = u_0 \in C^{\infty}(\overline{\Omega})$. However, the underlying physical problem demands that the trailing edge Γ_T be non-empty. This is assured by condition (1.2) if we make the interpretation $p = \overline{p} = p_0$ when $\Gamma_T = \emptyset$. Indeed by the Hopf maximum principle (Gilbarg and Trudinger [1]) $\frac{\partial u_0}{\partial \nu}(p_0) > 0$ so that (1.2) is not satisfied.

2. LINEARIZATIONS

Tuck considered the linearization obtained by setting $v = u-x$ and taking $|\text{grad } v|^2 \approx 0$ on Γ_T. Dropping the supplementary condition and setting $A = \operatorname{div} a \operatorname{grad}$, this gives

$$(2.1) \qquad \begin{cases} Av = -a_x & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_L \\ \dfrac{\partial v}{\partial x} = 0 & \text{on } \Gamma_T \ . \end{cases}$$

Pryde and van der Hoek [3], [5] have looked at problem (2.1) with p treated as a parameter, and obtained various existence, uniqueness and regularity results.

In this paper we consider a different linearization and we use it to obtain an existence result related to problem (1.1), (1.2). Again treat p as a parameter, and for sufficiently smooth functions f,g,h consider the more general problem

$$(2.2) \qquad \begin{cases} Au = f & \text{in } \Omega \\ u = g & \text{on } \Gamma_L \\ |\nabla u| = h & \text{on } \Gamma_T \ . \end{cases}$$

We shall see later (proposition 2.16) that under certain natural conditions on f,g there is a unique non-zero constant κ such that the following problem has a solution $\mu \in H^\sigma(\Omega)$ for sufficiently small $\sigma > 2$:

$$(2.3) \qquad \begin{cases} A\mu = f & \text{in } \Omega \\ \mu = g & \text{on } \Gamma_L \\ \dfrac{\partial \mu}{\partial \nu} = \kappa & \text{on } \Gamma_T \ . \end{cases}$$

Setting $v = u-\mu$, $B = 2\kappa \frac{\partial}{\partial \nu} + 2 \frac{\partial \mu}{\partial \tau} \frac{\partial}{\partial \tau}$ and $\hat{h} = h - |\nabla \mu|^2$, we find that u satisfies (2.2) if and only if v satisfies

$$(2.4) \qquad \begin{cases} Av = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_L \\ Bv = -|\nabla v|^2 + \hat{h} & \text{on } \Gamma_T \ . \end{cases}$$

In section 3, for appropriate choices of f,g,h, we shall use the following linearization of (2.4) or (2.2)

$$(2.5) \qquad \begin{cases} Av = f_1 & \text{in } \Omega \\ v = g_1 & \text{on } \Gamma_L \\ Bv = h_1 & \text{on } \Gamma_T \ . \end{cases}$$

Associated with (2.5) is an adjoint problem. Indeed, define $Cw = a \frac{\partial w}{\partial \nu} - \frac{\partial}{\partial \tau} (\frac{a}{\kappa} \frac{\partial \mu}{\partial \tau} w)$. Then for $v,w \in H^2(\Omega)$ the following Green's formula is valid

$$(2.6) \qquad (Av,w) - (v,Aw) = \langle Bv, \frac{a}{2\kappa} w \rangle - \langle v, Cw \rangle$$

where $(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle$ denote the $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products respectively. The associated (homogeneous) adjoint problem to (2.5) is

$$(2.7) \qquad \begin{cases} Aw = 0 & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_L \\ Cw = 0 & \text{on } \Gamma_T \ . \end{cases}$$

To see more closely the relationship between (2.5) and (2.7) consider the operator

$(A,\gamma_L,B_T)_\sigma : H^\sigma(\Omega) \to H^{\sigma-2}(\Omega) \times H^{\sigma-\frac{1}{2}}(\Gamma_L) \times H^{\sigma-1\frac{1}{2}}(\Gamma_T)$ defined by $(A,\gamma_L,B_T)_\sigma u = (Au,\gamma_L u,B_T u)$ for $u \in H^\sigma(\Omega)$, $\sigma > 2$, where $\gamma_L u = u/\Gamma_L$ and $B_T u = Bu/\Gamma_T$. The spaces $H^\sigma(\partial\Omega)$, for σ real, are defined in [2] and the spaces $H^\sigma(\Gamma)$, for $\Gamma = \Gamma_L$ or Γ_T, are the spaces of restrictions to Γ of distributions in $H^\sigma(\partial\Omega)$ together with the natural infimum norm. The following result is proved in Pryde [4] :

THEOREM 2.8. *For sufficiently small* $\sigma > 2$ *the operator* $(A,\gamma_L,B_T)_\sigma$ *is injective with closed range of codimension 2.*

The range of (A,γ_L,B_T) is identified via the adjoint problem (2.7). For this, let $H_A^{2-\sigma}(\Omega)$ denote the space of $u \in H^{2-\sigma}(\Omega)$ such that $Au \in L^2(\Omega)$, provided with the graph norm. Then $C^\infty(\overline{\Omega})$ is dense in $H_A^{2-\sigma}(\Omega)$ for all real σ. Moreover, γ_L and C_T, defined by $\gamma_L w = w/\Gamma_L$ and $C_T w = Cw/\Gamma_T$ on smooth functions, extend to bounded operators

γ_L: $H_A^{2-\sigma}(\Omega) \to H^{1\frac{1}{2}-\sigma}(\Gamma_L)$ and C_T: $H_A^{2-\sigma}(\Omega) \to H^{\frac{1}{2}-\sigma}(\Gamma_T)$. We use these operators to give meaning to w/Γ_L and Cw/Γ_T when $w \notin C^1(\overline{\Omega})$. It is also proved in [4] that

$(A,\gamma_L,C_T) : H_A^{2-\sigma}(\Omega) \to L^2(\Omega) \times H^{1\frac{1}{2}-\sigma}(\Gamma_L) \times H^{\frac{1}{2}-\sigma}(\Gamma_T)$ is surjective with kernel of dimension 2 for sufficiently small $\sigma > 2$. In particular

PROPOSITION 2.9 *For sufficiently small* $\sigma > 2$ *the space of solutions* $w \in H^{2-\sigma}(\Omega)$ *of problem (2.7) has dimension 2.*

It is also the case that the null spaces of the proposition all coincide for sufficiently small $\sigma > 2$, and that their members all belong to $L^2(\Omega)$. See [4].

For σ real and $\Gamma = \Gamma_L$ or Γ_T, we introduce the spaces $\overset{\circ}{H}{}^{\sigma}(\Gamma)$ defined as the subspaces of $H^{\sigma}(\partial\Omega)$ consisting of distributions with support in $\overline{\Gamma}$. Then $H^{\sigma}(\Gamma)$ and $\overset{\circ}{H}{}^{-\sigma}(\Gamma)$ are mutually dual with respect to the pairing given by the extension of the $L^2(\Gamma)$ inner product on the dense subspaces of smooth functions. Let $\langle \cdot\, , \cdot \rangle_{\Gamma_T}$ denote this natural pairing on $H^{\sigma-1\frac{1}{2}}(\Gamma_T) \times \overset{\circ}{H}{}^{1-\sigma}(\Gamma_T)$ and $\langle \cdot\, , \cdot \rangle_{\Gamma_L}$ the pairing on $H^{\sigma-\frac{1}{2}}(\Gamma_L) \times \overset{\circ}{H}{}^{\frac{1}{2}-\sigma}(\Gamma_L)$. It follows from Green's formula (2.6), theorem 2.8 and proposition 2.9 that

COROLLARY 2.10 *For sufficiently small* $\sigma > 2$, *and* $(f_1,g_1,h_1) \in H^{\sigma-2}(\Omega) \times H^{\sigma-\frac{1}{2}}(\Gamma_L) \times H^{\sigma-1\frac{1}{2}}(\Gamma_T)$, *the problem (2.5) has a solution* $v \in H^{\sigma}(\Omega)$ *if and only if* $(f_1,w) = \langle h_1, \frac{a}{2K} w\rangle_{\Gamma_T} - \langle g_1, Cw\rangle_{\Gamma_L}$ *for all solutions* $w \in L^2(\Omega)$ *of problem (2.7). Moreover, when a solution exists it is unique.*

We now take account of the symmetry of our original problem. A continuous function v on $S = \Omega$, $\partial\Omega$, Γ_L or Γ_T will be called even if $v(x,y) = v(x,-y)$, odd if $v(x,y) = -v(x,-y)$, for all $(x,y) \in S$. So a continuous function v on $\overline{S}$ is even (odd) if and only if $\int_S v\varphi = 0$ for all smooth odd (even) functions φ on S. We define a distribution v on S to be even (odd) if $\langle v,\varphi\rangle = 0$ for all odd (even) test functions φ on S, where $\langle \cdot,\cdot \rangle$ here denotes the pairing between distributions and test functions.

The height function a is even and the operators A, γ_L, B_T,

C_T all preserve even-ness and odd-ness. In particular, if f_1, g_1, h_1 are even, so is the unique solution of (2.5) when it exists. Moreover, it is proved in [4] that

PROPOSITION 2.11 *For sufficiently small* $\sigma > 2$ *, the space of solutions* $w \in H^{2-\sigma}(\Omega)$ *of problem (2.7) has a basis consisting of an even function* w_C *and an odd function* $\tilde{w}_C$.

COROLLARY 2.12 *For sufficiently small* $\sigma > 2$*, and an even triple* $(f_1, g_1, h_1) \in H^{\sigma-2}(\Omega) \times H^{\sigma-\frac{1}{2}}(\Gamma_L) \times H^{\sigma-1\frac{1}{2}}(\Gamma_T)$ *, the problem (2.5) has a solution* $v \in H^{\sigma}(\Omega)$ *if and only if* $(f_1, w_C) = \langle h_1, \frac{a}{2\kappa} w_C \rangle_{\Gamma_T} - \langle g_1, Cw_C \rangle_{\Gamma_L}$ *where* w_C *is a non-zero even solution in* $L^2(\Omega)$ *of problem (2.7).*

We return to problem (2.3). It can be considered as a special case of (2.5) with coefficients 2κ and $2 \frac{\partial \mu}{\partial \tau}$ in B replaced by 1 and 0 respectively. In place of the dual problem (2.7) we have

$$(2.13) \qquad \begin{cases} Aw = 0 & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_L \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_T \end{cases} .$$

From corollary 2.12 we obtain

COROLLARY 2.14 *For sufficiently small* $\sigma > 2$ *, and an even triple* $(f, g, \kappa) \in H^{\sigma-2}(\Omega) \times H^{\sigma-\frac{1}{2}}(\Gamma_L) \times H^{\sigma-1\frac{1}{2}}(\Gamma_T)$ *, the problem (2.3) has a solution* $\mu \in H^{\sigma}(\Omega)$ *if and only if* $(f, w_e) = \langle \kappa, aw_e \rangle_{\Gamma_T} - \langle g, a \frac{\partial w_e}{\partial \nu} \rangle_{\Gamma_L}$ *where* w_e *is a non-zero even solution in* $L^2(\Omega)$ *of problem (2.13).*

Consider now problem (2.3) with $(f,g,\kappa) = (0,0,1)$. By the Hopf maximum principle, for $\sigma > 2$ there is no solution $\mu \in H^{\sigma}(\Omega) \subset C^1(\overline{\Omega})$. Indeed if $\mu \in C^1(\overline{\Omega})$ were a solution then $\frac{\partial\mu}{\partial\nu} > 1$ at a point where μ achieves its minimum. By corollary 2.14 we conclude that $\langle 1, a\, w_e \rangle_{\Gamma_T} \neq 0$.

In particular, for a general even pair $(f,g) \in H^{\sigma-2}(\Omega) \times H^{\sigma-\frac{1}{2}}(\Gamma_L)$ we can set $\kappa = \langle 1, a\, w_e \rangle_{\Gamma_T}^{-1} [(f,w_e) + \langle g, a \frac{\partial w_e}{\partial\nu} \rangle_{\Gamma_L}]$ and by corollary 2.14, problem (2.3) has a unique solution $\mu \in H^{\sigma}(\Omega)$. Moreover, κ is the unique constant for which there is a solution in $H^{\sigma}(\Omega)$.

Finally, suppose in addition that (f,g) satisfies the condition

$$\text{(2.15)} \qquad \begin{array}{lll} \text{either} & f \geq 0 & \text{and} \quad \max\{g(q) : q \in \overline{\Gamma}_L\} = g(p) \\ \text{or} & f \leq 0 & \text{and} \quad \min\{g(q) : q \in \overline{\Gamma}_L\} = g(p) \ . \end{array}$$

For example, $(f,g) \equiv (0,x)$, as in the introduction, satisfies condition (2.15). Then, by the maximum principle, μ achieves its maximum (or minimum) or the boundary $\partial\Omega$ and therefore at a point $q \in \overline{\Gamma}_T$. By the Hopf maximum principle $\kappa = \frac{\partial\mu}{\partial\nu}(q) > 0$ (or < 0) . So

PROPOSITION 2.16 *For sufficiently small* $\sigma > 2$ *and an even pair* $(f,g) \in H^{\sigma-2}(\Omega) \times H^{\sigma-\frac{1}{2}}(\Gamma_L)$ *satisfying condition (2.15), there is a unique constant* $\kappa \neq 0$ *for which problem (2.3) has a solution* $\mu \in H^{\sigma}(\Omega)$. *Moreover, the solution* μ *, when it exists, is unique and even.*

3. AN EXISTENCE THEOREM

We obtain an existence result for a non-linear problem closely related to problems (2.2) and (2.4). This is done using the contraction mapping theorem and knowledge of the solvability of the linearized problem (2.5) as given by corollary 2.12.

For sufficiently small $\sigma' > 2$, let $f \in H^{\sigma'-2}(\Omega)$ and $g \in H^{\sigma'-\frac{1}{2}}(\Gamma_L)$ be even functions satisfying condition (2.15). Apply proposition 2.16 to obtain a constant $\kappa \neq 0$ such that (2.3) has a unique solution $\mu \in H^{\sigma'}(\Omega)$. Let B and C be the boundary operators determined by κ as in section 2, and $w_C \in L^2(\Omega)$ a non-zero even solution of (2.7). Let $\sigma \in (0,\sigma']$ be sufficiently small that the conclusions of corollary 2.12 are valid. Let $h \in H^{\sigma-1\frac{1}{2}}(\Gamma_T)$ be an even function and set $\hat{h} = h - |\nabla\mu|^2$.

Recall that $H^s(\Omega)$ is an algebra for $s > 1$ with $\|vw\| \leq c\|v\|\ \|w\|$. In particular, if $v \in H^\sigma(\Omega)$ with $\sigma > 2$, then $|\nabla v|^2 / \Gamma_T \in H^{\sigma-1\frac{1}{2}}(\Gamma_T)$. So $\hat{h} \in H^{\sigma-1\frac{1}{2}}(\Gamma_T)$.

With constant β to be chosen, we define a closed subspace V of $H^\sigma(\Omega)$ by $V = \{v \in H^\sigma(\Omega) : v$ is even, Av is constant, $\gamma_L v = 0$, $\|v;H^\sigma(\Omega)\| \leq \beta\}$ and a (non-linear) mapping $T : V \to V$ by $Tv = w$, where

$$
(3.1) \qquad \begin{cases} Aw = \lambda \quad \text{(constant) in } \Omega \\ w = 0 \quad \text{on } \Gamma_L \\ Bw = -|\nabla v|^2 + \hat{h} \quad \text{on } \Gamma_T . \end{cases}
$$

By corollary 2.12 and the Hopf maximum principle $(1,w_C) \neq 0$. Hence by corollary 2.12 there is a unique constant, namely $\lambda = (1,w_C)^{-1} \langle \hat{h} - |\nabla v|^2 , \frac{a}{2K} w_C \rangle_{\Gamma_T}$, for which (3.1) has a solution $w \in H^\sigma(\Omega)$. Moreover, w is even and unique.

To show that $w \in V$ it only remains to verify the norm condition. For this

$$\|w\,;\,H^\sigma(\Omega)\| \leq C_1 \left(\|Aw;H^{\sigma-2}(\Omega)\| + \|\gamma_L w\,;\,H^{\sigma-\frac{1}{2}}(\Gamma_L)\| + \|B_T w\,;\,H^{\sigma-1\frac{1}{2}}(\Gamma_T)\|\right)$$

(by corollary 2.12, since the set of even functions in any of these Sobolev spaces is a closed subspace)

$$\leq C_2 \left(|\lambda| + \|\hat{h} - |\nabla v|^2;\, H^{\sigma-1\frac{1}{2}}(\Gamma_T)\|\right)$$

$$\leq C_3 \|\hat{h} - |\nabla v|^2;\, H^{\sigma-1\frac{1}{2}}(\Gamma_T)\|$$

$$\leq C_4\ \beta^2$$

(provided $\|\hat{h}\,;\,H^{\sigma-1\frac{1}{2}}(\Gamma_T)\| \leq \beta^2$)

$$\leq \beta$$

(provided $\beta \leq \frac{1}{C_4}$).

Furthermore, T is a contraction for sufficiently small β. Indeed suppose $v_j \in V$ for $j = 1,2$ with $Tv_j = w_j$ and $Aw_j = \lambda_j$.

Then

$$\|Tv_1 - Tv_2 \; ; \; H^{\sigma}(\Omega)\|$$

$$= \|w_1 - w_2 \; ; \; H^{\sigma}(\Omega)\|$$

$$\leq C_2(|\lambda_1 - \lambda_2| + \||\nabla v_1|^2 - |\nabla v_2|^2 \; ; \; H^{\sigma-1\frac{1}{2}}(\Gamma_T)\|)$$

$$\leq C_3 \; \||\nabla v_1|^2 - |\nabla v_2|^2 \; ; \; H^{\sigma-1\frac{1}{2}}(\Gamma_T)\|$$

$$\leq C_5 \; \|\nabla(v_1 - v_2) \; ; \; H^{\sigma-1}(\Omega)\| \; \|\nabla(v_1 + v_2) \; ; \; H^{\sigma-1}(\Omega)\|$$

$$\leq C_6 \; \beta \; \|v_1 - v_2 \; ; \; H^{\sigma}(\Omega)\|$$

$$\leq \tfrac{1}{2} \; \|v_1 - v_2 \; ; \; H^{\sigma}(\Omega)\|$$

(provided $\beta \leq \dfrac{1}{2C_6}$) .

By the contraction mapping theorem, T has a unique fixed point in V . So we have proved

THEOREM 3.2 *Given* $\sigma' > 2$ *and even functions* $f \in H^{\sigma'-2}(\Omega)$, $g \in H^{\sigma'-\frac{1}{2}}(\Gamma_L)$ *satisfying condition (2.15) there exists* $\beta > 0$, $\sigma \in (2,\sigma']$ *and* $\mu \in H^{\sigma}(\Omega)$ *such that if* $h \in H^{\sigma-1\frac{1}{2}}(\Gamma_T)$ *is an even function with* $\|h - |\nabla\mu|^2 \; ; \; H^{\sigma-1\frac{1}{2}}(\Gamma_T)\| \leq \beta^2$ *than there exists a unique constant* λ *for which the problem*

$$\begin{cases} Au = f+\lambda & \text{on } \Omega \\ u = g & \text{on } \Gamma_L \\ |\nabla u|^2 = h & \text{on } \Gamma_T \end{cases}$$

has a solution $u \in H^{\sigma}(\Omega)$ *satisfying* $\|u-\mu\ ;\ H^{\sigma}(\Omega)\| \leq \beta$. *Moreover, the solution* u *when it exists is unique.*

Of course this theorem is only a preliminary step towards our goal of proving existence, uniqueness, and regularity for problem (1.1), (1.2). In it, the supplementary condition has been suppressed. But it is our conjecture that for certain choices of (f,g,h) , including $(f,g,h) = (0,x,1)$, the constant $\lambda = \lambda(p)$ will be 0 for appropriate choice of $p \in \partial\Omega$, and that the supplementary condition will then hold. Numerical evidence for this conjecture is provided in Tuck [6].

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