# Smooth Foliations Generated by Functions of Least Gradient

by

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The work that is outlined below has been done jointly with Harold Parks, Oregon State University.

Let $\Omega \subset R^n$ be a bounded open set and suppose $u \in BV(\Omega)$. The function is said to be of least gradient with respect to $\Omega$ if for each $v \in BV(\Omega)$ such that $u = v$ outside some compact subset of $\Omega$,

$$\int_\Omega |\nabla u| \leq \int_\Omega |\nabla v| .$$

A function of least gradient need not be continuous. Indeed, for any subset $A \subset \Omega$, the portion of the reduced boundary of $A$ which lies in $\Omega$ is area minimizing if and only if the characteristic function of $A$ is of least gradient.

In this work we consider the question of regularity of functions of least gradient subject to boundary constraints. Thus, we consider an open, bounded set $\Omega \subset R^n$ that is uniformly convex. We also assume that $\Omega$ is smoothly $(C^\infty)$ bounded. Let $\phi : \text{bdry } \Omega \to R^1$ be smooth and consider the variational problem

(1) $$\inf \left\{ \int_\Omega |\nabla u| : u = \phi \text{ on bdry } \Omega \right\}$$

where the infimum is taken over all Lipschitzian $u$. It was shown in [PH1], [PH2] that the variational problem (1) admits a unique extremal. The Euler-

Lagrange equation associated with (1) is

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \ .$$

Unfortunately, this equation is useless in investigating the regularity of $u$ for it falls outside the scope of the usual elliptic theory. In fact, the following example shows that solutions to (1) are not smooth everywhere. Let

$$\Omega = R^2 \cap \{(x,y) : x^2 + y^2 \le 1\}$$

and for $(\cos\theta, \sin\theta) \in \text{bdry}\ \Omega$, let

$$\phi(\cos\theta, \sin\theta) = \cos(2\theta) \ .$$

The function $u$ defined by

$$u(x,y) = \begin{cases} 2x^2 - 1 & \text{if} \quad x \ge 1/\sqrt{2}, \quad y \le 1/\sqrt{2} \\ 0 & \text{if} \quad x \le 1/\sqrt{2}, \quad y \le 1/\sqrt{2} \\ 1-2y^2 & \text{if} \quad x \le 1/\ 2, \quad y \ge 1/\sqrt{2} \end{cases}$$

is easily seen to be a solution to (1). However, $u$ is not smooth on $\Omega$ as $\nabla u$ does not exist on

$$\Omega \cap \{(x,y) : |x| = 1/\sqrt{2} \text{ or } |y| = 1/\sqrt{2}\} \ .$$

However, we do obtain a result concerning the partial regularity of $u$ .

*Theorem 1.* *Let* $2 \le n \le 7$ . *If* $u$ *is a solution of the variational problem* (1), *then* $u$ *is smooth on an open dense subset of* $\Omega$.

The proof of Theorem 1 will be sketched below. The reason for the restriction $2 \le n \le 7$ is that then it is known that for all but countably many $t$

$\Omega \cap u^{-1}(t)$ is a smooth area-minimizing hypersurface. If $n > 7$, then $\Omega \cap u^{-1}(t)$ may admit singularities. An essential fact underlying the proof of Theorem 1 is that the behavior of $\nabla u$ at one point of $\Omega \cap u^{-1}(t)$ determines the behavior of $\nabla u$ on all of $\Omega \cap u^{-1}(t)$. Indeed, if $\nabla u(x_0) = 0$ for some $x_0 \in \Omega \cap u^{-1}(t)$, then $\nabla u(x) = 0$ for all $x \in \Omega \cap u^{-1}(t)$. In this case we do not know of any method to prove smoothness of $u$ near $\Omega \cap u^{-1}(t)$. If, instead, $\nabla u(x_0) = 0$ is not true, i.e., if $\nabla u(x_0) \neq 0$ or $\nabla u(x_0)$ does not exist for some $x_0 \in \Omega \quad u^{-1}(t)$ and hence for every $x \in \Omega \cap u^{-1}(t)$, then it is possible to construct a solution of Jacobi's equation on $\Omega \cap u^{-1}(t)$ which has a positive lower bound. Jacobi's equation is an elliptic equation which a flow of minimal surfaces starting at $\Omega \cap u^{-1}(t)$ must initially satisfy. Once such a solution to Jacobi's equation is assured, then it follows that minimal surfaces near $\Omega \cap u^{-1}(t)$ vary smoothly as a function of their boundaries, i.e., the surfaces $\Omega \cap u^{-1}(s)$ generate a smooth foliation, for $s$ close to $t$.

We now give a few details. Let $\Gamma$ denote bdry $\Omega$. Consider a value of $t$, say 0, such that $\Omega \cap u^{-1}(0)$ satisfies the following conditions:

(i) $|\bar{\Omega} \cap u^{-1}(0)| = 0$, $H^{n-1}[\Gamma \cap \phi^{-1}(0)] = 0$; here $H^{n-1}$ denotes Hausdorff (n-1)-measure.

(ii) $\nabla\phi(x) \neq 0$ for all $x \in \Gamma \cap u^{-1}(0)$

(iii) $\Omega \cap u^{-1}(0)$ is connected

(iv) there exist $x_0 \in \Omega \cap u^{-1}(0)$, a sequence $\{t_i\} \to 0$, and a sequence $\{x_i\}$ with $x_i \in \Omega \cap u^{-1}(t_i)$ and $\lim x_i = x_0$ such that

$$0 < \liminf_{i \to \infty} \frac{|u(x_i) - u(x_0)|}{|x_i - x_0|}$$

For each $x \in \Omega \cap u^{-1}(0)$, let $N(x)$ denote the unit normal to $\Omega \cap u^{-1}(0)$ and let $w_r(x)$ be that number such that

$$x + w_r(x)N(x) \in \Omega \cap u^{-1}(r) .$$

If $\eta$ is a test function on $\Omega \cap u^{-1}(0)$, then the area of the surface

$$(2) \qquad x + (w_r(x) + t\eta(x))N(x)$$

is minimized when $t = 0$. A calculation of the first variation yields an equation, when written in local coordinates, of the form

$$(3) \qquad D_i(a^{ij}(x,w_r,\nabla w_r)D_j w_r) = w_r B_1(x,w_r,\nabla w_r) + B_2(x,\nabla w_r) .$$

Because of the estimates in [AW] and [SL], the terms $a^{ij}(x,w_r,\nabla w_r)$, $B_1(x,w_r,\nabla w_r)$ and $B_2(x,\nabla w_r)$ are uniformly bounded relative to $r$.

We now wish to investigate Jacobi's equation. By definition, it is the second variation of (2) or equivalently, the equation of variation of (3). A straightforward calculation shows that Jacobi's equation is linear. If we let

$$\omega_r = w_r/r ,$$

then Harnack's inequality applied to (3) along with (iv) above imply that on each compact subset $K$ of $\Omega \cap u^{-1}(0)$, $\omega_r$ is uniformly bounded above for all sufficiently small $r > 0$. Appealing to Harnack's inequality again, we find that $\omega_r$ is Hölder continuous of order $\alpha$, where $\alpha$ is independent of $r$. Therefore, it follows that, for a suitable subsequence, and for each compact subset $K \subset \Omega \cap u^{-1}(0)$, $\omega_r$ converges uniformly to a function $\zeta$. Because the extremal $u$ to problem (1) is Lipschitz (with constant $M$) it follows that

$$\zeta(x) \geq 1/M > 0$$

for each $x \in \Omega \cap u^{-1}(0)$ . Moreover, we have already seen that $\omega_r$, and therefore $\zeta$, is bounded above on each compact subset of $\Omega \cap u^{-1}(0)$.

The essential feature of $\zeta$ is that it can be shown to be a solution of Jacobi's equation. The fact that $\zeta$ is bounded above and away from 0 is critical for it implies the following

Theorem 2. *If* $\zeta^*$ *is a solution of Jacobi's equation on* $\Omega \cap u^{-1}(0)$ *with*

$$\zeta^* | \Gamma \cap u^{-1}(0) = 0$$

*then* $\zeta^* \equiv 0$ .

Proof. Suppose there is a point $x^1 \in \Omega \cap u^{-1}(0)$ such that $\zeta^*(x_1) > 0$ . then there is $c \in R^1$ and $x_2 \in \Omega \cap u^{-1}(0)$ such that

$$c\zeta^*(x) \leq \zeta(x)$$

for all $x \in \Omega \cap u^{-1}(0)$ and

$$c\zeta^*(x_2) = \zeta(x_2) .$$

But then $\zeta - c\zeta^* \geq 0$ is a solution of Jacobi's equation that vanishes at $x_2$. Hence, Harnack's inequality implies that $\zeta - c\zeta^* \equiv 0$ which is impossible since

$$\zeta \geq 1/M \quad \text{and} \quad \zeta^* | \Gamma \cap u^{-1}(0) = 0 .$$

This result along with assumption (ii) above now yield the following, which is our main result. The proof follows essentially from [WB, 3.1] or from an adaptation of the methods in [MC, §6.8.6].

*Theorem* 3. *There exists an open set* $W \subset \Omega$ *with*

$$\Omega \cap u^{-1}(0) \subset W$$

*such that* $u|W$ *is smooth*.

*Corollary*. *There exists an open, dense subset* $U \subset \Omega$ *such that* $u|U$ is *smooth*.

Proof. Let

$$N_1 = \text{bdry}\, \Omega \cap \{x : \nabla\phi(x) = 0\},$$
$$N_2 = \Omega \cap \{x : \nabla u(x) = 0\} .$$

It follows from Sard's theorem that $\phi(N_1)$ has Lebesgue measure 0 and because $u$ is Lipschitz the co-area formula [FH, §3.2.12] can be applied to conclude that

$$H^{n-1}[u^{-1}(t) \cap N_2] = 0 \quad \text{for a.e. } t.$$

Let $x \in \Omega$ and let $B \subset \Omega$ be an open ball containing $x$. If $u$ is constant on $B$, then of course $u$ is smooth on $B$. If not, then $u(B)$ is an interval. Choose $t \in u(B)$ such that $t \notin \phi(N_1)$ and $H^{n-1}[u^{-1}(t) \cap N_2] = 0$. Then it follows from Theorem 3 that there is an open set $W_t \supset \Omega \cap u^{-1}(t)$ such that $W_t \cap B \neq 0$ and $u|W_t$ is smooth. The result now follows if $U$ is defined as the union of all such $W_t$ and all open balls $B \subset \Omega$ such that $u|B$ is constant.

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