ISOLATED SINGULARITIES FOR EXTREMA OF GEOMETRIC VARIATIONAL PROBLEMS

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We here want to consider asymptotic behaviour on approach to an isolated singularity of an extremal u of a functional $\mathcal{F}(u)$ of the form

$$(*) \qquad \mathcal{F}(u) = \int_{B_1(0)} F(x,u,Du)\,dx \,,$$

where F is a given function and $B_1(0)$ is the open unit ball in $\mathbb{R}^n$. u is allowed to be vector-valued with values $u(x) = (u^1(x),\ldots,u^N(x)) \in \mathbb{R}^N$. What we have to say here has a natural generalization to the case when the domain of integration $B_1(0)$ in $(*)$ is replaced by a conical domain C_1 of the form $\{\lambda w : 0<\lambda<1,\ \omega\in\Sigma\}$, where Σ is some smooth embedded submanifold of S^{n-1}, and also to the case when $u(x) = u(r\omega)$ $(r=|x|, \omega=x/|x|)$ is a section of some vector bundle over Σ for each fixed r. For these generalizations (which are important, for example, for applications to minimal submanifolds) we refer to the paper [SL1]. In any case the essential ideas are the same in this less general setting.

Our main aim is to discuss asymptotic behaviour of an extremal $u = u(r\omega)$ of $(*)$ as $r\downarrow 0$, in case u has an isolated discontinuity at 0; notice that by an extremal of $\mathcal{F}(u)$ we mean a function u which satisfies the Euler-Lagrange system of $(*)$ in $B_1(0) \sim \{0\}$; thus u satisfies

$$(1) \qquad \mathcal{N}u = 0 \quad \text{in} \quad B_1(0) \sim \{0\} \,,$$

where $\mathcal{N}u$ is the second order quasilinear operator (with values in $\mathbb{R}^N$)

characterized by

$$(\mathcal{N}u,\zeta) = -\text{grad}\, \mathcal{F}(u)(\zeta)$$

$$= -\frac{d}{ds}\mathcal{F}(u+s\zeta)\Big|_{s=0} , \quad \zeta \in C_c^2(B_1(0)) .$$

We want to be able to say that such a extremal u satisfies

(2) $$\lim_{r\downarrow 0} u(r\omega) = \phi(\omega)$$

for some smooth function ϕ on S^{n-1} where the limit is relative to the $C^2(S^{n-1})$ norm. It is largely an open question when this is true. Notice that when it *is* true, we get quite a good picture of the discontinuity at 0, because (2) says

$$u(r\omega) = \phi(\omega) + \psi(r\omega)$$

where ψ is continous at 0 with $\psi(0) = 0$.

Here we discuss some conditions which are sufficient to guarantee (2). It is first necessary to impose some restrictions on the function F in (*): Specifically we assume that we can write

(3) $$F(x,\zeta,p) = r^{\gamma}\left[F_1(\omega,\zeta,rp) + F_2(r,\omega,\zeta,rp)\right]$$

where γ is a constant greater than $-n$, F_1, F_2 are smooth functions on $S^{n-1} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ and $\mathbb{R} \times S^{n-1} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ respectively, with

(4) $$\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \frac{\partial^2 F_1}{\partial p_i^\alpha\, \partial p_j^\beta}(\omega,\zeta,p)\, \xi^\alpha \xi^\beta \lambda_i \lambda_j > 0 , \quad \lambda \in \mathbb{R}^n \sim \{0\} , \quad \xi \in \mathbb{R}^N \sim \{0\} ,$$

(5) $$\sum_{\alpha=1}^{N} q^\alpha \frac{\partial F_1}{\partial q^\alpha}(\omega,\zeta, q\otimes\omega + p) > 0 , \quad q \in \mathbb{R}^N \sim \{0\} , \quad p \in \mathbb{R}^N \otimes T_\omega S^{n-1} ,$$

and

(6) $\quad \sup_{0<r<1,\ \omega\in S^{n-1},\ |\zeta|+|p|\le R}\ r^{-\varepsilon_0}(|F_2| + r|\bar{D}F_2| + r^2|\bar{D}^2F_2|)\ (r,\omega,\zeta,p) < \infty$

for each $R > 0$, where $\varepsilon_0 > 0$ is independent of R and where $\bar{D}$ denotes the full gradient in $(0,1) \times S^{n-1} \times \mathbb{R}^N \times \mathbb{R}^{nN}$.

We also need to impose a *real-analyticity* hypothesis on $F_1(\omega,\zeta,p)$ with respect to the ζ,p variables; specifically we assume that for each $(\zeta_0,p_0) \in \mathbb{R}^N \times \mathbb{R}^{nN}$

(7) $$F_1(\omega,\zeta,p) = \sum_{(\alpha,\beta)\in \mathbb{Z}_+^N \times \mathbb{Z}_+^{nN}} a_{\alpha,\beta}(\omega)(\zeta-\zeta_0)^{\alpha}(p-p_0)^{\beta}$$

where the series, together with the series obtained by twice differentiating the coefficients with respect to the ω variables, converge uniformly for $|\zeta-\zeta_0|$, $|p-p_0|$ sufficiently small (depending on ζ_0,p_0) and for $\omega \in S^{n-1}$. ($\mathbb{Z}_+$ denotes the set of non-negative integers.)

Notice that all these hypotheses are satisfied for the *energy functional* $E(u)$ of maps $u:(B_1(0),g) \to (\mathbb{R}^N,\gamma)$, where g,γ are smooth metrics on $B_1(0) \subset \mathbb{R}^N$ and on $\mathbb{R}^N$ respectively, and where γ is *real-analytic* and

(8) $\quad g_{ij}(0) = \delta_{ij}\ ,\ \partial g_{ij}(0)/\partial x^k = 0\ ,\ i,j,k=1,\ldots,n\ ,$

Recall that $E(u)$ is given by

$$E(u) = \frac{1}{2}\int_{B_1(0)} g^{ij}(x)\gamma_{\alpha\beta}(u(x))\,\frac{\partial u^{\alpha}}{\partial x^i}\,\frac{\partial u^{\beta}}{\partial x^j}\,\sqrt{g}\,dx\ ,$$

where $\quad g = \det(g_{ij})\ ,\ (g^{ij}) = (g_{ij})^{-1}$.

Because of (8) it is easy to check that this function can be written in the form of $F(u)$ of (*) with (4), (5), (6), (7) all holding. For more discussion (and also for discussion of how the area functional over a cone can be treated by using modifications of the above) we refer to [SL1, I§3].

We are now ready to state the main theorem

THEOREM 1 *Suppose* u *is a* $C^2(\bar{B}_1(0) \sim \{0\})$ *solution of* (1) *with*

(**) $\sup_{0<r<1,\ \omega\in S^{n-1}} (|u(r\omega)| + r|Du(r\omega)| + r^2|D^2u(r\omega)|) < \infty$,

and suppose (4), (5), (6), (7) *all hold. Then* (2) *holds.*

Perhaps the most unsatisfactory aspect of this theorem is the assumption (**). It can be significantly relaxed in certain cases - see the discussion in [SL1, II§5]. In case $n=3$, in case $\mathcal{F}(u)$ is the energy functional $E(u)$ described above, and in case u is actually *minimizing* $E(u)$ relative to all $W^{1,2}(B_1(0)\ ; \mathbb{R}^N)$ maps which agree with u outside a compact subset of $B_1(0)$, then (**) holds automatically, hence (2) holds in this case. (Of course in this case (2) implies that ϕ is a *harmonic* map $S^{n-1} \to (\mathbb{R}^N, \gamma)$, where S^{n-1} is equipped with the standard metric.)

The proof of Theorem 1 is rather lengthy and we do not have space to discuss it here. Instead we refer the reader to [SL1, Part II)] (or [SL2], where there is also discussion of how the appropriately modified version of Theorem 1 gives good information about asymptotic behaviour of minimal submanifolds on approach to isolated singular points. (Notice that we need to make the change of variable $t=-\log r$ to bring $\mathcal{F}(u)$ into the form considered in [SL1,2]).

REFERENCES

[SL1] L. Simon, *Isolated singularities of extrema of geometric variational problems*, To appear in Springer Lecture Notes (C.I.M.E. subseries).

[SL2] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Annals of Math. 118 (1983), 525-571.