THE MALLIAVIN CALCULUS AND LONG TIME ASYMPTOTICS OF CERTAIN WIENER INTEGRALS

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1. INTRODUCTION

The asymptotic behavior of stochastic oscillatory integrals has recently received much attention in the probabilistic literatures and is closely related to various problems in the analysis and applied mathematics, (cf. [3],[4],[6],[5],[10],[14]~[17] and [19]). In particular, in order to study asymptotic properties of stochastic oscillatory integrals, Malliavin [17] has used the stochastic calculus of variation. Gaveau and Moulinier [5] have also been interested in similar problems. The main purpose of this paper is to complete in detail the proof of Malliavin's results which was sketched in [17]. To do this, as is shown in the section 5, we need some considerations which are not discussed in [17], (see Propositions 5.1 and 5.2). We will also give a slight extension of some results in Malliavin [17].

Let us consider a smooth Riemannian metric g on R^d which is uniformly elliptic and bounded, (see Assumption 2.2). Then there exists the diffusion process $\{X(t),P_x,x \in R^d\}$ generated by half the Laplace-Beltrami operator $\Delta_g/2$ with respect to g. For every smooth differential 1-form θ, we set

$$K_t(x,y;\theta) = E_x[\exp\{\sqrt{-1}\int_{X[0,t]} \theta\}|X(t) = y] \quad \text{for } (t,x,y) \in (0,\infty) \times R^d \times R^d \tag{1.1}$$

where $E_x[\cdot|X(t) = y]$ denotes the conditional expectation with respect to the probability P_x given $X(t) = y$ and

$$\int_{X[0,t]} \theta$$

is the stochastic line integral of θ along the curve $X[0,t] = \{X(s) ; 0 \leq s \leq t\}$, (see [8] and [9], for details of the definition). We are interested in the asymptotic behavior of $a(t;\theta)$ defined by

$$a(t;\theta) = \sup_{x,y\in R^d} |K_t(x,y;\theta)| \tag{1.2}$$

as $t \longrightarrow \infty$. In this paper, from now on, we always assume the following:

ASSUMPTION 1.1 (i) The derivatives of all orders (≥ 1) of $\theta_i(x)$, $i = 1,2,\cdots,d$ are bounded where $\theta_i(x)$, $i = 1,2,\cdots,d$ are the coefficients of θ with respect to the basis dx^i, $i = 1,2,\cdots,d$.

(ii) There exists a positive constant C such that

$$\|d\theta\|^2(x) \geq C \qquad \text{for} \quad x \in R^d \tag{1.3}$$

where $\|\cdot\|(x)$ denotes the norm on $T_x^*(R^d) \otimes T_x^*(R^d)$ with respect to g.

ASSUMPTION 1.2 Let δ be the adjoint operator of the exterior differential operator d with respect to g. We assume that

$$\delta\theta = 0 . \tag{1.4}$$

The physical meaning of this assumption combining the integral (1.1) in the theory of electro-magnetic fields can be found in [4] and [23].

The result of Malliavin in [17] is as follows: In case when g is the standard flat metric on R^d, under some appropriate conditions, $a(t;\theta)$ decays exponentially as $t \longrightarrow \infty$, i.e.,

$$\lim_{t\to\infty} \frac{1}{t} \log a(t;\theta) < 0 . \tag{1.5}$$

By using the stochastic calculus of variation called the Malliavin calculus, Malliavin gave a program of the proof of (1.5) and a sketch of details. In this paper, we will show how one can obtain an extension of Malliavin's result based on ideas of the partial Malliavin calculus, (see Section 3). We will also give full details of the proof of Malliavin's result in [17]. Our analysis relies heavily on the theory of partial Malliavin calculus. In particular, we will use the integration by parts formula with respect to the conditional expectation on the Wiener space.

The organization of this paper is as follows: In Section 2, we will reformulate our problem in terms of Wiener integrals and state main results. In Section 3, we summarize the basic notations and results in the theory of the partial Malliavin calculus. It should also be noted that these are useful themselves in the study of various problems. In Section 4, for every positive N, we will evaluate the quantity

$$\sup_{x,y\in R^d,\ |x-y|\leq N} |K_t(x,y;\theta)| .$$

The section 5 will be devoted to the proof of (1.5) in case when g is the standard flat metric on R^d. Finally, in Section 6, we will prove Theorem 2.1, by combining the results obtained in Sections 4 and 5.

2. MAIN RESULTS

Let $(x^1,x^2,\cdots,x^d)$ be the standard coordinate of x in R^d and we denote by ∂_j, $j = 1,2,\cdots,d$, the vector fields on R^d

$$\frac{\partial}{\partial x^j} \quad , \quad j = 1,2,\cdots,d \ .$$

We set

$$g_{ij}(x) = g_x(\partial_i,\partial_j) \quad , \quad x \in R^d \quad , \quad i,j = 1,2,\cdots,d \ .$$

In this paper, we always assume the following:

ASSUMPTION 2.1 (i) The Riemannian metric g is uniformly elliptic and bounded in the following sense: For some positive constant K,

$$(2.1) \qquad \frac{1}{K}|\xi|^2 \leqq \sum_{i,j=1}^{d} g_{ij}(x)\xi^i\xi^j \leqq K|\xi|^2$$

for $x \in R^d$ and $\xi = (\xi^1,\xi^2,\cdots,\xi^d) \in R^d$.

(ii) $g_{ij}(x)$, $i,j = 1,2,\cdots,d$ and their derivatives of all orders are bounded.

In order to apply the Malliavin calculus to our problem, we have to rewrite the quantity $K_t(x,y;\theta)$ given by (1.1) in terms of Wiener integrals. To do this we first summarize some of basic facts in the stochastic analysis, (see [9], for details). We denote by $O(R^d)$ the bundle of orthonormal frames on the Riemannian manifold (R^d,g) and let $\pi : O(R^d) \longrightarrow R^d$ be the natural projection. Let $\{L_1,L_2,\cdots,L_d\}$ be the system of standard horizontal vector fields on $O(R^d)$ with respect to the Riemannian connection ∇ on (R^d,g) . We now consider the following stochastic differential equation on $O(R^d)$ of Stratonovich's type defined on the d-dimensional Wiener space $\{W_0^d,\mathcal{B}(W_0^d),P^W\}$:

$$(2.2) \qquad dr(t) = \sum_{\alpha=1}^{d} L_\alpha(r(t))\circ dw^\alpha(t)$$

where W_0^d denotes the space of all functions $w : [0,\infty) \longrightarrow R^d$ with $w(0) = 0$ and P^W is the d-dimensional Wiener measure. Let $\{r(t,r,w);\ t \geqq 0\}$ be the unique solution of (2.2) with initial value $r(\in O(R^d))$. We set

$$X(t,r,w) = \pi(r(t,r,w)) . \tag{2.3}$$

The stochastic line integral of θ along the curve $X([0,t];r,w)$, (given by $\{X(s,r,w);0 \leqq s \leqq t\}$),

$$\int_{X([0,t];r,w)} \theta$$

can be decomposed in the following form:

$$\int_{X([0,t];r,w)} \theta = \sum_{\alpha=1}^{d} \int_0^t \bar{\theta}_\alpha(r(s,r,w))dw^\alpha(s) - \frac{1}{2}\int_0^t \delta\theta(X(s,r,w))ds \tag{2.4}$$

where $\bar{\theta} = (\bar{\theta}_1,\bar{\theta}_2,\cdots,\bar{\theta}_d)$ denotes the scalarization of θ, i.e.,

$$\bar{\theta}_i(r) = \theta(e_i) \quad \text{for} \quad r = (x,[e_1,e_2,\cdots,e_d]) \in O(R^d) ,$$

(see [8] and [9]). By the assumption 1.2 and (2.4) we have

$$\int_{X([0,t];r,w)} \theta = \sum_{\alpha=1}^{d} \int_0^t \bar{\theta}_\alpha(r(s,r,w))dw^\alpha(s) . \tag{2.5}$$

By using similar arguments as those in [9], Chapter V, it is easy to see that the image measures on W^d and W^{d+1} of P^W by the mappings:

$$W_0^d \ni w \longrightarrow \{X(t,r,w);0 \leqq t < \infty\} \in W^d \tag{2.6}$$

and

$$W_0^d \ni w \longrightarrow \{(X(t,r,w),u + \int_{X([0,t];r,w)} \theta);0 \leqq t < \infty\} \in W^{d+1} \tag{2.7}$$

depend only on $x = \pi(r)$ and $(\pi(r),u)$ respectively. Here W^n denotes the space of all continuous functions $w : [0,\infty) \longrightarrow R^n$. Letting

$x = \pi(r)$, we denote above measures by P_x and $Q_{(x,u)}$ respectively. Then we have the system of diffusion measures $\{P_x; x \in R^d\}$ generated by half the Laplace-Beltrami operator $\Delta_g/2$. It should be also noted that the quantities which we will evaluate depend only on the laws of stochastic processes given by (2.6) or (2.7) and those of stochastic processes induced by their functionals. Since these laws depend only on $(\pi(r), u)$, for simplicity we use the notations $\{X(t,x,w); 0 \leqq t < \infty\}$ and $\{(X(t,x,w), u + \int_{X([0,t];x,w)} \theta)\}$ to denote the stochastic processes given by (2.6) and (2.7) respectively. Now the kernel $K_t(x,y;\theta)$, $t \geqq 0$, $x,y \in R^d$ defined by (1.1) can be rewritten in the following form:

$$K_t(x,y;\theta) = E^W[\exp\{\sqrt{-1}\int_{X([0,t];x,w)} \theta\} | X(t,x,w) = y] \tag{2.8}$$

where $E^W[\,\cdot\, | X(t,x,w) = y]$ means the conditional expectation with respect to the Wiener measure P^W given $X(t,x,w) = y$. Furthermore it holds that

(i) for every smooth exact differential 1-form α ,

$$|K_t(x,y;\theta)| = |K_t(x,y;\theta + \alpha)| \quad , \quad x,y \in R^d , \tag{2.9}$$

(ii) for every $t,s \geqq 0$

$$\begin{aligned} &K_{t+s}(x,y;\theta)p(t+s,x,y) \\ &\quad = \int_{R^d} K_t(x,z;\theta)p(t,x,z)K_s(z,y;\theta)p(s,z,y)m(dz) \end{aligned} \tag{2.10}$$

where $p(t,x,y)$, $(t > 0, x,y \in R^d)$, denotes the fundamental solution with respect to the Riemannian volume m of the heat equation on the Riemannian manifold (R^d, g)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_g u . \tag{2.11}$$

REMARK 2.1 We note that the semi-group relation (6.7) in Ikeda-Watanabe [10] (also the line 16 of page 25 in Malliavin [17]) should be read as (2.10) mentioned above.

Before stating the main result, we have to introduce an assumption.

ASSUMPTION 2.2 There exists a compact set D in R^d satisfying the following:

(i) The Riemannian metric g coincides with the standard flat metric in R^d outside of D, i.e.,

$$g_{ij}(x) = \delta_{ij} \quad \text{for} \quad x \notin D \;,\; i,j = 1,2,\cdots,d \;. \tag{2.12}$$

(ii) There are positive constants a_i, $i = 1,2$, $(a_1 \leqq a_2)$, such that for every $\eta \in S^{d-1}$ we can find a number β satisfying the following

$$0 < a_1 \leqq \langle\gamma_\beta,\eta\rangle^2(z) \leqq a_2 \quad \text{for every} \quad z \notin D \;, \tag{2.13}$$

where

$$\begin{aligned} \gamma_{\alpha,\beta}(z) &= (\partial_\alpha\theta_\beta - \partial_\beta\theta_\alpha)(z) \quad , \quad \alpha,\beta = 1,2,\cdots,d \\ \langle\gamma_\alpha,\eta\rangle(z) &= \sum_{\beta=1}^{d} \gamma_{\alpha,\beta}(z)\eta^\beta \quad , \quad \eta = (\eta^1,\eta^2,\cdots,\eta^d) \in S^{d-1} \end{aligned} \tag{2.14}$$

and S^{d-1} denotes the (d-1)-dimensional unit sphere, i.e.,

$$S^{d-1} = \{\eta \;;\; \eta \in R^d \;,\; |\eta| = 1\} \;.$$

REMARK 2.2 It is clear that

$$d\theta(x) = \sum_{1\leqq i<j\leqq d} \gamma_{ij}(x)\,dx^i \wedge dx^j \quad , \quad x \in R^d \;.$$

For every $\eta = (\eta^1, \eta^2, \cdots, \eta^d) \in S^{d-1}$, we consider the vector field A_η defined by

$$(A_\eta)_x = \sum_{i=1}^{d} \eta^i (\partial_i)_x .$$

Then, letting $i(A_\eta)d\theta$ be the interior product of the vector field A_η and the differential 2-form $d\theta$, we have

$$i(A_\eta)d\theta = -\frac{1}{2} \sum_{j=1}^{d} \langle \gamma_j, \eta \rangle (x) dx^j ,$$

(cf. [20]).

We now are in a position to state our main result.

THEOREM 2.1 *Under Assumptions* 1.1 , 1.2 , 2.1 *and* 2.2 , *it holds that*

(2.15) $$\lim_{t\to\infty} \frac{1}{t} \log a(t;\theta) < 0 .$$

REMARK 2.3 As shown in Section 4, we use only the assumptions 1.1 , 1.2 and 2.1 in some parts of the proof of Theorem 2.1.

REMARK 2.4 Under the assumption 2.1 , the condition (ii) of Assumption 1.1 is equivalent to the following: There exists a positive constant C' such that

(2.16) $$\sum_{\alpha,\beta=1}^{d} \gamma_{\alpha,\beta}(x)^2 \geqq C' \quad \text{for} \quad x \in R^d .$$

In fact, this can be proved as follows: By the assumption 2.1, we can choose a positive definite symmetric matrix $(a^i_j(x))$ such that

$$g^{ij}(x) = \sum_{k=1}^{d} a^i_k(x) a^j_k(x) \quad \text{for} \quad x \in R^d$$

where $(g^{ij}(x))$ is the inverse matrix of $(g_{ij}(x))$. Then it follows from (2.1) that for some positive constant K ,

$$(2.17)\quad \frac{1}{K}|\xi|^2 \leqq \sum_{k=1}^{d} \Big| \sum_{i=1}^{d} a_k^i(x)\xi_i \Big|^2 \leqq K|\xi|^2 \quad \text{for} \quad \xi = (\xi_1,\xi_2,\cdots,\xi_d) \in R^d .$$

Therefore (2.16) implies that for $x \in R^d$,

$$\begin{aligned} \|d\theta\|^2(x) &= \sum_{m=1}^{d} \sum_{n=1}^{d} \Big(\sum_{i=1}^{d} \sum_{j=1}^{d} a_m^i(x) a_n^j(x) \gamma_{ij}(x) \Big)^2 \\ &\geqq \frac{1}{K} \sum_{m=1}^{d} \sum_{j=1}^{d} \Big(\sum_{i=1}^{d} a_m^i(x)\gamma_{ij}(x) \Big)^2 && \text{(by (2.17))} \\ &\geqq \frac{1}{K^2} \sum_{i=1}^{d} \sum_{j=1}^{d} \gamma_{ij}(x)^2 && \text{(by (2.17))} \\ &\geqq C'/K^2 . && \text{(by (2.16))} \end{aligned}$$

This implies (1.3). Conversely we now assume (1.3). Letting $(b_j^i(x)) = (a_j^i(x))^{-1}$, we have

$$(2.18)\quad \frac{1}{K}|\xi|^2 \leqq \sum_{k=1}^{d} \Big| \sum_{i=1}^{d} b_k^i(x)\xi_i \Big|^2 \leqq K|\xi|^2 \quad \text{for} \quad \xi = (\xi_1,\xi_2,\cdots,\xi_d) \in R^d .$$

Hence we have, for $x \in R^d$,

$$\begin{aligned} \sum_{i,j=1}^{d} \gamma_{ij}(x)^2 &= \sum_{i,j=1}^{d} \Big(\sum_{n=1}^{d} \sum_{m=1}^{d} \sum_{k=1}^{d} \sum_{\ell=1}^{d} b_i^n(x) b_j^m(x) a_n^k(x) a_m^\ell(x) \gamma_{k,\ell}(x) \Big)^2 \\ &\geqq \frac{1}{K^2} \sum_{i=1}^{d} \sum_{j=1}^{d} \Big(\sum_{m=1}^{d} \sum_{n=1}^{d} a_i^m(x) a_j^n(x) \gamma_{mn}(x) \Big)^2 , && \text{(by (2.18))}, \\ &\geqq C/K^2 && \text{, (by (1.3))}, \end{aligned}$$

which implies (2.16).

REMARK 2.5 In 2-dimensional case, the condition (ii) of Assumption 1.1 implies the condition (ii) of Assumption 2.2. In this case, it holds by (2.13) that

$$\gamma_{1,2}(x)^2 = \gamma_{2,1}(x)^2 \geqq \frac{C'}{2} \quad \text{for} \quad x \in R^d .$$

Hence, for every $\eta \in S^1$,

$$\langle \gamma_1, \eta \rangle (x) \geqq \frac{C'}{4} \quad \text{for} \quad x \in R^d$$

or

$$\langle \gamma_2, \eta \rangle (x) \geqq \frac{C'}{4} \quad \text{for} \quad x \in R^d ,$$

which implies (2.13). In general case, the condition (ii) of Assumption 2.2 is not natural in a sense and is technical. In case when g is the standard flat metric in R^d , Malliavin [17] has discussed the above problem under Assumptions 1.1 and 1.2. As will be shown in Section 6, in many cases, without assuming the condition (ii) of Assumption 2.2, (2.15) still holds. However, in such cases, in order to prove (2.15) we need different arguments from those given in the section 5. For example, the following shows that in case $d \geqq 3$, the condition (ii) of Assumption 1.1 does not imply, in general, the condition (ii) of Assumption 2.2.

EXAMPLE 2.1 Let $d \geqq 3$ and $2 \leqq d_1 < d$. Let us consider a smooth differential 1-form θ on R^{d_1} satisfying the assumptions 1.1 , 1.2 and the condition (ii) of Assumption 2.2 in case of the d_1-dimensional Euclidean space R^{d_1} . θ can also be regarded as a smooth differential 1-form on R^d and it satisfies the assumptions 1.1 and 1.2 on the d-dimensional Euclidean space R^d . However it does not satisfy the condition (ii) of the assumption 2.2 on R^d . It is not hard to show (2.15) in this case. The proof can be reduced to the estimate in the case of d_1-dimensional Euclidean space. For details, see Section 5.

Before closing this section, we give one more remark. It is easy to see that the system $\{Q_{(x,u)}; (x,u) \in R^{d+1}\}$ defined above is a diffusion on R^{d+1}. It is also not hard to show that the generator of this diffusion is given by, for every $f \in C_0^{(2)}(R^d \times R^1)$,

$$(2.19) \qquad Af(x,u) = \frac{1}{2}\{\Delta_g f(x,u) + 2\frac{\partial}{\partial u}(b(\theta)(x)f(x,u)) + \|\theta\|^2(x)\frac{\partial^2}{\partial u^2} f(x,u) - \delta\theta(x)\frac{\partial}{\partial u} f(x,u)\} \qquad \text{for } (x,u) \in R^d \times R^1$$

where $b(\theta)$ is the vector field defined by

$$\theta_x(B) = g_x(b(\theta)_x, B) \qquad \text{for every} \quad B \in T_x(R^d) \ , \ x \in R^d$$

(see [20]) and $\|\theta\|(x)$ denotes the norm of θ in $T_x^*(R^d)$. As shown in Section 4, under the assumptions 1.1, 1.2 and 2.1, the transition probabilities $Q(t,(x,u),\cdot)$, $t > 0$, $(x,u) \in R^d \times R^1$, of the diffusion $\{Q_{(x,u)}; (x,u) \in R^d \times R^1\}$ have the smooth positive densities $q(t,(x,u),(y,v))$, $t > 0$, $(x,u),(y,v) \in R^d \times R^1$, with respect to the measure $\mu(dydv) = m(dy)dv$ on $R^d \times R^1$. The diffusion $\{Q_{(x,u)}; (x,u) \in R^d \times R^1\}$ is symmetric with respect to the measure μ, i.e., for every $t > 0$,

$$(2.20) \qquad q(t,(x,u),(y,v)) = q(t,(y,v),(x,u)) \qquad \text{for every} \quad (x,u),(y,v) \in R^d \times R^1 .$$

Then the kernel $K_t(x,y;\theta)$ can be written in the form

$$(2.21) \qquad K_t(x,y;\theta) = (p(t,x,y))^{-1}\int_{R^1} e^{\sqrt{-1}v} q(t,(x,0),(y,v))dv \qquad \text{for } t > 0 \ , \ x,y \in R^d .$$

Hence, by the Feynman-Kac formula, it is easily seen that $K_t(x,y;\theta)p(t,x,y)$ is the kernel function associated to the operator

$$(2.22) \qquad Hf = \frac{1}{2}\Delta_g f - \frac{1}{2}\|\theta\|^2 f + \sqrt{-1}\langle df,\theta\rangle - \frac{1}{2}\sqrt{-1}\delta\theta f\ .$$

For the physical meaning of these related to the electro-magnetic field, see [4], [6] and [23].

3. THE PARTIAL MALLIAVIN CALCULUS

Before proceeding to the proof of Theorem 2.1, in this section, we prepare the basic notation and results in the theory of Malliavin's calculus which we need. These are a slight modification of some results in [21], [12] and [10].

Let H be a separable Hilbert space with inner product $\langle\cdot,*\rangle_H$ and we identify H^*, the dual space of H, with H. We now fix an abstract Wiener space $\{W,H,\mu\}$, that is, H is included in a separable Banach space W as a subspace and the injection $i : H \longrightarrow W$ is continuous with the dense range. Also μ is a Gaussian measure with zero mean carried on W such that

$$\int_W \exp\{\sqrt{-1}h(w)\}\mu(dw) = \exp\{-|h|_H^2/2\} \quad \text{for} \quad h \in W^* \subset H^* = H \subset W$$

where $h(w)$ is the canonical bilinear form between W^* (= the dual space of W) and W and $|\cdot|_H^2 = \langle\cdot,\cdot\rangle_H$, (cf. [7], [11] and [27]). In general, let E be a separable Hilbert space. Then we denote by $L_p(W \longrightarrow E,\mu) = L_p(E)$, $1 \leqq p < \infty$, the usual L_p-space of μ-measurable functions $F : W \longrightarrow E$ with

$$\|F\|_p = \{\int_W |F(w)|_E^p \mu(dw)\}^{1/p} < \infty\ .$$

$L_p(W \longrightarrow R^1, \mu)$ is simply denoted by $L_p(W,\mu) = L_p$. A Wiener functional $F: W \longrightarrow R^1$ is called a polynomial if there exist an integer n, elements $\ell_1, \ell_2, \cdots, \ell_n$ of W^* and a real polynomial $p(\xi_1, \xi_2, \cdots, \xi_n)$ in n variables such that

$$F(w) = p(\ell_1(w), \ell_2(w), \cdots, \ell_n(w)) \quad \text{for any} \quad w \in W .$$

In this expression, we can always assume that

$$\langle \ell_i, \ell_j \rangle_H = \delta_{ij} \qquad i,j = 1,2,\cdots,n .$$

We denote by P the set of such polynomials. For a dense subspace E^0 of a separable Hilbert space E, we set

$$P(E^0) = \{F; F: W \longrightarrow E^0 \text{ which is expressed in the form}$$

$$F(w) = \sum_{i=1}^{n} F_i(w) e_i \quad \text{for some } n, \ F_i \in P \text{ and } e_i \in E^0\} .$$

An element F of $P(E^0)$ is called an E^0-valued polynomial. As usual in the theory of Malliavin's calculus we define the weak derivative D, the adjoint operator δ of D and the Ornstein-Uhlenbeck operator L on P by (3.2),(3.4) and (3.6) in Ikeda-Watanabe [10] respectively. For details, see also Watanabe [27] and Sugita [25]. We define the Sobolev norm $\|\cdot\|_{p,s}$, $1 \leqq p < \infty$, $s \in R^1$ by

$$\|F\|_{p,s} = \|(I - L)^{s/2} F\|_p \quad \text{for} \quad F \in P(E) .$$

Set

$$D_p^s(E) = \text{the completion of } P(E) \text{ by the norm } \|\cdot\|_{p,s} .$$

We use the same notation $\|\cdot\|_{p,s}$ to denote the norm of $D_p^s(E)$. We define the Sobolev space $D^\infty(E)$ over the Wiener space by

$$\mathcal{D}^{\infty}(E) = \bigcap_{p\geq 1, s\in R^1} \mathcal{D}_p^s(E)$$

and

$$\mathcal{D}^{-\infty}(E) = \bigcup_{p\geq 1, s\in R^1} \mathcal{D}_p^s(E) .$$

Then $\mathcal{D}^{-\infty}(E)$ is the dual space of $\mathcal{D}^{\infty}(E)$, (cf. [25] and [27]). For $E = R^1$, $\mathcal{D}_p^s(E)$, $\mathcal{D}^{\infty}(E)$ and $\mathcal{D}^{-\infty}(E)$ are denoted simply by $\mathcal{D}_p^s$, $\mathcal{D}^{\infty}$ and $\mathcal{D}^{-\infty}$ respectively. Then we obtain that the operators D, δ and L on $\mathcal{P}(E)$ can be extended to the continuous linear operators:

$$D : \mathcal{D}^{\infty}(E) \longrightarrow \mathcal{D}^{\infty}(H \otimes E)$$

$$\delta : \mathcal{D}^{\infty}(H \otimes E) \longrightarrow \mathcal{D}^{\infty}(E)$$

$$L : \mathcal{D}^{\infty}(E) \longrightarrow \mathcal{D}^{\infty}(E) ,$$

respectively. For detailed properties of the Sobolev norm and the Sobolev space, see Watanabe [27] and Sugita [25].

We now proceed to introducing the notion of partial weak derivatives. Let us consider a family $\mathcal{H}$ of closed subspaces $H(w)$ of H, i.e.,

(3.1) $$\mathcal{H} = \{H(w) ; H(w) : \text{ closed subspace of } H , w \in W\} .$$

Take a separable Hilbert space E. We denote by $P_{\mathcal{H}}^E(w)$ the projection: $H \otimes E \longrightarrow H(w) \otimes E$. For every $F : W \longrightarrow H \otimes E$, we define $P_{\mathcal{H}}^E F : W \longrightarrow H \otimes E$ by

(3.2) $$P_{\mathcal{H}}^E F(w) = P_{\mathcal{H}}^E(w)(F(w)) \qquad \text{for every} \quad w \in W .$$

For simplicity, we often denote this mapping by $P_{\mathcal{H}}F$. In this section, from now on, we always assume the following:

ASSUMPTION 3.1 The family $\mathcal{H}$ satisfies the following conditions: For every separable Hilbert space E,

(i) $P_{\mathcal{H}}(\mathcal{B}(W\,;H\otimes E)) \subset \mathcal{B}(W\,;H\otimes E)$,

(ii) $P_{\mathcal{H}}(\mathcal{D}^{\infty}(H\otimes E)) \subset \mathcal{D}^{\infty}(H\otimes E)$.

Here $\mathcal{B}(E_1,E_2)$ denotes the space of all measurable mappings: $F : E_1 \longrightarrow E_2$ where E_i, $i = 1,2$, are separable Banach spaces.

DEFINITION 3.1 For $F \in \mathcal{P}(E)$ and $w \in W$, we define $D_{\mathcal{H}}F(w) \in H \otimes E$ by

$$(3.3)\qquad \langle D_{\mathcal{H}}F(w), h\otimes e\rangle_{H\otimes E} = \lim_{t\downarrow 0}(\langle F(w + tP_{\mathcal{H}}(w)h) - F(w), e\rangle_E/t)$$

for $w \in W$, $h \in H$ and $e \in E$.

By the definition of $D_{\mathcal{H}}$ we have

LEMMA 3.1 (i) *It holds that*

$$(3.4)\qquad D_{\mathcal{H}}F = P_{\mathcal{H}}(DF) \qquad \textit{for} \quad F \in \mathcal{P}(E) .$$

(ii) *Let* $p \geqq 1$. *If* $F_n \in \mathcal{P}(E)$, $F_n \longrightarrow 0$ *in* $L_p(E)$ *and* $D_{\mathcal{H}}F_n \longrightarrow G$ *in* $L_p(H\otimes E)$, *then* $G = 0$.

Proof. Since, for $F \in \mathcal{P}(E)$,

$$\langle DF(w), h\otimes e\rangle_{H\otimes E} = \lim_{t\downarrow 0}(\langle F(w + th) - F(w), e\rangle_E/t)$$

for $w \in W$, $h \in H$, $e \in E$,

(3.4) follows from (3.3).

We take $F_n \in \mathcal{P}(E)$ such that $F_n \longrightarrow 0$ in $L_p(E)$ and $D_{\mathcal{H}}F_n \longrightarrow G$ in $L_p(H \otimes E)$. Then, for every $K \in \mathcal{D}^{\infty}(H \otimes E)$,

$$\begin{aligned} E[\langle G,K\rangle_{H\otimes E}] &= \lim_{n\to\infty} E[\langle D_{\mathcal{H}}F_n, K\rangle_{H\otimes E}] \\ &= \lim_{n\to\infty} E[\langle F_n, \delta(P_{\mathcal{H}}K)\rangle_E] \\ &= 0 , \end{aligned}$$

where $E[\cdot]$ denotes the expectation with respect to the measure μ. Here we use the assumption 3.1. Hence $G = 0$ which completes the proof.

The lemma 3.1 implies that for every $F \in \mathcal{D}^{\infty}(E)$, there exists a sequence $\{F_n\}$, $F_n \in \mathcal{P}(E)$ satisfying the following:

(i) $F_n \longrightarrow F$ in $\mathcal{D}^{\infty}(E)$,

(ii) $\{D_{\mathcal{H}}F_n\}$ consists of a Cauchy sequence in $\mathcal{D}^{\infty}(E)$,

(iii) the limit

$$D_{\mathcal{H}}F = \lim_{n\to\infty} D_{\mathcal{H}}F_n$$

is uniquely determined.

For every $F \in \mathcal{D}^{\infty}(E)$, $D_{\mathcal{H}}F$ determined above is called the $\mathcal{H}$-partial weak derivative of F. Furthermore it is easy to see that for every $F \in \mathcal{D}^{\infty}(H \otimes E)$, there exists an element $\Phi \in \mathcal{D}^{\infty}(E)$ satisfying

$$E[\langle F, D_{\mathcal{H}}G\rangle_{H\otimes E}] = E[\langle \Phi, G\rangle_E] \quad \text{for every} \quad G \in \mathcal{D}^{\infty}(E) .$$

We set

$$\delta_{\mathcal{H}}F = \Phi \tag{3.5}$$

and $\delta_H : \mathcal{D}^\infty(H \otimes E) \longrightarrow \mathcal{D}^\infty(E)$ is called the adjoint operator of the H-partial weak derivative D_H. We also define the H-partial Ornstein-Uhlenbeck operator $L_H : \mathcal{D}^\infty(E) \longrightarrow \mathcal{D}^\infty(E)$ by

$$L_H F = -\delta_H D_H F \qquad \text{for} \quad F \in \mathcal{D}^\infty(E) . \tag{3.6}$$

We now have the following:

LEMMA 3.2 *It holds that*

$$\begin{aligned} D_H F &= P_H(DF) && \text{for} \quad F \in \mathcal{D}^\infty(E) \\ \delta_H F &= \delta(P_H F) && \text{for} \quad F \in \mathcal{D}^\infty(H \otimes E) \\ L_H F &= -\delta D_H F && \text{for} \quad F \in \mathcal{D}^\infty(E) . \end{aligned} \tag{3.7}$$

It should be emphasized that in common with D the operator D_H has the following properties:

LEMMA 3.3 (i) *(Derivation property). For* $F_1, F_2 \in \mathcal{D}^\infty$,

$$D_H(F_1 F_2) = F_1 D_H F_2 + F_2 D_H F_1 . \tag{3.8}$$

(ii) *(Chain rule). Let us consider* $F = (F^1, F^2, \cdots, F^d)$, $F^i \in \mathcal{D}^\infty(R^1)$, $i = 1, 2, \cdots, d$. *Then, for every* $\phi \in C^\infty_\uparrow(R^d)$,

(a) $\phi(F) \in \mathcal{D}^\infty$,

(b)

$$D_H(\phi(F)) = \sum_{i=1}^{d} (\partial_i \phi)(F) D_H F^i , \tag{3.9}$$

where $C^\infty_\uparrow(R^d)$ *denotes the space of all slowly increasing* C^∞*-functions* $f : R^d \longrightarrow R^1$.

Proof. Combining Lemma 3.2, the derivation property and the chain rule for D we can complete the proof.

Before turning to typical examples of the partial weak derivative, it will be useful to introduce some notation. We consider the n-dimensional Wiener space $\{W_0^n, \mathcal{B}(W_0^n), P^W\}$. In this section, we will restrict ourselves to the case of a finite interval $[0,T]$ where $T > 0$ is an arbitrary but fixed number: We use the same notation W_0^n to denote the space of all continuous functions $w : [0,T] \longrightarrow R^n$ such that $w(0) = 0$. W_0^n is a Banach space with norm

$$|w| = \max_{0 \le t \le T} |w(t)| .$$

If H is the Hilbert space given by

$$\begin{aligned} H = \{w \in W_0^n \ ; \ &\text{each component of } w \text{ is absolutely continuous} \\ &\text{with square integrable derivatives}\} , \end{aligned} \tag{3.10}$$

$$\langle h_1, h_2 \rangle_H = \sum_{i=1}^{n} \int_0^T \dot{h}_1^i(t) \dot{h}_2^i(t)\,dt \ , \ h_i \in H \ , \ \dot{h}_j^i(t) = \frac{d}{dt} h_j^i(t) \ ,$$

then $\{W_0^n, H, P^W\}$ is an abstract Wiener space which we call the n-dimensional Wiener space.

EXAMPLE 3.1 (Partial weak derivatives, [21] and [12])

Let d_i, $i = 1,2$, be positive integers and $d = d_1 + d_2$. Let us consider the d-dimensional Wiener space $\{W_0^d, H, P^W\}$. Set

$$H_{(2)} = \{h \ ; h = (\underbrace{0,0,\cdots,0}_{d_1}, h^{d_1+1}, \cdots, h^d) \in H\}$$

and we consider a family $\mathcal{H} = \{H_{(2)}\}$, i.e., letting $H(w) = H_{(2)}$, $w \in W_0^n$, we set $\mathcal{H} = \{H(w) \ ; w \in W_0^n\}$. Then $\mathcal{H}$ satisfies the assumption 3.1. Then it is not hard to see that

$$D_{\mathcal{H}}F = D_{(2)}F \qquad \text{for every} \quad F \in \mathcal{D}^{\infty}(E)$$

where $D_{(2)}$ denotes the operator introduced in Section 1 of Kusuoka-Stroock [12]. For details, see [12] and [21].

EXAMPLE 3.2 (Conditional weak derivatives, [10])

Let $\{W,H,\mu\}$ be an abstract Wiener space. We fix a Wiener functional $F = (F^1,F^2,\cdots,F^d) \in \mathcal{D}^{\infty}(R^d)$ such that

$$(3.11) \qquad (\det \sigma_F)^{-1} \in \bigcap_{p\geq 1} L_p$$

where $\sigma_F = (\langle DF^i,DF^j\rangle_H)$ denotes the Malliavin covariance of F. Following Ikeda-Watanabe [10], we define the conditional weak derivative $D_F : \mathcal{D}^{\infty}(R^1) \longrightarrow \mathcal{D}^{\infty}(H)$ given F by

$$(3.12) \qquad D_F G = DG - \sum_{i=1}^{d} \sum_{j=1}^{d} (\gamma_F)_{ij} \langle DG,DF^j\rangle_H DF^i$$

where $\gamma_F = ((\gamma_F)_{ij})$ is the inverse matrix of σ_F. We set

$$H(w) = \{h \in H \,;\, \langle h,DF^i(w)\rangle_H = 0 \;,\; 1 \leqq i \leqq d\} \;,\; w \in W$$

and

$$\mathcal{H} = \{H(w) \,;\, w \in W\} \;.$$

It should be noted that here we have to fix nice versions of $DF^i(w)$, $1 \leqq i \leqq d$. Then $\mathcal{H}$ satisfies the assumption 3.1. Furthermore, it is clear that for every $G \in \mathcal{D}^{\infty}(H \otimes R^1)$

$$(3.13) \qquad P_{\mathcal{H}}G(w) = G(w) - \sum_{i=1}^{d} \sum_{j=1}^{d} (\gamma_F)_{ij}(w) \langle G(w),DF^j(w)\rangle_H DF^i(w)$$

$$\text{on} \quad \{w \,;\, \det \sigma_F(w) \neq 0\} \;.$$

Hence, by the lemma 3.2,

$$(3.14) \qquad D_F = D_H \quad \text{on} \quad \mathcal{D}^\infty .$$

Since, $D_H F^i = 0$, $1 \leqq i \leqq d$, it follows from (3.9) that for every $f \in C_\uparrow^\infty(R^d)$,

$$(3.15) \qquad D_H(f \circ F) = 0 .$$

We will now point out the different properties of D_H and δ_H from the those of D and δ respectively.

LEMMA 3.4 *Let* $F = (F^1, F^2, \cdots, F^d) \in \mathcal{D}^\infty(R^d)$ *and*

$$(3.16) \qquad D_H F^i = 0 \quad , \quad i = 1,2,\cdots,d .$$

Then, for every $\phi \in C_\uparrow^\infty(R^d)$ *and* $K \in \mathcal{D}^\infty(H)$

(i) $D_H(\phi(F)) = 0$,

(ii) $\phi(F)K \in \mathcal{D}^\infty(H)$ *and*

$$(3.17) \qquad \delta_H(\phi(F)K) = \phi(F)\delta_H K .$$

Proof. It is clear that (i) follows from the lemma 3.3. Next we note that

$$\delta(GK) = -\langle DG, K\rangle_H + G\delta K$$

(see [27]). By combining this with Lemma 3.2, we have, for $G \in \mathcal{D}^\infty$ and $K \in \mathcal{D}^\infty(H)$,

$$\delta_H(GK) = \delta(GP_H K) = -\langle DG, P_H K\rangle_H + G\delta_H K .$$

Then setting $G = \phi(F)$ and using (i) we can complete the proof of (3.17).

We now turn to studying properties of the pull-back of Schwartz distributions on R^d as elements in $\mathcal{D}^{-\infty}$ related to the partial weak derivative D_H. Let $\mathcal{S}'(R^d)$ be the Schwartz space of tempered distributions on R^d.

PROPOSITION 3.1 *Let us consider a Wiener functional* $F = (F^1, F^2, \cdots, F^d) \in \mathcal{D}^\infty(R^d)$ *such that*

(i) $$D_H F^i = 0 \quad , \quad i = 1,2,\cdots,d \, ,$$

(ii)

$$(\det \sigma_F)^{-1} \in \bigcap_{p\geq 1} L_p \tag{3.18}$$

where σ_F *is the Malliavin covariance of* F.

Then, for every $G \in \mathcal{D}^\infty$, $\Phi \in \mathcal{D}^\infty(H)$ and $T \in \mathcal{S}'(R^d)$,

$$<<D_H G, \Phi>_H, T(F)> = <G\delta_H \Phi, T(F)> \tag{3.19}$$

where $T(F)$ *denotes the pull-back of* T *under the mapping* $F : W \longrightarrow R^d$ (cf. [27] *and* [10]) *and* $<\cdot,*>$ *denotes the canonical pairing between* $\mathcal{D}^\infty$ *and* $\mathcal{D}^{-\infty}$.

Proof. Let $\{\phi_n\}$ be a sequence of $\phi_n \in \mathcal{S}(R^d)$ such that

$$\phi_n \longrightarrow T \quad \text{in} \quad \mathcal{S}'(R^d)$$

where $\mathcal{S}(R^d)$ denotes the Schwartz space of rapidly decreasing C^∞-functions on R^d. Then we have

$$\begin{aligned} <<D_H G, \Phi>_H, T(F)> &= \lim_{n\to\infty} E[<D_H G, \Phi>_H \phi_n(F)] \\ &= \lim_{n\to\infty} E[<D_H G, \phi_n(F)\Phi>_H] \\ &= \lim_{n\to\infty} E[G\delta_H(\phi_n(F)\Phi)] \\ &= \lim_{n\to\infty} E[G(\delta_H(\Phi))\phi_n(F)] \qquad \text{(by (3.17))} \end{aligned}$$

$$= \langle G\delta_{H}(\Phi), T(F)\rangle$$

which completes the proof (3.19).

COROLLARY 3.1 *(Integration by parts formula). Let* $F \in D^{\infty}(R^{d})$ *satisfy the same condition as in the proposition 3.1. Let* $G = (G^{1}, G^{2}, \cdots, G^{n}) \in D^{\infty}(R^{n})$ *and*

$$(\det \sigma_{G,H})^{-1} \in \bigcap_{p\geq 1} L_{p} \tag{3.20}$$

where $\sigma_{G,H}$ *denotes the H-partial Malliavin covariance of* G *defined by*

$$\sigma_{G,H} = (\langle D_{H}G^{i}, D_{H}G^{j}\rangle_{H}) . \tag{3.21}$$

Then, for every $K \in D^{\infty}$, $i_1, i_2, \cdots, i_k \in \{1,2,\cdots,n\}$ *and* $\phi \in C_{\uparrow}^{\infty}(R^{n})$,

$$\begin{aligned} &E[(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}\phi)\circ G(w)K(w)\,|F(w) = y] \\ &\quad = E[\phi\circ G(w)\,\Phi_{i_1 i_2\cdots i_k}(K;G,F)(w)\,|F(w) = y] \end{aligned} \tag{3.22}$$

if $\langle 1, \delta_{y}(F)\rangle \neq 0$ *, where*

$$\begin{aligned} &\Phi_{i}(K;G,F)(w) = \sum_{j=1}^{n} \delta_{H}((\gamma_{G,H})_{ij}(w)K(w)D_{H}G^{j}(w)) \\ &\Phi_{i_1 i_2\cdots i_m}(K;G,F)(w) = \Phi_{i_m}(\Phi_{i_1\cdots i_{m-1}}(K;G,F);G,F)(w) \\ &\qquad\qquad m = 1,2,\cdots,k . \end{aligned} \tag{3.23}$$

Here $E[\cdot\,|F(w) = y]$ *denotes the conditional expectation with respect to the measure* μ *given* $F(w) = y$ *and* $\gamma_{G,H} = ((\gamma_{G,H})_{ij})$ *is the inverse matrix of* $\sigma_{G,H}$.

Proof. By the assumption (3.18), there exists a continuous version of $\langle 1, \delta_{x}(F)\rangle$ and

$$(3.24)\quad \begin{aligned} &<(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}\phi)\circ G(w)K(w),\delta_y(F)> \\ &= E[(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}\phi)\circ G(w)K(w)\,|F(w) = y]<1,\delta_y(F)>\ , \\ &<\phi\circ G(w)\Phi_{i_1\cdots i_k}(K;G,F)(w),\delta_y(F)> \\ &= E[\phi\circ G(w)\Phi_{i_1\cdots i_k}(K;G,F)(w)\,|F(w) = y]<1,\delta_y(F)>\ , \end{aligned}$$

(cf. [27] and [10]). On the other hand, by using (3.9) and (3.20), we have

$$\partial_i\phi\circ G(w) = \sum_{j=1}^{n}(\gamma_{G,H})_{ij}<D_H(\phi\circ G),D_HG^j>_H\ .$$

Hence

$$\begin{aligned} &<\partial_i\phi\circ G(w)K(w),\delta_y(F)> \\ &= <<D_H(\phi\circ G),\sum_{j=1}^{d}(\gamma_{G,H})_{ij}K(w)D_HG^j>_H,\delta_y(F)>\ . \end{aligned}$$

Combining this with (3.19) we have

$$\begin{aligned} &<\partial_i\phi\circ G(w)K(w),\delta_y(F)> \\ &= <\phi\circ G(w)\delta_H(\sum_{j=1}^{d}(\gamma_{G,H})_{ij}(w)K(w)D_HG^j(w)),\delta_y(F)> \\ &= <\phi\circ G(w)\Phi_i(K;G,F)(w),\delta_y(F)>\ . \end{aligned}$$

By using this and (3.24), we can conclude (3.22) in case $k = 1$. Now by repeating the argument of the induction on k, we can easily complete the proof.

REMARK 3.1 We again consider a Wiener functional $F = (F^1,F^2,\cdots,F^d) \in \mathcal{D}^\infty(R^d)$ satisfying the same condition as in the proposition 3.1. Let $G = (G^1,G^2,\cdots,G^n) \in \mathcal{D}^\infty(R^n)$ and we assume that

$$(3.25) \qquad (DG^j - D_{\mathcal{H}}G^j)(w) \in \mathrm{span}\{DF^i(w); 1 \leq i \leq d\} \quad \text{a.e.}, \ 1 \leq j \leq n \ .$$

Then the condition

$$(3.26) \qquad (\det \sigma_K)^{-1} \in \bigcap_{p \geq 1} L_p$$

is necessary and sufficient for (3.20) where $K = (G,F)$ and σ_K denotes the Malliavin covariance of K. In fact, this can be proved as follows. Since $D_{\mathcal{H}}F^i = 0$, $i = 1,2,\cdots,d$,

$$\langle DF^i, DG^j \rangle_H = \langle DF^i, DF^j - D_{\mathcal{H}}G^j \rangle_H, \ i = 1,2,\cdots,d, \ j = 1,2,\cdots,n \ .$$

It also holds that, by (3.7),

$$(3.27) \qquad \langle DG^i, DG^j \rangle_H = \langle D_{\mathcal{H}}G^i, D_{\mathcal{H}}G^j \rangle_H + \langle DG^i - D_{\mathcal{H}}G^i, DG^j - D_{\mathcal{H}}G^j \rangle_H$$

$$i,j = 1,2,\cdots,n \ .$$

By the assumption (3.25), there exists the matrix $C(w) = (C_k^j(w))$ such that

$$(3.28) \qquad DG^i(w) - D_{\mathcal{H}}G^i(w) = \sum_{k=1}^{d} C_k^i(w) DF^k(w) \ , \ i = 1,2,\cdots,n \ .$$

Hence, setting

$$A_{F,G}(w) = (\langle DF^i(w), (DG^j - D_{\mathcal{H}}G^j)(w) \rangle_H) \ ,$$

we have, by (3.27),

$$\langle DG^i, DG^j \rangle_H = \langle D_{\mathcal{H}}G^i, D_{\mathcal{H}}G^j \rangle_H + \sum_{k=1}^{d} C_k^i(w) (A_{F,G}(w))_j^k \ .$$

Combining (3.27), (3.28) with this, we have

$$\det \sigma_K(w) = \det \begin{pmatrix} \sigma_{G,\mathcal{H}}(w) + C(w)A_{F,G}(w) & C(w)\sigma_F(w) \\ & \\ A_{F,G}(w) & \sigma_F(w) \end{pmatrix} \tag{3.29}$$

$$= \det \sigma_{G,\mathcal{H}}(w) \det \sigma_F(w) \ .$$

This implies that (3.20) and (3.26) are equivalent.

Before closing this section, we show a relation which is useful in the evaluation of $a(t;\theta)$, (see, Section 5). From now on, in this section, we always fix $\ell_1, \ell_2, \cdots, \ell_d \in W^*$ such that

$$\langle \ell_i, \ell_j \rangle_H = \delta_{ij} \ , \quad i,j = 1,2,\cdots,d \ .$$

We also set

$$H_0 = \{h \in H \ ; \langle h, \ell_i \rangle_H = 0 \ , \ 1 \leqq i \leqq d\}$$

$$\mathcal{H} = \{H_0\} \quad (\equiv \{H(w) \ ; w \in W\} \ , \ H(w) = H_0)$$

and

$$F(w) = (\ell_1(w), \ell_2(w), \cdots, \ell_d(w)) \in \mathcal{D}^\infty(R^d) \ .$$

PROPOSITION 3.2 *For every* $G \in \mathcal{D}^\infty$ *and* $x \in R^d$,

$$E[(L_{\mathcal{H}}G)^2 | F = x] = E[|D_{\mathcal{H}}G|_H^2 + |D_{\mathcal{H}}^2 G|_{H\otimes H}^2 | F = x] \ . \tag{3.30}$$

In order to prove, we will prepare some notation and a lemma. Let W_1 be the linear span of $\ell_1, \ell_2, \cdots, \ell_d$ and

$$W_2 = \{w \in W \ ; \ell(w) = 0 \quad \text{for every} \quad \ell \in W_1 \subset W^*\} \ .$$

Then we have the pseudo orthogonal decomposition:

$$W = W_1 \oplus W_2 \quad \text{(direct sum)} \ ,$$

(cf. [18],[22] and [10]). We write $\pi_i w = w_i$, if $w = w_1 + w_2$, $w_i \in W_i$, $i = 1,2,$. Set $\mu_i = (\pi_i)_*(\mu)$, the image measure of μ under the mapping $\pi_i : W \longrightarrow W_i$, $(i = 1,2,)$. Then we have an abstract Wiener space $\{W_2, H_0, \mu_2\}$. For every $f \in C^\infty(R^n)$, $z \in R^n$, we set

$$f_z(y) = f(y+z) \qquad y \in R^n .$$

LEMMA 3.5 *Take an element* $K \in P$ *such that*

$$K(w) = p(q_1(w), q_2(w), \cdots, q_n(w)) , \quad w \in W$$

where $q_1, q_2, \cdots, q_n \in W^*$. *Then for every* $x \in R^d$,

$$E[K|F = x] = E^{\mu_2}[K^x] \tag{3.31}$$

where

$$K^x(w) = p_{z_q(x)}(q_1(w), q_2(w), \cdots, q_n(w))$$

$$z_q(x) = \Big(\sum_{i=1}^{d} \langle q_1, \ell_i \rangle_H x^i, \cdots, \sum_{i=1}^{d} \langle q_n, \ell_i \rangle_H x^i\Big) \in R^n$$

$$\textit{for} \quad x = (x^1, x^2, \cdots, x^d) \in R^d .$$

Here E^{μ_2} *denotes the expectation with respect to* μ_2 .

Proof. The mapping $q : W \longrightarrow R^n$ is defined by

$$q(w) = (q_1(w), q_2(w), \cdots, q_n(w)) .$$

Then

$$E[K|F = x] = \langle K, \delta_x(F) \rangle (\sqrt{2\pi})^d \exp\{|x|^2/2\}$$

$$= \langle p(q(\pi_2 w) + q(\pi_1 w)), \delta_x(F) \rangle (\sqrt{2\pi})^d \exp\{|x|^2/2\}$$

$$= \langle p(q(\pi_2 w) + z_q(x)), \delta_x(F) \rangle (\sqrt{2\pi})^d \exp\{|x|^2/2\}$$

$$= E[p(q(\pi_2 w) + z_q(x))]<1, \delta_x(F)>(\sqrt{2\pi})^d \exp\{|x|^2/2\}$$

$$= E^{\mu_2}[K^x]$$

which completes the proof.

Proof of Proposition 3.2 We first note that F satisfies the condition of Proposition 3.1. Since the mappings: $D_H : \mathcal{D}^\infty \longrightarrow \mathcal{D}^\infty(H)$, $D_H^2 : \mathcal{D}^\infty \longrightarrow \mathcal{D}^\infty(H \otimes H)$ and $L_H : D^\infty \longrightarrow D^\infty$ are continuous, it is sufficient to show (3.30) for $G \in \mathcal{P}$. If we take $G(w) \in \mathcal{P}$ such that

$$G(w) = p(q_1(w), q_2(w), \cdots, q_n(w)) \ , \quad q_1, q_2, \cdots, q_n \in W^* \ ,$$

then

$$(3.32) \quad \begin{aligned} D_H G(w) &= \sum_{i=1}^{n} \partial_i p(q(w)) \pi_2 q_i && w \in W \\ D_H^2 G(w) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i \partial_j p(q(w)) \pi_2 q_i \otimes \pi_2 q_j && w \in W \\ L_H G(w) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i \partial_j p(q(w)) <\pi_2 q_i, \pi_2 q_j>_H \\ &\quad - \sum_{i=1}^{n} \partial_i p(q(w)) q_i(\pi_2 w) && w \in W \ . \end{aligned}$$

Since $W_2 \subset W$ and $W^* \subset W_2^*$, G can be regarded as a polynomial on the abstract Wiener space $\{W_2, H_0, \mu_2\}$. Letting $\tilde{D}$ and $\tilde{L}$ be the weak derivative and the Ornstein-Uhlenbeck's operator defined on $\{W_2, H_0, \mu_2\}$ respectively, we have

$$(3.33) \quad \begin{aligned} \tilde{D}G(w) &= \sum_{i=1}^{n} \partial_i p(q(w)) q_i && w \in W_2 \ , \\ \tilde{D}^2 G(w) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i \partial_j p(q(w)) q_i \otimes q_j && w \in W_2 \ , \end{aligned}$$

$$\tilde{L}G(w) = \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_i\partial_j p(q(w))\langle q_i, q_j\rangle_{H_0}$$

$$- \sum_{i=1}^{n} \partial_i p(q(w))q_i(w) \qquad w \in W_2 .$$

By Lemma 3.5, (3.32) and (3.33), we have

$$\begin{aligned} &E[|D_H G|_H^2 | F = x] = E^{\mu_2}[|\tilde{D}G^x|^2] \\ (3.34)\qquad & \\ &E[|D_H^2 G|_{H\otimes H}^2 | F = x] = E^{\mu_2}[|\tilde{D}^2 G^x|^2] . \end{aligned}$$

Setting

$$k(y_1,\cdots,y_n,u_1,\cdots,u_d) = \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_i\partial_j p(y)\langle \pi_2 q_i, \pi_2 q_j\rangle_H$$

$$- \sum_{i=1}^{n} \partial_i p(y)(y_i - \sum_{j=1}^{d} \langle q_i, \ell_j\rangle_H u_j)$$

$$K(w) = k(q_1(w),\cdots,q_n(w),\ell_1(w),\cdots,\ell_d(w))$$

we have, by Lemma 3.5,

$$\begin{aligned} &E[(L_H G)^2 | F = x] = E^{\mu_2}[(K^x)^2] \\ (3.35)\qquad & \\ &\qquad = E^{\mu_2}[(\tilde{L}G^x)^2] . \end{aligned}$$

On the other hand, it holds that

$$E^{\mu_2}[(\tilde{L}G^x)^2] = E^{\mu_2}[|\tilde{D}G^x|_{H_0}^2 + |\tilde{D}^2 G^x|_{H_0\otimes H_0}^2] ,$$

(see Shigekawa [22]). By combining (3.34), (3.35) with this we obtain (3.30).

4. LOCAL ESTIMATES

In this section, we assume only Assumptions 1.1, 1.2 and 2.1. Hence we do not assume the assumption 2.2. The purpose of this section is to prove the following estimate:

PROPOSITION 4.1 *Under Assumptions* 1.1, 1.2 *and* 2.1, *for every positive constants* t *and* L, *there exists a positive constant* $\beta(t,L)$ *which may depend on* t *and* L *such that*

$$(4.1) \qquad \sup_{x,y \in R^d,\, |x-y| \le L} |K_t(x,y;\theta)| \leqq \beta(t,L) < 1 .$$

We first begin to introduce some notation. Let us consider the 1-dimensional torus

$$\mathcal{T} = \{z \in C ; |z| = 1\} .$$

Then by the mapping: $[0,2\pi) \ni u \longrightarrow e^{\sqrt{-1}u} \in \mathcal{T}$ we can identify $\mathcal{T}$ with the interval $[0,2\pi)$. For every positive ℓ, let M_ℓ be the totality of probabilities λ such that

$$(4.2) \qquad \left| \int_{\mathcal{T}} \frac{d\phi}{du}(e^{\sqrt{-1}u}) \lambda(du) \right| \leqq \ell \|\phi\|_\infty \quad \text{for every} \quad \phi \in C^\infty(\mathcal{T}) ,$$

where

$$\|\phi\|_\infty = \sup_u |\phi(e^{\sqrt{-1}u})|$$

and $\frac{d\phi}{du}$ denotes the derivative of the mapping: $u \longrightarrow \phi(e^{\sqrt{-1}u})$. Then we obtain the following:

LEMMA 4.1 *For every positive* ℓ,

$$\sup_{\lambda \in M_\ell} \left| \int_{\mathcal{T}} e^{\sqrt{-1}u} \lambda(du) \right| < 1 .$$

Proof. Let M be the totality of all probabilities on T. Since T is compact, M also is compact in the weak $*$-topology. Furthermore, for every $\phi \in C^{\infty}(T)$, the mapping

$$M \ni \lambda \longrightarrow \int_T \frac{d\phi}{du}(e^{\sqrt{-1}u})\lambda(du)$$

is continuous. Hence M_ℓ is closed. Therefore M_ℓ is compact in the weak $*$-topology. It follows from this that if

$$\sup_{\lambda \in M_\ell} \left| \int_T e^{\sqrt{-1}u}\lambda(du) \right| = 1 ,$$

there exists an element $\lambda_0 \in M_\ell$ such that

$$\left| \int_T e^{\sqrt{-1}u}\lambda_0(du) \right| = 1 .$$

Since $|e^{\sqrt{-1}u}| = 1$, this implies that λ_0 is the Dirac measure. This contradicts (4.2). Therefore

$$\sup_{\lambda \in M_\ell} \left| \int_T e^{\sqrt{-1}u}\lambda(du) \right| < 1$$

which completes the proof.

Next we set

$$F(t,x,w) = X(t,x,w) \quad \text{and} \quad G(t,x,w) = \int_{X([0,t];x,w)} \theta .$$

Letting $\lambda_{t,x,y}$ be the image measure of $P^W[\cdot \,|X(t,x,w) = y]$ under the mapping: $w \longrightarrow \exp\{\sqrt{-1}G(t,x,w)\}$, we have

$$\text{(4.3)} \qquad K_t(x,y;\theta) = \int_T e^{\sqrt{-1}u}\lambda_{t,x,y}(du) , \quad x,y \in R^d , \quad t > 0 .$$

LEMMA 4.2 *For every positive constants* t *and* L, *there exists a positive constant* $\ell = \ell(t,L)$ *which may depend on* t *and* L *such that*

$$(4.4) \qquad \lambda_{t,x,y} \in M_\ell \qquad \textit{for} \quad x,y \in R^d \ , \ |x - y| \leqq L \ .$$

Proof. It is well-known that for every $t > 0$ and $x \in R^d$,

$$(4.5) \qquad F(t,x,w) \in \mathcal{D}^\infty(R^d) \ , \ G(t,x,w) \in \mathcal{D}^\infty \ ,$$

(cf. [10]). Furthermore, by using the assumptions 1.1 and 2.1, we can show that for every $p \geqq 1$

$$(4.6) \qquad \sup_{x \in R^d} \| (\det \sigma_{F(t,x,w)})^{-1} \|_p < \infty$$

where $\sigma_{F(t,x,w)}$ denotes the Malliavin covariance of $F(t,x,w)$. In order to obtain the uniform estimate like as (4.6) we need a delicate lower bounds on the Malliavin covariance in terms of vector fields $\{L_1, L_2, \cdots, L_d\}$ in (2.2). For details, see Kusuoka-Stroock [13]. It also holds, by the assumptions 1.1 and 2.1, that for every integer $p \geqq 1$ and $s \in R^1$, there is a positive constant C_1 satisfying the following:

$$(4.7) \qquad \|F(t,x,w) - x\|_{p,s} \leqq C_1 \ , \ \|G(t,x,w)\|_{p,s} \leqq C_1 \quad \text{for} \quad x \in R^d \ .$$

Letting

$$H(w) = \{h \in H \ ; \ \langle h, DX^i(t,x,w)\rangle_H = 0 \ , \ 1 \leqq i \leqq d\} \ ,$$

we consider a system $\mathcal{H}$ of closed subspaces of H given by

$$\mathcal{H} = \{H(w) \ ; \ w \in W_0^d\} \ .$$

Then $\mathcal{H}$ satisfies the assumption 3.1 and it holds that

$$(4.8) \qquad D_{\mathcal{H}}(\phi(e^{\sqrt{-1}G(t,x,w)})) = \frac{d\phi}{du}(e^{\sqrt{-1}G(t,x,w)}) D_{\mathcal{H}}G(t,x,w) \ , \ \phi \in C^\infty(\mathcal{T}) \ .$$

Here and from now on, we identify a function ψ on T with the periodic extended function on R^1. Setting $K(t,x,w) = (G(t,x,w), F(t,x,w))$, we have, by (3.29),

$$(4.9) \qquad \det \sigma_{K(t,x,w)} = (\det \sigma_{G(t,x,w),H})(\det \sigma_{F(t,x,w)})$$

where $\sigma_{K(t,x,w)}$ and $\sigma_{G(t,x,w),H}$ are the Malliavin covariance of $K(t,x,w)$ and the H-partial Malliavin covariance of $G(t,x,w)$ respectively. We also have

$$(4.10) \qquad \sigma_{G(t,x,w),H} = \langle D_H G(t,x,w), D_H G(t,x,w)\rangle_H .$$

We will now show that for every $p \geqq 1$

$$(4.11) \qquad \sup_{x\in R^d} \|(\det \sigma_{G(t,x,w),H})^{-1}\|_p < \infty .$$

By (4.7) and (4.9), this can be reduced to show that for every $p \geqq 1$

$$(4.12) \qquad \sup_{x\in R^d} \|(\det \sigma_{K(t,x,w)})^{-1}\|_p < \infty .$$

Let us consider the system $\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_d\}$ of vector fields on $O(R^d) \times R^1$ given by

$$\tilde{A}_\alpha(r,u) = L_\alpha(r) + \bar{\theta}_\alpha(r)\frac{\partial}{\partial u} , \quad \alpha = 1,2,\cdots,d .$$

Then it holds that for every $\alpha, \beta = 1,2,\cdots,d$

$$[\tilde{A}_\alpha, \tilde{A}_\beta](r,u) = [L_\alpha, L_\beta](r) + \overline{d\theta}_{\alpha,\beta}(r)\frac{\partial}{\partial u} , \quad \alpha,\beta = 1,2,\cdots,d$$

where $[\ ,\ ]$ means the Lie bracket and $\overline{d\theta} = (\overline{d\theta}_{\alpha,\beta})$ denotes the scalarization of $d\theta$, ([9]). Letting $\tilde{\pi}$ be the mapping defined by

$$\tilde{\pi} = \pi \times id : O(R^d) \times R^1 \longrightarrow R^d \times R^1$$

and using $\pi_*([L_\alpha, L_\beta]) = 0$, $\alpha,\beta = 1,2,\cdots,d$, we have

$$\tilde{\pi}_*([\tilde{A}_\alpha,\tilde{A}_\beta]) = \overline{d\theta}_{\alpha,\beta}\,\frac{\partial}{\partial u} \qquad \alpha,\beta = 1,2,\cdots,d$$

where π_* and $\tilde{\pi}_*$ are the differentials of the mappings π and $\tilde{\pi}$ respectively. Therefore, in the standard coordinate $((x^1,x^2,\cdots,x^d)$, $(e^i_j),u)$ in $O(R^d)\times R^1$, we have

$$\det(\tilde{\pi}_*\tilde{A}_1,\tilde{\pi}_*\tilde{A}_2,\cdots,\tilde{\pi}_*\tilde{A}_d,\tilde{\pi}_*([\tilde{A}_\alpha,\tilde{A}_\beta]))(r,u)$$

$$= \begin{vmatrix} e^i_j & 0 \\ \bar{\theta}_j(r) & \overline{d\theta}_{\alpha,\beta}(r) \end{vmatrix} = \det(e^i_j)\,\overline{d\theta}_{\alpha,\beta}(r)$$

for $(r,u)\in O(R^d)\times R^1$.

Hence

$$\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}(\det(\tilde{\pi}_*\tilde{A}_1,\tilde{\pi}_*\tilde{A}_2,\cdots,\tilde{\pi}_*\tilde{A}_d,\tilde{\pi}_*([\tilde{A}_\alpha,\tilde{A}_\beta]))^2$$

$$(4.13) \qquad = \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\det(e^i_j)^2\,\overline{d\theta}_{\alpha,\beta}^{\,2} = \det(g^{ij}(x))\,\|d\theta\|^2(x) .$$

On the other hand, by the assumptions 1.1 and 2.1,

$$\det(g^{ij}(x))\,\|d\theta\|^2(x) \geqq \left(\frac{1}{K}\right)^d C .$$

By combining (4.13) with this, we obtain (4.12), (cf. [26]).

By using (4.11), we obtain that for every $\phi\in C^\infty(T)$,

$$\int_T \frac{d\phi}{du}(e^{\sqrt{-1}u})\lambda_{t,x,y}(du)$$

$$(4.14) \qquad = \left\langle\frac{d\phi}{du}(e^{\sqrt{-1}G(t,x,w)}),\delta_y(F(t,x,w))\right\rangle(p(t,x,y)m(y))^{-1}$$

where $m(y) = \sqrt{\det(g^{ij}(y))}$ i.e., the density of the Riemannian volume with respect to the Lebesgue measure. By Proposition 3.1, (4.8) and (4.11), it holds that for every $\phi\in C^\infty(T)$,

$$E^W[\frac{d\phi}{du}(e^{\sqrt{-1}G(t,x,w)})\,|F(t,x,w) = y]$$

$$= E^W[\phi(e^{\sqrt{-1}G(t,x,w)})\delta_H(\frac{D_H G(t,x,w)}{<D_H G(t,x,w), D_H G(t,x,w)>_H})\,|F(t,x,w) = y] .$$

Hence, for $k > d$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$|E^W[\frac{d\phi}{du}(e^{\sqrt{-1}G(t,x,w)})\,|F(t,x,w) = y]|$$

$$\leqq \|\phi\|_\infty \{\|\delta_y(F(t,x,w))\|_{q,-k} \tag{4.15}$$

$$\times \|\{\delta_H(\frac{D_H G(t,x,w)}{<D_H G(t,x,w), D_H G(t,x,w)>_H})\}^2\|_{p,k}(<1,\delta_y(F(t,x,w))>)^{-1}\}^{1/2}$$

(see Watanabe [27]). By (4.6), there exists a constant C_2 independent of x and y such that

$$\|\delta_y(F(t,x,w))\|_{q,-k} \leqq C_2\|\delta_0\|_{C_{-k}} \tag{4.16}$$

where C_{-k} denotes the completion of $S(R^d)$ with respect to the norm

$$\|\cdot\|_{-k} = \|(1 + |x|^2 - \Delta)^{-k/2}\cdot\|_\infty ,$$

(cf. Watanabe [25]). We also note that, by (3.7),

$$\delta_H(\frac{D_H G(t,x,w)}{<D_H G(t,x,w), D_H G(t,x,w)>_H})$$

$$= \frac{<D_H<D_H G(t,x,w), D_H G(t,x,w)>_H, DG(t,x,w)>_H}{(<D_H G(t,x,w), D_H G(t,x,w)>_H)^2} \tag{4.17}$$

$$- \frac{L_H G(t,x,w)}{<D_H G(t,x,w), D_H G(t,x,w)>_H} .$$

Combining (4.7), (4.10), (4.11) with this, we obtain that

$$\sup_{x\in R^d} \|\{\delta_H(\frac{D_H G(t,x,w)}{<D_H G(t,x,w), D_H G(t,x,w)>_H})\}^2\|_{p,k} < \infty . \tag{4.18}$$

Furthermore it is well-known that for every $L > 0$,

$$(4.19) \qquad \sup_{x,y\in R^d,\,|x-y|\le L} (p(t,x,y)m(y))^{-1} < \infty .$$

By (4.15), (4.16), (4.18) and (4.19),

$$(4.20) \qquad \sup_{x,y\in R^d,\,|x-y|\le L} \left|E^W\left[\frac{d\phi}{du}\left(e^{\sqrt{-1}G(t,x,w)}\right) \middle| F(t,x,w) = y\right]\right| \leqq C_3\|\phi\|_\infty$$

where C_3 is a positive constant. Therefore, by (4.14), (4.19) and (4.20), we can conclude that for every $L > 0$

$$\sup_{x,y\in R^d,\,|x-y|\le L} \left|\int_T \frac{d\phi}{du}(e^{\sqrt{-1}u})\lambda_{t,x,y}(du)\right| \leqq C_4\|\phi\|_\infty$$

where C_4 is a positive constant. This completes the proof of the lemma.

Proof of Proposition 4.1 (4.1) is an easy consequence of the lemmas 4.1 and 4.2.

REMARK 4.1 By Proposition 4.1, we can show that

$$\sup \operatorname{spec} H < 0$$

where H is the self-adjoint operator in $L_2(m)$ defined by (2.22) and $\operatorname{spec} H$ is the set of spectrums of H. To see this, denoting by $\{Q_t; t \geqq 0\}$ the semi-group generated by H, it is enough to show that

$$\|Q_t\| < 1$$

for some $t > 0$ where $\|Q_t\|$ is the operator norm of Q_t.

First we remember that Q_t can be expressed as

$$Q_t f(x) = \int_{\mathbb{R}^d} K_t(x,y;\theta)p(t,x,y)f(y)m(dy) .$$

Hence we have

$$\|Q_t f\|^2_{L_2(m)} = \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} K_t(x,y;\theta)p(t,x,y)f(y)m(dy)\right|^2 m(dx)$$

$$\leqq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K_t(x,y;\theta)|^2 p(t,x,y)m(dy)\right.$$

$$\left.\times \int_{\mathbb{R}^d} p(t,x,y)|f(y)|^2 m(dy)\right)m(dx) \quad .$$

On the other hand, using Proposition 4.1, we obtain

$$\int_{\mathbb{R}^d} |K_t(x,y;\theta)|^2 p(t,x,y)m(dy)$$

$$\leqq \int_{|x-y|\leqq L} \beta(t,L)^2 p(t,x,y)m(dy) + \int_{|x-y|>L} p(t,x,y)m(dy)$$

$$= \beta(t,L)^2\left(1 - \int_{|x-y|>L} p(t,x,y)m(dy)\right) + \int_{|x-y|>L} p(t,x,y)m(dy)$$

$$= (1 - \beta(t,L)^2)\int_{|x-y|>L} p(t,x,y)m(dy) + \beta(t,L)^2.$$

But we can choose L so that

$$\int_{|x-y|>L} p(t,x,y)m(dy) \leqq \frac{1}{2} \ , \text{ for } \ x \in \mathbb{R}^d$$

(see the proof of Lemma 6.2). In this case we get

$$\|Q_t f\|^2_{L_2(m)} \leqq \frac{1}{2}(1+\beta(t,L)^2)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p(t,x,y)|f(y)|^2 m(dy)m(dx)$$

$$\leqq \frac{1}{2}(1+\beta(t,L)^2)\|f\|^2_{L_2(m)} \quad .$$

Thus we have

$$\|Q_t\| \leqq \{\tfrac{1}{2}(1+\beta(t,L)^2)\}^{1/2} < 1$$

which we desired. For related results, see [23].

5. THE CASE OF THE BROWNIAN MOTION

In this section, we always assume that g is the standard flat metric on R^d. Hence the assumption 2.1 and the condition (i) of the assumption 2.2 are automatically satisfied. Furthermore the corresponding diffusion process is the Brownian motion on R^d. Also we assume the assumptions 1.1 and 1.2. Furthermore we assume that the inequality (2.13) holds for any $x \in R^d$. Under these conditions we will show the estimate (2.15). To do this it is sufficient to show that for fixed $t_0 > 0$,

$$0 < a(t_0;\theta) < 1 . \tag{5.1}$$

In fact, it follows from (2.10) that

$$\log a(t + s;\theta) \leqq \log a(t;\theta) + \log a(s;\theta) \quad \text{for} \quad t,s > 0 .$$

Hence we have

$$\lim_{t\to\infty} \frac{1}{t} \log a(t;\theta) = \inf_{t>0} \frac{1}{t} \log a(t;\theta)$$

which implies (2.15).

Next, we note that

$$\lim_{t\to\infty} \frac{1}{t} \log a(t;\theta) \geqq \sup \operatorname{spec} H \tag{5.2}$$

where H is the operator defined by (2.22). To see this, note that $Q_t = e^{tH}$ is given by

$$Q_t f(x) = \int_{\mathbb{R}^d} K_t(x,y;\theta) p(t,x,y) f(y) m(dy) .$$

Therefore

$$\|Q_t f\|^2_{L_2(m)} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} K_t(x,y;\theta) p(t,x,y) f(y) m(dy) \right|^2 m(dx)$$

$$\leqq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |K_t(x,y;\theta)|^2 p(t,x,y) |f(y)|^2 m(dy) m(dx)$$

$$\leqq a(t;\theta)^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p(t,x,y) |f(y)|^2 m(dy) m(dx)$$

$$= a(t;\theta)^2 \|f\|^2_{L_2(m)} .$$

Thus it holds that

$$\|Q_t\| \leqq a(t;\theta) .$$

Hence we have

$$\sup \operatorname{spec} H = \lim_{t\to\infty} \frac{1}{t} \log \|Q_t\|$$

$$\leqq \lim_{t\to\infty} \frac{1}{t} \log a(t;\theta) .$$

Let $\{W_0^d, \mathcal{B}(W_0^d), P^W\}$ be the d-dimensional Wiener space. Then, as stated in Section 3, $\{W,H,P^W\}$ is an abstract Wiener space, where $W = W_0^d$ and H is the Hilbert space defined by (3.10) with $n = d$. Let us consider a system of vector fields V_α on R^{d+1} given by

$$V_\alpha \equiv \sum_{j=1}^{d+1} V_\alpha^j \partial_j = \partial_\alpha + \theta_\alpha \partial_{d+1} \qquad \alpha = 1,2,\cdots,d$$

where $\theta = \sum_{j=1}^{d} \theta_j dx^j$. Now we consider the following stochastic differential equation (S.D.E.) defined on $\{W_0^d, \mathcal{B}(W_0^d), P^W\}$

$$(5.3) \qquad d\bar{X}(t) = \sum_{\alpha=1}^{d} V_\alpha(\bar{X}(t)) \circ dw^\alpha(t) .$$

Let $\bar{X}(t,\bar{x},w)$, $\bar{x} = (x,x^{d+1}) \in R^d \times R^1$, be the solution of S.D.E. (5.3) with the initial condition $\bar{X}(0) = \bar{x}$. Set

$$\bar{X}(t,\bar{x},w) = (X(t,\bar{x},w), X^{d+1}(t,\bar{x},w)) \in R^d \times R^1 .$$

Then it is clear that $X(t,\bar{x},w)$, $\bar{x} = (x,x^{d+1}) \in R^d \times R^1$, is independent of x^{d+1} and depends only on x . Hence, for simplicity, we denote it by $X(t,x,w)$. Furthermore we note that

$$(5.4) \qquad X(t,x,w) = w(t) + x \qquad x \in R^d .$$

Also it is well-known that

$$\bar{X}(t,\bar{x},w) \in \mathcal{D}^\infty(R^{d+1}) .$$

Setting

$$G(t,x,w) = X^{d+1}(t,(x,0),w) ,$$

we have

$$(5.5) \quad G(t,x,w) = \int_{X([0,t];x,w)} \theta = \sum_{i=1}^{d} \int_0^t \theta_i(X(s,x,w)) \circ dX^i(s,x,w) ,$$

(see (2.5)). Hence, as stated in (2.8), $K_t(x,y;\theta)$ can be expressed in the form

$$(5.6) \qquad K_t(x,y;\theta) = E^W[\exp\{\sqrt{-1}G(t,x,w)\} | X(t,x,w) = y] .$$

For simplicity, without loss of generality, we set $t_0 = 1$. Now let

$$H(w) = \{h ; h \in H , \langle h, DX^i(1,x,w)\rangle = 0 , 1 \leq i \leq d\} .$$

Then the system $\mathcal{H} = \{H(w) ; w \in W_0^d\}$ satisfies the assumption 3.1. We can also define the operators $D_{\mathcal{H}}, \delta_{\mathcal{H}}$ and $L_{\mathcal{H}}$ as in Section 3. We will first prove the following lemmas:

LEMMA 5.1 *(Malliavin [17]). It holds that*

(5.7)
$$D_H G(1,x,w)[h] = -\sum_{\alpha=1}^{d}\int_0^1 (\eta_\alpha(s,x\ w) - \bar{\eta}_\alpha(x,w))\dot{h}^\alpha(s)ds$$
$$= -\sum_{\alpha=1}^{d}\int_0^1 \eta_\alpha(s,x,w)(\dot{h}^\alpha(s) - h^\alpha(1))ds \ , \ h \in H \ ,$$

where

(5.8)
$$\eta_\alpha(t,x,w) = \sum_{\beta=1}^{d}\int_0^t \gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w)$$
$$\bar{\eta}_\alpha(x,w) = \int_0^1 \eta_\alpha(s,x,w)ds \ .$$

and $\dot{h}^\alpha(s) = \frac{d}{ds}h^\alpha(s)$, $0 \leqq s \leqq 1$, $\alpha = 1,2,\cdots,d$.

We now set, for a continuous function f on $[a,b]$,

(5.9)
$$V_{[a,b]}(f) = \frac{1}{b-a}\int_a^b (f(t) - \bar{f})^2 dt \ , \ \bar{f} = \frac{1}{b-a}\int_a^b f(t)dt \ .$$

LEMMA 5.2 *(Malliavin [17]). It holds that*

(5.10)
$$\langle D_H G(1,x,w), D_H G(1,x,w)\rangle_H = \sum_{\alpha=1}^{d} V_{[0,1]}(\eta_\alpha) \ .$$

LEMMA 5.3 *(Malliavin [17]) . It holds that*

(5.11)
$$\operatorname{trace} DD_H G(1,x,w) = \sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(X(s,x,w))s(1-s)\circ dX^\alpha(s,x,w) \ .$$

LEMMA 5.4 *(Malliavin* [17]*). It holds that*

$$
\begin{aligned}
L_H G(1,x,w) &= \sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(X(s,x,w))(1-s)s\circ dX^\alpha(s,x,w) \\
&\quad - \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\{\int_0^1 X^\alpha(s,x,w)\gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w) \\
&\quad - [X^\alpha(0,x,w)\int_0^1 \gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w) \\
&\quad + (X^\alpha(1,x,w) - X^\alpha(0,x,w))\int_0^1 s\gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w)]\} .
\end{aligned}
\tag{5.12}
$$

Although Malliavin [17] has formulated the lemmas mentioned above in terms of the pinned Wiener space, we use the partial Malliavin calculus. This allows us to handle the results of Sections 4 and 5 in the same framework. Then, in order to prove the lemmas, we can use the general theory of Malliavin calculus in case of Wiener functionals obtained by Itô's calculus.

Proof of Lemma 5.1 We consider the Jacobian matrix $Y(t)$ at $\bar{x} = (x,0)$ given by

$$
Y(t) = (Y^i_j(t,\bar{x},w)) = (\partial_j X^i(s,\bar{x},w))
$$

and set

$$
Z(t) = (Z^i_j(t,\bar{x},w)) = Y(t)^{-1} ,
$$

(cf. [9]). Then we have, by (5.4) and (5.5),

$$(5.13)\qquad Y(t) = \begin{pmatrix} 1 & \ddots & 0 & 0 \\ & 0 & \ddots & 1 \\ * & \cdots & * & 1 \end{pmatrix} \quad \text{and} \quad z(t) = \begin{pmatrix} 1 & \ddots & 0 & 0 \\ & 0 & \ddots & 1 \\ * & \cdots & * & 1 \end{pmatrix}$$

and

$$(5.14)\qquad \begin{aligned} Y_i^{d+1}(t,\bar{x},w) &= \sum_{\alpha=1}^{d} \int_0^t \partial_i \theta_\alpha (X(s,x,w)) \circ dX^\alpha(s,x,w) \\ Z_i^{d+1}(t,\bar{x},w) &= -Y_i^{d+1}(t,\bar{x},w) \end{aligned} \quad , \ i = 1,2,\cdots,d .$$

Hence, by (5.6) in [10], we have

$$\begin{aligned} &DG(1,x,w)[h] \\ &= \sum_{\alpha=1}^{d} \int_0^1 \{Y_\alpha^{d+1}(1,\bar{x},w) - Y_\alpha^{d+1}(t,\bar{x},w) + \theta_\alpha(X(t,x,w))\}\dot{h}^\alpha(t)dt . \end{aligned}$$

Combining (5.14) with Itô's formula, we obtain

$$\begin{aligned} &Y_\alpha^{d+1}(1,\bar{x},w) - Y_\alpha^{d+1}(t,\bar{x},w) + \theta_\alpha(X(t,x,w)) \\ &= \sum_{\beta=1}^{d} \int_t^1 \gamma_{\alpha,\beta}(X(s,x,w)) \circ dX^\beta(s,x,w) + \theta_\alpha(X(1,x,w)) \quad , \ \alpha = 1,2,\cdots,d . \\ &= -\eta_\alpha(t,x,w) + \eta_\alpha(1,x,w) + \theta_\alpha(X(1,x,w)) \end{aligned}$$

Therefore, noting

$$(5.15)\qquad \begin{aligned} D_{H}G(1,x,w)[h] &= DG(1,x,w)[h] \\ &\quad - \sum_{i=1}^{d} \langle DG(1,x,w), DX^i(1,x,w)\rangle_H DX^i(1,x,w)[h] \\ DX^i(1,x,w)[h] &= \int_0^1 \dot{h}^i(s)ds \qquad \text{for} \quad h \in H \ , \ i = 1,2,\cdots,d , \end{aligned}$$

we can complete the proof of (5.7) with (5.8).

Proof of Lemma 5.2 (5.10) is an easy consequence of (5.7) with (5.8) and so details are omitted.

Proof of Lemma 5.3 It follows from (5.7) with (5.8) that

$$DD_{H}G(1,x,w)[h,g] = -\sum_{\alpha=1}^{d}\int_0^1 D\eta_\alpha(t,x,w)[g](\dot{h}^\alpha(t) - h^\alpha(1))dt , \quad h,g \in H .$$

Since

$$-\eta_\alpha(t,x,w) = -\sum_{\beta=1}^{d}\int_0^t \partial_\alpha\theta_\beta(X(s,x,w))\circ dX^\beta(s,x,w)$$

$$+ (\theta_\alpha(X(t,x,w)) - \theta_\alpha(X(0,x,w))) ,$$

we have, by (5.15),

$$DD_{H}G(1,x,w)[h,g]$$

$$= \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\sum_{k=1}^{d}\int_0^1 \partial_k\partial_\alpha\theta_\beta(X(s,x,w))g^k(s)[h^\alpha(s) - sh^\alpha(1)]\circ dX^\beta(s,x,w)$$

$$+ \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\int_0^1 \partial_\alpha\theta_\beta(X(s,x,w))[h^\alpha(s) - sh^\alpha(1)]\dot{g}^\beta(s)ds$$

$$+ \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\int_0^1 \partial_\beta\theta_\alpha(X(s,x,w))g^\beta(s)[\dot{h}^\alpha(s) - h^\alpha(1)]ds .$$

Let $\{\xi_k\}_{k=0,1,\cdots}$ be a complete orthonormal system on $L_2([0,1];dt)$ such that

$$\xi_0(t) = 1 , \quad 0 \leqq t \leqq 1 .$$

Setting

$$e^\alpha = (0,\cdots,0,\overset{\overset{\alpha}{\vee}}{1},0,\cdots,0)$$

$$h_k^\alpha(t) = \eta_k(t)e^\alpha \qquad \alpha = 1,2,\cdots,d , k = 0,1,\cdots ,$$

$$\eta_k(t) = \int_0^t \xi_k(s)ds$$

we obtain an orthonormal system $\{h_k^\alpha(t)\}_{\alpha=1,2,\cdots,d,k=0,1,\cdots}$ in H .

Then

$$\operatorname{trace} DD_H G(1,x,w)$$

$$= \sum_{\alpha=1}^{d} \sum_{k=1}^{\infty} \{ \sum_{j=1}^{d} \int_0^1 \partial_\alpha \partial_\alpha \theta_j (X(s,x,w))(\eta_k(s))^2 \circ dX^j(s,x,w)$$

$$+ 2\int_0^1 \partial_\alpha \theta_\alpha (X(s,x,w))\eta_k(s)\xi_k(s)ds\} \ .$$

Noting

$$\sum_{\alpha=1}^{d} \partial_\alpha \theta_\alpha = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} (\eta_k(s))^2 = s(1-s) \ ,$$

we obtain

$$\operatorname{trace} DD_H G(1,x,w) = \sum_{\alpha=1}^{d} \int_0^1 \Delta\theta_\alpha (X(s,x,w))s(1-s) \circ dX^\alpha(s,x,w)$$

which completes the proof.

Proof of Lemma 5.4 We first note that

$$L_H G(1,x,w) = \operatorname{trace} DD_H G(1,x,w) - D_H G(1,x,w)[w] \ .$$

Hence it is sufficient, by Lemma 5.3, to show that

$$D_H G(1,x,w)[w]$$

$$= \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \{ \int_0^1 X^\alpha(s,x,w)\gamma_{\alpha,\beta}(X(s,x,w)) \circ dX^\beta(s,x,w)$$

(5.16)

$$- [X^\alpha(0,x,w)\int_0^1 \gamma_{\alpha,\beta}(X(s,x,w)) \circ dX^\beta(s,x,w)$$

$$+ (X^\alpha(1,x,w) - X^\alpha(0,x,w))\int_0^1 s\gamma_{\alpha,\beta}(X(s,x,w)) \circ dX^\beta(s,x,w)]\} \ .$$

By Lemma 5.1, we have

$$D_H G(1,x,w)[w]$$

$$= - \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \{ \int_0^1 (\int_0^t \gamma_{\alpha,\beta}(X(s,x,w)) \circ dX^\beta(s,x,w)) \circ dX^\alpha(t,x,w)$$

$$- \int_0^1\int_0^t \gamma_{\alpha,\beta}(X(s,x\ w))\circ dX^\beta(s,x,w)dt(X^\alpha(1,x,w) - X^\alpha(0,x,w))\}$$

$$= - \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \{\int_0^1 \gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w)X^\alpha(1,x,w)$$

$$- \int_0^1 X^\alpha(s,x,w)\gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w)$$

$$- \int_0^1 (1 - s)\gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w)(X^\alpha(1,x,w) - X^\alpha(0,x,w))\}$$

which implies (5.16).

We now consider the following stochastic differential equation defined on $\{W_0^d, \mathcal{B}(W_0^d), P^W\}$:

$$dB(t) = dw(t) - \frac{B(t)}{1 - t} dt \quad , \quad 0 \leqq t < 1 \ , \tag{5.17}$$

and let $B(t,w)$ be the solution of (5.17) with $B(0) = 0$. We set

$$\xi(t,x,y,w) = B(t,w) + (x + t(y - x)) \ . \tag{5.18}$$

Then, as is well-known ([9]), conditional Wiener integrals can be re-written as follows:

LEMMA 5.5 (i)

$$E^W[|\sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \{\int_0^1 X^\alpha(s,x,w)\gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w)$$

$$- [X^\alpha(0,x,w)\int_0^1 \gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x\ w) + (X^\alpha(1,x,w)$$

$$- X^\alpha(0,x,w))\int_0^1 s\gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w)]\}|^2 | X(1,x,w) = y] \tag{5.19}$$

$$= E^W[|\sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \int_0^1 B^\alpha(s,w)\gamma_{\alpha,\beta}(\xi(s,x,y,w))\circ d\xi^\beta(s,x,y,w)|^2] ,$$

where E^W *denotes the expectation with respect to* P^W .

(ii)

$$
\begin{aligned}
(5.20)\quad & E^W[|\sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(X(s,x,w))(1-s)s\circ dX^\alpha(s,x,w)|^2|X(1,x,w)=y] \\
& = E^W[|\sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(\xi(s,x,y,w))(1-s)s\circ d\xi^\alpha(s,x,y,w)|^2] .
\end{aligned}
$$

(iii)

$$
\begin{aligned}
(5.21)\quad & E^W[(\langle D_H G(1,x,w), D_H G(1,x,w)\rangle_H)^{-p}|X(1,x,w)=y] \\
& = \sum_{\alpha=1}^{d} E^W[(\int_0^1(\zeta_\alpha(t,x,y,w)-\bar{\zeta}_\alpha(x,y,w))^2 dt)^{-p}]
\end{aligned}
$$

where

$$
\begin{aligned}
(5.22)\quad & \zeta_\alpha(t,x,y,w) = \sum_{\beta=1}^{d}\int_0^t \gamma_{\alpha,\beta}(\xi(s,x,y,w))\circ d\xi^\beta(s,x,y,w) , \\
& \bar{\zeta}_\alpha(x,y,w) = \int_0^1 \zeta_\alpha(t,x,y,w)\,dt .
\end{aligned}
$$

Proof. The image measure of P^W under the mapping: $W_0^d \ni w \longrightarrow \xi(t,x,y,w) \in W^d_{0,x;1,y}$ is the d-dimensional conditional Wiener measure where

$$
W^d_{0,x;1,y} = \{w \in W^d ; w(0) = x , w(1) = y\} .
$$

Hence (5.19) and (5.20) are clear. Futhermore (5.21) is an easy consequence of (5.7) and (5.8).

Before proceeding to estimating several conditional Wiener integrals, we note the following.

LEMMA 5.6 *There exists a positive constant* C_1 *such that*

$$
(5.23)\quad E^W[|B(t,w)|^2] \leqq C_1 t(1-t) , E^W[|B(t,w)|^4] \leqq C_1\{(t(1-t)\}^2 \qquad 0 \leqq t \leqq 1 .
$$

Proof. The proof is easy and is omitted.

LEMMA 5.7 *There exist positive constants* C_2 *and* C_3 *such that*

$$
\begin{aligned}
(5.24)\qquad & E^W[|\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\{\int_0^1 X^\alpha(s,x,w)\gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w) \\
& - [X^\alpha(0,x,w)\int_0^1 \gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w) + (X^\alpha(1,x,w) \\
& - X^\alpha(0,x,w))\int_0^1 s\gamma_{\alpha,\beta}(X(s,x,w))\circ dX^\beta(s,x,w)]\}|^2 | X(1,x,w) = y] \\
& \leqq C_2\{1 + (\sum_{\alpha=1}^{d}\{E^W[\int_0^1 |<\gamma_\alpha, \frac{y-x}{|y-x|}>(\xi(s,x,y,w))|^4 ds\}^{1/2})|y-x|^2\} \\
& \leqq C_3(1 + |y-x|^2) \ .
\end{aligned}
$$

Proof. By (5.17) and (5.18), we have

$$
\begin{aligned}
& \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\int_0^1 B^\alpha(s,w)\gamma_{\alpha,\beta}(\xi(s,x,y,w))\circ d\xi^\beta(s,x,y,w) \\
& = \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\int_0^1 B^\alpha(s,w)\gamma_{\alpha,\beta}(\xi(s,x,y,w))dw^\beta(s) \\
(5.25)\qquad & - \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\int_0^1 B^\alpha(s,w)\gamma_{\alpha,\beta}(\xi(s,x,y,w))\ \frac{B^\beta(s,w)}{1-s}\ ds \\
& + \sum_{\alpha=1}^{d}\int_0^1 B^\alpha(s,w)<\gamma_\alpha, \frac{y-x}{|y-x|}>(\xi(s,x,y,w))ds|y-x| \\
& + \frac{1}{2}\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\int_0^1 B^\alpha(s,w)\partial_\beta\gamma_{\alpha,\beta}(\xi(s,x,y,w))ds \ .
\end{aligned}
$$

By Lemma 5.6, there exist positive constants K_1 and K_2 independent of x and y such that

$$
\begin{aligned}
& \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\int_0^1 B^\alpha(s,w)\gamma_{\alpha,\beta}(\xi(s,x,y,w))\frac{B^\beta(s,w)}{1-s}\ ds = 0 \\
(5.26)\qquad & E^W[|\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\int_0^1 B^\alpha(s,w)\gamma_{\alpha,\beta}(\xi(s,x,y,w))dw^\beta(s)|^2] \leqq K_1
\end{aligned}
$$

$$E^W[|\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\int_0^1 B^\alpha(s,w)\partial_\beta\gamma_{\alpha,\beta}(\xi(s,x,y,w))ds|^2] \leqq K_2 .$$

Furthermore

$$E^W[|\sum_{\alpha=1}^{d}\int_0^1 B^\alpha(s,w)<\gamma_\alpha,\frac{y-x}{|y-x|}>(\xi(s,x,y,w))ds|y-x||^2]$$

(5.27)
$$\leqq d \times |y-x|^2 \sum_{\alpha=1}^{d}\int_0^1 (\sqrt{E^W[|B^\alpha(s,w)|^4]}\sqrt{E^W[|<\gamma_\alpha,\frac{y-x}{|y-x|}>(\xi(s,x,y,w))|^4]})ds$$

$$\leqq K_3|y-x|^2 \sum_{\alpha=1}^{d}\{\int_0^1 E^W[|<\gamma_\alpha,\frac{y-x}{|y-x|}>(\xi(s,x,y,w))|^4]ds\}^{1/2} ,$$

where K_3 is a positive constant independent of x and y. By combining (5.19),(5.25),(5.26) and (5.27) we can complete the proof of (5.24).

LEMMA 5.8 *There exist positive constants* c_4 *and* c_5 *such that*

$$E^W[|\sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(X(s,x,w))(1-s)s\circ dX^\alpha(s,x,w)|^2|X(1,x,w) = y]$$

(5.28)
$$\leqq c_4\{1 + \sum_{\alpha=1}^{d} E^W[\int_0^1 |\partial_\alpha<\gamma_\alpha,\frac{y-x}{|y-x|}>(\xi(s,x,y,w))|^2 ds]|y-x|^2\}$$

$$\leqq c_5(1 + |y-x|^2)$$

where $\partial_\alpha<\gamma_\alpha,\frac{y-x}{|y-x|}>(z) = \frac{\partial}{\partial z^\alpha}<\gamma_\alpha,\frac{y-x}{|y-x|}>(z)$.

Proof. Since $\delta\theta = 0$, we have

$$\sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(\xi(s,x,y,w))(1-s)s\circ d\xi^\alpha(s,x,y,w)$$

$$= \sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(\xi(s,x,y,w))(1-s)s d\xi^\alpha(s,x,y,w) .$$

Furthermore

(5.29)
$$\Delta\theta_\alpha = -\sum_{\beta=1}^{d}\partial_\beta\gamma_{\alpha,\beta} \qquad \alpha = 1,2,\cdots,d ,$$

and

$$(5.30)\qquad \sum_{\alpha=1}^{d} \Delta\theta_\alpha(z)(y^\alpha - x^\alpha) = \sum_{\alpha=1}^{d} \partial_\alpha \langle \gamma_\alpha, \frac{y-x}{|y-x|}\rangle (z)\,|y-x| \;.$$

Hence

$$E^W[|\sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(\xi(s,x,y,w))(1-s)s\circ d\xi^\alpha(s,x,y,w)|^2]$$

$$(5.31)\qquad \begin{aligned} &= E^W[|\sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(\xi(s,x,y,w))(1-s)s dw^\alpha(s) \\ &+ \sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(\xi(s,x,y,w))sB^\alpha(s,w)ds \\ &+ \sum_{\alpha=1}^{d}\int_0^1 \partial_\alpha\langle\gamma_\alpha,\frac{y-x}{|y-x|}\rangle(\xi(s,x,y,w))s(1-s)ds\,|y-x|\,|^2] \;. \end{aligned}$$

It is easy to see that for some positive constants K_4, K_5, K_6, K_7,

$$E^W[|\sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(\xi(s,x,y,w))(1-s)s dw^\alpha(s)|^2] \leqq K_4 \;,$$

$$E^W[|\sum_{\alpha=1}^{d}\int_0^1 \Delta\theta_\alpha(\xi(s,x,y,w))sB^\alpha(s,w)ds|^2] \leqq K_5 \;,$$

$$(5.32)\qquad \begin{aligned} &E^W[|\sum_{\alpha=1}^{d}\int_0^1 \partial_\alpha\langle\gamma_\alpha,\frac{y-x}{|y-x|}\rangle(\xi(s,w,y,w))s(1-s)ds\,|y-x|\,|^2] \\ &\leqq K_6 \sum_{\alpha=1}^{d} E[\int_0^1 |\partial_\alpha\langle\gamma_\alpha,\frac{y-x}{|y-x|}\rangle(\xi(s,x,y,w))|^2 ds]\,|y-x|^2 \\ &\leqq K_7 |y-x|^2 \;. \end{aligned}$$

By combining (5.20),(5.31) and (5.32) we can complete the proof.

By using the lemmas 5.4,5.7 and 5.8 we obtain the following.

PROPOSITION 5.1 *There exists a positive constant* C_6 *such that*

$$(5.33)\qquad E^W[|L_H G(1,x,w)|^2 | X(1,x,w) = y] \leqq C_6(1 + |y-x|^2) \;.$$

REMARK. 5.1 Let $d = 2$ and θ be the differential 1-form,

$$\theta = \frac{1}{2}(x^1 dx^2 - x^2 dx^1) . \tag{5.34}$$

Then θ satisfies the assumption 2.1 and the condition (ii) of the assumption 2.2. In this case, by (5.12),

$$\begin{aligned} L_H G(1,x,w) = &- \int_0^1 X^1(s,x,w) \circ dX^2(s,x,w) + [X^1(0,x,w)\int_0^1 dX^2(x,s,w) \\ &+ (X^1(1,x,w) - X^1(0,x,w))\int_0^1 s dX^2(s,x,w)] \\ &+ \int_0^1 X^2(s,x,w) \circ dX^1(s,x,w) - [X^2(0,x,w)\int_0^1 dX^1(s,x,w) \\ &+ (X^2(1,x,w) - X^2(0,x,w))\int_0^1 s dX^1(s,x,w)] . \end{aligned}$$

Therefore we have

$$\begin{aligned} &E^W[|L_H G(1,x,w)|^2 | X(1,x,w) = y] \\ &= E^W[|\int_0^1 B^1(s,w) dw^2(s) - \int_0^1 B^2(x,w) dw^1(s) \\ &\quad + ((y^2 - x^2)\int_0^1 B^1(s,w) ds - (y^1 - x^1)\int_0^1 B^2(s,w) ds)|^2] \\ &= E^W[|\int_0^1 B^1(s,w) dw^2(s) - \int_0^1 B^2(s,w) dw^1(s)|^2] \\ &\quad + E^W[|(y^2 - x^2)\int_0^1 B^1(s,w) ds - (y^1 - x^1)\int_0^1 B^2(s,w) ds|^2] \\ &= E^W[|\int_0^1 B^1(s,w) dw^2(s)|^2] + E^W[|\int_0^1 B^2(s,w) dw^1(s)|^2] \\ &\quad + |y^2 - x^2|^2 E[|\int_0^1 B^1(s,w) ds|^2] + |y^1 - x^1|^2 E[|\int_0^1 B^2(s,w) ds)|^2] . \end{aligned}$$

Hence

$$E^W[|L_H G(1,x,w)|^2 | X(1,x,w) = y] = \frac{1}{3} + \frac{1}{12}|x - y|^2 . \tag{5.35}$$

This means that under the assumption 2.2, the estimate (5.33) is sufficiently sharp.

Next we prepare a general lemma.

LEMMA 5.9 *(cf. D. Stroock [24]). For every continuous function* $f : [0,T] \longrightarrow R^1$ *with*

$$f(t) \geqq \varepsilon > 0 \qquad \text{for} \quad 0 \leqq t \leqq T ,$$

$$V_{[0,T]}(g) \geqq \frac{\varepsilon^2}{12} T^2 \tag{5.36}$$

where

$$g(t) = \int_0^t f(s)\,ds , \qquad 0 \leqq t \leqq T .$$

Proof. For some t_0, $0 \leqq t_0 \leqq T$,

$$\bar{g} = \int_0^{t_0} f(s)\,ds .$$

Hence

$$V_{[0,T]}(g) = \frac{1}{T}\left\{\int_0^{t_0} dt\left(\int_t^{t_0} f(s)\,ds\right)^2 + \int_{t_0}^{T} dt\left(\int_{t_0}^{t} f(s)\,ds\right)^2\right\}$$

$$\geqq \varepsilon^2 \frac{1}{T}\int_0^T (t - t_0)^2 dt \geqq \varepsilon^2 V_{[0,T]}(h) = \frac{\varepsilon^2}{12} T^2$$

where

$$h(t) = t \qquad , \qquad 0 \leqq t \leqq T .$$

In order to prove (5.1) the following proposition plays an important role.

PROPOSITION 5.2 *For every positive integer* $p \geqq 2$ *, there exists a positive constant* c_7 *independent of* x *and* y *such that*

$$E^W[|D_{\mathcal{H}}G(1,x,w)|_H^{-p}|X(1,x,w) = y] \leqq C_7(1 + |x - y|^2)^{-p/2} . \tag{5.37}$$

Proof. Since (5.29) holds, we obtain

$$\begin{aligned}\zeta_\alpha(t,x,y,w) &= \sum_{\beta=1}^{d}\int_0^t \gamma_{\alpha,\beta}(\xi(s,x,y,w))\,dw^\beta(s) \\ &\quad - \sum_{\beta=1}^{d}\int_0^t \gamma_{\alpha,\beta}(\xi(s,x,y,w))\frac{B^\beta(s,w)}{1 - s}\,ds \\ &\quad + |y - x|\int_0^t \langle\gamma_\alpha, \frac{y - x}{|y - x|}\rangle(\xi(s,x,y,w))\,ds + \frac{1}{2}\int_0^t \Delta\theta_\alpha(\xi(s,x,y,w))\,ds\end{aligned} \tag{5.38}$$

$$\alpha = 1,2,\cdots,d .$$

We set

$$\zeta(t,x,y,w) = \sum_{\alpha=1}^{d}\zeta_\alpha(t,x,y,w) \tag{5.39}$$

and

$$A(t,w) = \sum_{\beta=1}^{d}\int_0^t (\langle\gamma_\beta,\underline{1}\rangle(\xi(s,x,y,w)))^2\,ds$$

where $\underline{1} = (1,1,\cdots,1)$. Then, by the assumption 2.2, there exist positive constants C_8, C_9 independent of x and y such that

$$\begin{aligned}&C_8 \leqq \sum_{\beta=1}^{d}(\langle\gamma_\beta,\underline{1}\rangle(z))^2 \leqq C_9 \\ &C_8 t \leqq A(t,w) \leqq C_9 t \qquad , \quad 0 \leqq t \leqq 1 .\end{aligned} \tag{5.40}$$

We also consider

$$a(t,x,y,w) = \zeta(A^{-1}(t,w),x,y,w) \qquad , \quad 0 \leqq t < A(t,w)$$

where $A^{-1}(t,w)$ is the inverse function of $t \longrightarrow A(t,w)$. Then we have

$$V_{[0,1]}(\zeta(\cdot,x,y,w)) = \int_0^1 (a(A(s,w),x,y,w) - \bar{\zeta}(x,y,w))^2\,ds \tag{5.41}$$

where

$$\bar{\zeta}(x,y,w) = \int_0^1 \zeta(s,x,y,w)\,ds \ .$$

Setting

$$\sigma_N(w) = \inf\{s\ ;\ |B(s,w)| \geqq 1\} \wedge (1/N^2) \ ,$$

we obtain that for some C_{10} and C_{11} ,

$$(5.42) \qquad P^W[\sigma_N(w) \neq \frac{1}{N^2}] \leqq 2d \exp[-C_{10}N^2] \qquad \text{for} \quad N^2 \geqq C_{11} \ ,$$

([9]). Next, if $1/N^3 \leqq C_8$, by (5.40) and (5.41),

$$V_{[0,1]}(\zeta(\cdot,x,y,w)) \geqq \frac{1}{C_9}\int_0^{C_8} (a(s,x,y,w) - \bar{\zeta}(s,y,w))^2 ds$$

$$\geqq \frac{1}{C_9}\frac{1}{N^3} V_{[0,1/N^3]}(a(\cdot,x,y,w))$$

([27]). It follows from this that for $1/N^3 \leqq C_8$,

$$(5.43) \qquad \begin{aligned} &\{w; V_{[0,1]}(\zeta(\cdot,x,y,w)) \leqq \frac{1}{C_9}\frac{1}{N^7}\} \\ &\subseteq \{w;\sigma_N(w) \neq \frac{1}{N^2}\} \cup \{w; V_{[0,1/N^3]}(a(\cdot,x,y,w)) \leqq \frac{1}{N^4}\ ,\ \sigma_N(w) = \frac{1}{N^2}\} \ . \end{aligned}$$

On the other hand, by the general theory of martingales, there exists a Brownian motion $\{\beta(t,w)\}$ such that

$$(5.44) \qquad a(t,x,y,w) = \beta(t,w) + m(t,x,y,w) \quad , \quad 0 \leqq t \leqq A(1,w)$$

where

$$\begin{aligned} m(t,x,y,w) = &-\int_0^{A^{-1}(t,w)} \sum_{\alpha=1}^{d} \langle\gamma_\alpha, B(s)\rangle(\xi(s,x,y,w))\,\frac{ds}{1-s} \\ &+ |y-x|\int_0^{A^{-1}(t,w)} \sum_{\alpha=1}^{d} \langle\gamma_\alpha,\frac{y-x}{|y-x|}\rangle(\xi(s,x,y,w))\,ds \\ &+ \frac{1}{2}\int_0^{A^{-1}(t,w)} \sum_{\alpha=1}^{d} \Delta\theta_\alpha(\xi(s,x,y,w))\,ds \ . \end{aligned}$$

Then, if $\sigma_N(w) = 1/N^2$, $N \geqq \frac{1}{C_8} \vee 1$,

$$|m(t,x,y,w)| \leqq C_{12}(1 + |y - x|)t \ , \quad \text{for} \quad t \leqq \frac{1}{N^3}$$

where C_{12} is a positive constant independent of x and y . Hence, if $\sigma_N(w) = 1/N^2$, $N \geqq \frac{1}{C_8} \vee 1$, then

$$V_{[0,1/N^3]}(m(\cdot,x,y,w)) \leqq \left(\frac{C_{12}}{\sqrt{3}}(1 + |y - x|)\frac{1}{N^3}\right)^2 . \tag{5.45}$$

Since

$$V^{1/2}_{[0,1/N^3]}(\beta) \leqq V^{1/2}_{[0,1/N^3]}(a) + V^{1/2}_{[0,1/N^3]}(m)$$

([27]), if $N \geqq (C_8)^{-1} \vee 1$ and

$$\frac{1}{\sqrt{3}} C_{12}(1 + |x - y|) \leqq N ,$$

then

$$\begin{aligned} &\{w \,;\, V^{1/2}_{[0,1/N^3]}(a(\cdot,x,y,w)) \leqq \frac{1}{N^2} , \sigma_N(w) = \frac{1}{N^2}\} \\ &\subset \{w \,;\, V^{1/2}_{[0,1/N^3]}(\beta(\cdot,w)) \leqq \frac{2}{N^2}\} . \end{aligned} \tag{5.46}$$

By combining (5.42),(5.43),(5.46) with Lemma V-8.6 in [9], we obtain that if $N \geqq (C_8)^{-1} \vee (1 \vee \sqrt{C_{11}} \vee C_9)$ and

$$N \geqq \frac{\sqrt{2}}{\sqrt{3}} C_{12}(1 + |x - y|) ,$$

then

$$P^W[V_{[0,1]}(\zeta(\cdot,x,y,w)) \leqq \frac{1}{N^8}] \leqq C_{13}e^{-C_{14}N} \tag{5.47}$$

where C_{13} and C_{14} are positive constants independent of x and y . This implies that for some positive constant C_{15} ,

$$E^W[V_{[0,1]}(\zeta(\cdot,x,y,w))^{-p}] \leqq C_{15}(1+|x-y|)^{8p+1}$$

(5.48)

$$E^W[(\sum_{\alpha=1}^{d} V_{[0,1]}(\zeta_\alpha(\cdot,x,y,w)))^{-p}] \leqq C_{15}(1+|x-y|)^{8p+1} .$$

Next we will show (5.37) by using (5.48). It follows from the assumption 2.2 that for $(y-x)/|y-x| \in S^{d-1}$ we can choose an integer $\alpha = \alpha(x,y)$ satisfying

(5.49) $$a_1 \leqq (\langle\gamma_\alpha, \frac{y-x}{|y-x|}\rangle(z))^2 \leqq a_2 , \quad \text{for } z \in R^d .$$

We set

$$F(x,y;w) = \sum_{\beta=1}^{d} V_{[0,1]}(\zeta_\beta(\cdot,x,y,w))$$

$$f_\alpha^{(1)}(t,x,y,w) = \int_0^t \langle\gamma_\alpha, \frac{y-x}{|y-x|}\rangle(\xi(s,x,y,w))ds\,|y-x|$$

$$f_\alpha^{(2)}(t,x,y,w) = \int_0^t \frac{1}{1-s}\langle\gamma_\alpha, B(s,w)\rangle(\xi(s,x,y,w))ds$$

(5.50)

$$f_\alpha^{(3)}(t,x,y,w) = \int_0^t \langle\gamma_\alpha, dw(s)\rangle(\xi(s,x,y,w))$$

$$= \sum_{\beta=1}^{d}\int_0^t \gamma_{\alpha,\beta}(\xi(s,x,y,w))dw^\beta(s)$$

$$f_\alpha^{(4)}(t,x,y,w) = \frac{1}{2}\int_0^t \Delta\theta_\alpha(\xi(s,x,y,w))ds .$$

By Lemma 5.9 and (5.49), we have

(5.51) $$V_{[0,1/2]}(f_\alpha^{(1)}(\cdot,x,y,w)) \geqq \frac{a_1}{48}|x-y|^2 .$$

Setting

$$K_8 = \sup\{|\gamma_{ij}(z)|, |\Delta\theta_i(z)|, z \in R^d, i,j = 1,2,\cdots,d\} ,$$

we have

$$(5.52)\qquad \begin{aligned} &V_{[0,1/2]}(f_\alpha^{(2)}(\cdot,x,y,w)) \leqq \frac{d^2}{3} K_8^2 \Big(\sup_{0\le s\le \frac{1}{2}} |B(s,w)|\Big)^2 \\ &V_{[0,1/2]}(f_\alpha^{(4)}(\cdot,x,y,w)) \leqq \frac{K_8^2}{48} . \end{aligned}$$

Next we note that there exists a 1-dimensional Brownian motion $\{\tilde\beta(t,w)\}$ and the time change function $\tau(t,w)$ such that

$$f_\alpha^{(3)}(t,x,y,w) = \tilde\beta(\tau(t,w),w) \quad , \ \tau(t,w) = \int_0^t \Big(\sum_{\beta=1}^d \gamma_{\alpha,\beta}(\xi(s,x,y,w))^2\Big) ds .$$

Hence, for some positive constant K_9 (independent of x and y)

$$\tau(t,w) \leqq K_9 t$$

and

$$\sup_{0\le t\le 1} |f_\alpha^{(3)}(t,x,y,w)| \leqq \sup_{0\le t\le K_9} |\tilde\beta(t,w)| .$$

Therefore we have

$$(5.53)\qquad V_{[0,1/2]}(f_\alpha^{(3)}(\cdot,x,y,w)) \leqq \Big(\sup_{0\le t\le K_9} |\tilde\beta(t,w)|\Big)^2 .$$

We now set

$$\begin{aligned} \Omega(x,y) = \{w ; \ &\sup_{0\le t\le 1/2} |B(s,w)| \leqq \frac{1}{6}\frac{\sqrt{3}}{dK_8}\sqrt{\frac{a_1}{48}}\,|x-y| , \\ &\sup_{0\le t\le K_1} |\tilde\beta(t,w)| \leqq \frac{1}{6}\sqrt{\frac{a_1}{48}}\,|x-y|\} . \end{aligned}$$

Then, noting

$$\begin{aligned} (V_{[0,1/2]}(\zeta_\alpha(\cdot,x,y,w)))^{1/2} \geqq\ &(V_{[0,1/2]}(f_\alpha^{(1)}(\cdot,x,y,w)))^{1/2} \\ &- \sum_{k=2}^4 (V_{[0,1/2]}(f_\alpha^{(k)}(\cdot,x,y,w)))^{1/2} , \end{aligned}$$

([27]), we obtain that if

$$|x - y| \geqq \frac{K_8}{\sqrt{a_1}} \times 6$$

then for $w \in \Omega(x,y)$,

$$(V_{[0,1/2]}(\zeta_\alpha(\cdot,x,y,w)))^{1/2}$$

$$\text{(5.54)} \quad \begin{aligned} &\geqq \sqrt{\frac{a_1}{48}}|x-y| - [\frac{K_8 d}{\sqrt{3}} \times \frac{\sqrt{3}}{K_8 d} \frac{1}{6} \sqrt{\frac{a_1}{48}}|x-y| + \frac{1}{6}\sqrt{\frac{a_1}{48}}|x-y| + \frac{K_8}{\sqrt{48}}] \\ &\geqq \sqrt{\frac{a_1}{48}}|x-y| - [\frac{1}{3}\sqrt{\frac{a_1}{48}}|x-y| + \frac{1}{6}\sqrt{\frac{a_1}{48}}|x-y|] \\ &= \frac{1}{2}\sqrt{\frac{a_1}{48}}|x-y| \ . \end{aligned}$$

By combining this with

$$F(x,y,w) \geqq V_{[0,1]}(\zeta_\alpha(\cdot,x,y,w)) \geqq \frac{1}{2} V_{[0,1/2]}(\zeta_\alpha(\cdot,x,y,w)) \ ,$$

we obtain that if

$$|x - y| \geqq \frac{K_8}{\sqrt{a_1}} \times 6$$

$$\text{(5.55)} \qquad E^W[F(x,y,w)^{-\frac{p}{2}} ; \Omega(x,y)] \leqq (2\sqrt{\frac{96}{a_1}})^p |x-y|^{-p} \ .$$

On the other hand, there exist positive constants C_{16} and C_{17} independent of x and y such that

$$\text{(5.56)} \qquad P^W[\Omega(x,y)^c] \leqq C_{16} \exp[-C_{17}|x - y|^2] \ .$$

By (5.48),(5.55) and (5.56), if

$$|x - y| \geqq \frac{K_8}{\sqrt{a_1}} \times 6$$

then

$$E^W[|D_H G(1,x,w)|_H^{-p} | X(1,x,w) = y]$$

$$= E^W[F(x,y,w)^{-p/2}]$$

$$\leqq E^W[F(x,y,w)^{-p/2} ; \Omega(x,y)] + (E^W[F(x,y,w)^{-p}])^{1/2}(P^W[\Omega(x,y)^c])^{1/2}$$

$$\leqq (2\sqrt{\frac{96}{a_1}})^p |x-y|^{-p} + C_{15}^{1/2}(1+|x-y|)^{(8p+1)/2}\sqrt{C_{16}}\exp[-C_{17}|x-y|^2/2]$$

which implies (5.37).

We now return to the proof of (5.1). By using (3.22), (3.30), (4.8) and (4.17) we obtain that for every $\phi \in C^\infty(T)$,

$$|E^W[\frac{d\phi}{du}(e^{\sqrt{-1}G(1,x,w)}) | X(1,x,w) = y]|$$

$$\leqq \|\phi\|_\infty \{E^W[\frac{2\langle D_H^2 G(1,x,w), D_H G(1,x,w) \otimes D_H G(1,x,w)\rangle_H}{|D_H G(1,x,w)|_H^4} | X(1,x,w) = y]$$

$$+ E^W[\frac{|L_H G(1,x,w)|}{|D_H G(1,x,w)|_H^2} | X(1,x,w) = y]\}$$

$$\leqq 3\|\phi\|_\infty (E^W[|L_H G(1,x,w)|^2 | X(1,x,w) = y])^{1/2}$$

$$\times (E^W[|D_H G(1,x,w)|_H^{-4} | X(1,x,w) = y])^{1/2} .$$

Hence, by Propositions 5.1 and 5.2, if

$$|x - y| \geqq \frac{K_8}{\sqrt{a_1}} \times 6$$

then, for $\phi \in C^\infty(T)$,

$$|E^W[\frac{d\phi}{du}(e^{\sqrt{-1}G(1,x,w)}) | X(1,x,w) = y]|$$

$$\leqq 3\|\phi\|_\infty (C_6(1 + |y - x|^2))^{1/2}(C_7(1 + |x - y|^2)^{-2})^{1/2}$$

$$= 3\sqrt{C_6 C_7}\|\phi\|_\infty (1 + |x - y|^2)^{-1/2} .$$

By combining this with Lemma 4.2, for $\phi \in C^{\infty}(T)$,

$$(5.57) \qquad \left|E^{W}\left[\frac{d\phi}{du}\left(e^{\sqrt{-1}G(1,x,w)}\right)\middle|X(1,x,w) = y\right]\right| \leqq C_{18}\|\phi\|_{\infty}$$

where C_{18} is a positive constant independent of x and y . By Lemma 4.1 and (5.57), we have

$$a(1;\theta) < 1$$

which completes the proof of (5.1).

Before closing this section, we give some remarks. Setting

$$e_i = (0,\cdots,\overset{\overset{i}{\vee}}{1},\cdots,0) \qquad i = 1,2,\cdots,d ,$$

if, for $1 < m \leqq i \leqq d$

$$(5.58) \qquad \langle\gamma_{\alpha},e_i\rangle(z) = 0 \quad , \quad z \in R^d \quad , \quad \alpha = 1,2,\cdots,d ,$$

then the assumption 2.2 does not hold. In this case,

$$\gamma_{\alpha,\beta}(z) = 0 \quad , \quad z \in R^d \quad , \quad \alpha = 1,2,\cdots, d \, , \, \beta = m,m+1,\cdots,d .$$

Hence, by (5.8),

$$\eta_{\alpha}(t,x,y,w) = \sum_{\beta=1}^{m-1}\int_0^t \gamma_{\alpha,\beta}(X(s,x,w))\circ dX^{\beta}(s,x,w) \quad , \quad \alpha = 1,2,\cdots,d .$$

Furthermore if we assume that

$$\gamma_{\alpha,\beta}(z) \, , \alpha = 1,2,\cdots,d \, , \beta = 1,2,\cdots,m-1 \, , z = (z^1,\cdots,z^{m-1},z^m,\cdots,z^d) \in R^d$$

depend only on $(z^1,z^2,\cdots,z^{m-1})$, then $D_H G(1,x,w)[h]$ is independent of $\{(X^m(t,x,w),X^{m+1}(t,x,w),\cdots,X^d(t,x,w)), 0 \leqq t < \infty\}$. Therefore

$$E^W[|D_H G(1,x,w)|^{-p}|X(1,x,w) = y]$$

$$E^W[|L_H G(1,x,w)|^2|X(1,x,w) = y]$$

are independent of $(x^m, x^{m+1}, \cdots, x^d)$ and $(y^m, y^{m+1}, \cdots, y^d)$. Under these assumptions, in many cases, (5.57) (also (2.15)) still holds. In fact, in such cases, the problem can be reduced to one in case of R^{m-1}. Next we return to the remark 5.1. In this case, we have

$$<D_H^2 G(1,x,w), h \otimes g> = 2\{\int_0^1 (\int_0^t (\dot{h}^1(s) - h^1(1))ds)(\dot{g}^2(t) - g^2(1))dt$$

$$+ \int_0^1 (\int_0^t (\dot{g}^2(s) - g^2(1))ds)(\dot{h}^1(t) - h^1(1))dt\} .$$

It follows from this that for some $C > 0$,

$$|D_H^2 G(1,x,w)|_{H \otimes H} \leqq C$$

and

$$|\delta_H(\{<D_H G(1,x,w), D_H G(1,x,w)>_H\}^{-1} D_H G(1,x,w))|$$

$$\geqq (|L_H G(1,x,w)| - C)|D_H G(1,x,w)|_H^{-2} .$$

This means that in order to show (5.1) we need the estimate (5.37).

6. THE PROOF OF THEOREM 2.1

First we choose a positive constant R_0 such that

$$D \subset B(0,R_0) \quad \text{where} \quad B(a,r) = \{z \in R^d ; |a - z| < r\} .$$

We now show the follwing two lemmas.

LEMMA 6.1 *For every* $t > 0$, *there exist* $R(t) \geqq R_0$ *and* $0 \leqq \beta(t) < 1$ *such that if* $x,y \in R^d$ *and* $\ell(x,y) \cap \overline{B(0,R(t))} = \phi$,

(6.1) $$|K_t(x,y;\theta)| \leqq \beta(t) ,$$

where

$$\ell(x,y) = \{z \in R^d \,;\, z = x + \eta(y-x) \,,\, 0 \leqq \eta \leqq 1\} \,.$$

Proof. We consider the first hitting time $\tau(w)$ for $B(0,R_0)$, i.e.,

$$\tau(w) = \inf\,\{t \geqq 0 \,;\, w(t) \in B(0,R_0)\} \,, \quad w \in W^d \,.$$

Then we have

$$K_t(x,y;\theta) = E^W[e^{\sqrt{-1}G(t,x,w)} \,;\, \tau(X(\cdot;x,w)) > t \,|\, X(t,x,w) = y]$$

$$+ E^W[e^{\sqrt{-1}G(t,x,w)} \,;\, \tau(X(\cdot;x,w)) \leqq t \,|\, X(t,x,w) = y]$$

(6.2)
$$= E^W[e^{\sqrt{-1}\int_{B([0,t];x,w)}\theta} \,;\, \tau(B(\cdot;x,w)) > t \,|\, B(t;x,w) = y] \frac{g(t,x,y)}{p(t,x,y)m(y)}$$

$$+ E^W[e^{\sqrt{-1}G(t,x,w)} \,;\, \tau(X(\cdot;x,w)) \leqq t \,|\, X(t,x,w) = y]$$

where $B(t;x,w) = x + w(t)$ and $g(t,x,y)$, $t > 0$, $x,y \in R^d$, denotes the heat kernel on R^d, i.e.,

$$g(t,x,y) = \left(\frac{1}{\sqrt{2\pi t}}\right)^d \exp\{-\frac{|x-y|^2}{2t}\} \,.$$

It follows from (6.2) that

$$|K_t(x,y;\theta)|$$

(6.3)
$$\leqq |E^W[\exp\{\sqrt{-1}\int_{B([0,t];x,w)}\theta\} \,;\, \tau(B(\cdot;x,w)) > t \,|\, B(t;x,w) = y]\,| \frac{g(t,x,y)}{p(t,x,y)m(y)}$$

$$+ |P^W[\tau(X(\cdot;x,w)) \leqq t \,|\, X(t;x,w) = y]\,| \quad .$$

In order to estimate the first term of the right hand in (6.3), we can assume that (2.13) holds for all $z \in R^d$. Hence we obtain, by using the result of Section 5, that there exists a positive constant $\gamma(t) < 1$

satisfying

$$|K_t(x,y;\theta)| \leqq \gamma(t)g(t,x,y)/(p(t,x,y)m(y))$$

(6.4) $$+ P^W[\tau(B(\cdot;x,w)) \leqq t | B(t;x,w) = y]\frac{g(t,x,y)}{p(t,x,y)m(y)}$$

$$+ P^W[\tau(X(\cdot;x,w)) \leqq t | X(t;x,w) = y] \ .$$

We also have

$$\frac{g(t,x,y)}{p(t,x,y)m(y)} = \frac{P^W[\tau(X(\cdot;x,w)) > t | X(t;x,w) = y]}{P^W[\tau(B(\cdot;x,w)) > t | B(t;x,w) = y]}$$

$$\leqq \frac{1}{P^W[\tau(B(\cdot;x,w)) > t | B(t;x,w) = y]} \ .$$

Hence for $\varepsilon = (1 - \gamma(t))/8$, there exists a positive constant $R(t)$ such that if $\ell(x,y) \cap \overline{B(0,R(t))} = \phi$,

$$\frac{g(t,x,y)}{p(t,x,y)m(y)} \leqq 1 + \varepsilon$$

$$P^W[\tau(B(\cdot;x,w)) \leqq t | B(t;x,w) = y] \leqq \varepsilon$$

$$P^W[\tau(X(\cdot;x,w)) \leqq t | X(t;x,w) = y] \leqq \varepsilon \ ,$$

([2]). Combining this with (6.4), we have

$$|K_t(x,y;\theta)| \leqq \gamma(t)(1 + \varepsilon) + \varepsilon(1 + \varepsilon) + \varepsilon \leqq \frac{1+\gamma(t)}{2} < 1 \ ,$$

which completes the proof of (6.1).

LEMMA 6.2 *There exist positive constants* $R_1 \geqq R(1)$ *and* $0 < \beta < 1$ *such that if* $|x - y| \geqq R_1$,

(6.5) $$|K_1(x,y;\theta)| \leqq \beta$$

where $R(t)$ *is the constant given by* Lemma 6.1.

Proof. It holds by the assumption 2.1 that

$$C_1\left(\frac{1}{\sqrt{2\pi t}}\right)^d \exp\left[-\frac{|x-y|^2}{2tm_1}\right]$$

$$\leqq p(t,x,y) \leqq C_2\left(\frac{1}{\sqrt{2\pi t}}\right)^d \exp\left[-\frac{|x-y|^2}{2tm_2}\right] \ , \ t > 0 \, , x,y \in R^d \ ,$$

where $0 < m_1 \leqq m_2 < \infty$, $0 < C_1 \leqq C_2 < \infty$, (for example, see [1]). We take a t_0 such that

$$\frac{3}{4} < t_0 < 1 \qquad \text{and} \qquad \frac{1}{2}\left(\frac{1}{m_1} - \frac{1}{m_2}\right) < \frac{(t_0 - \frac{3}{4})^2}{4m_2 t_0 (1-t_0)} \ .$$

We also choose an R_1 such that

$$R_1 > 4R(1 - t_0)$$

(6.6)
$$M\frac{(C_2)^2}{C_1}(\sqrt{2})^{d-2}(\sqrt{m_2})^d \ \exp\left\{R_1\left(\frac{1}{2}\left(\frac{1}{m_1} - \frac{1}{m_2}\right) - \frac{(t_0-\frac{3}{4})^2}{4m_2 t_0(1-t_0)}\right)\right\} \leqq \frac{1}{2}$$

where

$$M = \sup_{z \in R^d} m(z) \ .$$

We fix points x and y such that

$$r = |x - y| \geqq R_1 \ .$$

We also take $e_1, e_2, \cdots, e_{d-1} \in R^d$ such that

$$\langle e_i, e_j \rangle = \delta_{i,j} \quad , \quad \langle e_i, x - y \rangle = 0 \quad , \quad i,j = 1,2,\cdots,d-1 \ .$$

Setting

$$\Gamma_+(x,y) = \{x + a(y-x) + \sum_{i=1}^{d-1} c_i e_i \ ; a \geqq \frac{3}{4} \ , \ c_i \in R^1\}$$

$$\Gamma_-(x,y) = \{x + a(y-x) + \sum_{i=1}^{d-1} c_i e_i \ ; \ a \leqq \frac{1}{4} \ , \ c_i \in R^1\} \ ,$$

we have

$$\Gamma_+(x,y) \cap \overline{B(0,R_2)} = \phi \qquad \text{or} \qquad \Gamma_-(x,y) \cap \overline{B(0,R_2)} = \phi$$

where $R_2 = R(1-t_0)$. We first consider the case when

$$\Gamma_+(x,y) \cap \overline{B(0,R_2)} = \phi .$$

We now consider the standard coordinate with x the origin with respect to the basis

$$(\frac{y-x}{|y-x|}, e_1, \cdots, e_{d-1}) .$$

Then

$$x = (0,0,\cdots,0) , \; y = (r,0,\cdots,0) , \; r = |y-x| .$$

By (2.10) ,

$$|K_1(x,y;\theta)| \leqq p(1,x,y)^{-1}\int_{\Gamma_+(x,y)^c} p(t_0,x,z)p(1-t_0,z,y)m(dz)$$

$$+ p(1,x,y)^{-1}\beta(1-t_0)\int_{\Gamma_+(x,y)} p(t_0,x,z)p(1-t_0,z,y)m(dz)$$

(6.7)

$$= \beta(1-t_0) + (1-\beta(1-t_0))$$

$$\times p(1,x,y)^{-1}\int_{\Gamma_+(x,y)^c} p(t_0,x,z)p(1-t_0,z,y)m(dz)$$

where $\beta(t)$ is the constant given in Lemma 6.1. On the other hand

$$p(1,x,y)^{-1}\int_{\Gamma_+(x,y)^c} p(t_0,x,z)p(1-t_0,z,y)m(dz)$$

$$\leqq M\frac{(C_2)^2}{C_1}(\frac{1}{2\pi t_0(1-t_0)})^{d/2}\exp\{\frac{r^2}{2m_1}\}\int_{-\infty}^{3r/4}\exp\{-\frac{a^2}{2m_2t_0} - \frac{(a-r)^2}{2m_2(1-t_0)}\}da$$

$$\leqq \frac{M}{2}\frac{(C_2)^2}{C_1}(\sqrt{2m_2})^d\exp\{(\frac{1}{2}(\frac{1}{m_1} - \frac{1}{m_2}) - \frac{(t_0-\frac{3}{4})^2}{4m_2t_0(1-t_0)})r^2\} \leqq \frac{1}{2} , \text{ (by (6.6))}.$$

Combining this with (6.7), we have

$$|K_1(x,y;\theta)| \leqq \frac{1}{2}(1 + \beta(1 - t_0)) < 1 . \tag{6.8}$$

Next we consider the case when

$$\Gamma_-(x,y) \cap \overline{B(0,R_2)} = \phi .$$

Then we have, by (2.10) and Lemma 6.1,

$$|K_1(x,y;\theta)| \leqq \beta(1-t_0)$$
$$+ (1-\beta(1-t_0))p(1,x,y)^{-1}\int_{\Gamma_-(x,y)^c} p(1-t_0,x,z)p(t_0,z,y)m(dz) .$$

By using the argument mentiond above, we also have

$$p(1,x,y)^{-1}\int_{\Gamma_-(x,y)^c} p(1-t_0,x,z)p(t_0,z,y)m(dz) \leqq 1/2$$

which implies

$$|K_1(x,y;\theta)| \leqq \frac{1}{2}(1 + \beta(1 - t_0)) < 1 .$$

Combining this with (6.8), we can complete the proof.

THE PROOF OF THEOREM 2.1 By combining Lemma 6.2 and results in Section 4, there exists a positive constant β , $0 < \beta < 1$ such that

$$|K_1(x,y;\theta)| \leqq \beta \quad \text{for} \quad x,y \in R^d .$$

Hence

$$a(1,\theta) < 1 , \ a(k,\theta) \leqq a(1,\theta)^k , \ k = 1,2,\cdots .$$

As stated in Section 5, by combining these results with (2.10) we can complete the proof of (2.15).

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