OVERDETERMINED SYSTEMS DEFINED BY COMPLEX VECTOR FIELDS

Francois Treves

§1. FORMALLY INTEGRABLE STRUCTURE

Let Ω be a C^∞ manifold (Hausdorff, countable at infinity), dim $\Omega = N(\geq 1)$, and let $L_L....,L_n$ be n complex vector fields, of class C^∞, in Ω, linearly independent at every point (so that $n \leq N$). We would like to study the *homogeneous* equations

(1) $$L_j h = 0 \ , \ j = 1,...,n \ ,$$

as well as the *inhomogeneous* equations

(2) $$L_j u = f_j \ , \ j = 1,...,n \ ,$$

with right-hand sides $f_j \in C^\infty(\Omega)$. It is known from the study of a single vector field (*i.e.*,n = 1) that difficulties arise even at the *local* level. In this expository note I shall limit myself to the local viewpoint and Ω can be taken to be an open subset of Euclidean space $\mathbb{R}^N$. Yet it is perhaps advisable to continue thinking of Ω as a manifold lest the important consideration of invariance be forgotten.

The questions one begins by asking, about equations (1) and (2), are the standard ones: existence, uniqueness and approximation of solutions, their regularity, their representations (say, by means of integral operators), etc. Answering these questions with satisfactory generality seems to be very difficult. Here I shall briefly describe some of the results about existence, uniqueness and approximation of the solutions.

The considerations of invariance are of two kinds: invariance under coordinate changes - in the base manifold Ω ; invariance under linear substitutions of the vector fields L_j ,

$$L_j^{\#} = \sum_{k=1}^{n} a_j^k L_k , \quad j = 1,\ldots,n ,$$

with (a_j^k) a smooth $n\times n$ *nonsingular* matrix-valued function in Ω . The properties of the solutions of (1) and (2) that interest us obviously partake of both kinds of invariance. In modern terminology this means that we are not simply dealing with vector fields in Ω; in fact, we are dealing with (smooth) sections of a certain vector subbundle, henceforth denoted by $\mathcal{V}$, of the complex tangent bundle of Ω , $\mathbb{C}T\Omega$.

The first particular case that comes to mind is the one where the "generators" L_j can be taken to be *real*. In this case it is well known that we must take the system to be "involutive", i.e., to satisfy:

$$(3) \qquad [L_j,L_k] = \sum_{l=1}^{n} a_{jk}^l L_l , \quad j,k = 1,\ldots,n,$$

where the coefficients a_{jk}^l are smooth. Condition (3) is classically known as the *Frobenius* or the *integrability*, condition ($[X,Y] = XY - YX$ is the commutation bracket).

In the general case (when the L_j may be complex) we shall also make the hypothesis (3), which we shorten as

(3') $$[\mathcal{V},\mathcal{V}] \subset \mathcal{V} ,$$

and to which we refer as *formal integrability* (of the vector bundle $\mathcal{V}$). The adjective formal is there to distinguish this notion from that of integrability, or rather of *local integrability*, which is the assertion of the existence of "enough" local solutions to the homogeneous equations (1). Before discussing the latter concept let us give a brief list of important special classes of formally integrable structures (this refers to vector bundles like $\mathcal{V}$) on a manifold Ω :

Already alluded to, the *essentially real* structures: there are local systems of generators consisting of real vector fields.

In the complex case we must introduce the *complex conjugate*, $\bar{\mathcal{V}}$, of the vector bundle $\mathcal{V}$: the fibre of $\bar{\mathcal{V}}$ at a point of Ω is the complex conjugate (within the complex tangent space) of that of $\mathcal{V}$.

With this definition one then says that $\mathcal{V}$ is (or defines) a *complex structure* on the manifold Ω if

(4) $$\mathbb{C}T\Omega = \mathcal{V} \oplus \bar{\mathcal{V}}$$

($\oplus$: direct sum, to be understood,like eq (4), fibrewise).
Keep in mind that (3') is assumed to hold. In the literature, what I call here a complex structure is often called an almost complex

structure, but I shall omit the "almost", for a reason that will be clearer in a moment.

The next two classes of structures are obtained by weakening condition (4):

$\mathcal{V}$ is said to be *elliptic* if

$$\mathbb{C}T\Omega = \mathcal{V} + \bar{\mathcal{V}} \tag{5}$$

(+: fibrewise vector sum, not necessarily direct);

$\mathcal{V}$ is said to be a *Cauchy-Riemann* (or CR) *structure* on Ω if

$$\mathcal{V} \cap \bar{\mathcal{V}} = 0 \ . \tag{6}$$

Thus a complex structure is an elliptic CR structure. The preceding terminology has the following justification:

Suppose $L_1,\dots,L_n$ are local generators of $\mathcal{V}$; to say that $\mathcal{V}$ is elliptic is equivalent to saying that the second-order linear partial differential operator

$$\bar{L}_1 L_1 + \dots + \bar{L}_n L_n$$

is elliptic. Another way of saying it is by looking at the symbols $\sigma(L_j)$ of the vector fields L_j ; the common zeros of $\sigma(L_1),\dots,\sigma(L_n)$ in any real cotangent space to Ω (at a point of the domain of definition of the basis $L_1,\dots,L_n$) is the origin.

The motivation for the name CR comes from the theory of several complex variables: Let S be a (real, smooth) hypersurface in $\mathbb{C}^{n+1}$ and consider all the complex vector fields L defined on S that have the following two properties at every point of S : L is tangent to S; L is a linear combination of the Cauchy-Riemann operators $\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_{n+1}}$. Suppose for instance that S is defined by a real C^∞ equation:

(7) $$\rho(x,y) = 0$$

where $x = \mathcal{Re}\, z$, $y = \mathcal{Im}\, z$, and $d\rho \neq 0$. Suppose in fact that

(8) $$\frac{\partial \rho}{\partial y_{n+1}} = 0 \ .$$

Then the following vector fields

(9) $$L_j = \frac{\partial}{\partial \bar{z}_j} \left[\frac{\partial \rho}{\partial \bar{z}_j} \Big/ \frac{\partial \rho}{\partial \bar{z}_{n+1}}\right] \frac{\partial}{\partial \bar{z}_{n+1}} \ , \ j = 1, \ldots, n,$$

are tangent to S, and they are linearly independent. They span what is sometimes called the Cauchy-Riemann tangent bundle of S, and is often denoted by $T^{0,1}$. One also writes $\bar{T}^{0,1} = T^{1,0}$, the vector bundle spanned by

$$\bar{L}_j = \frac{\partial}{\partial z_j} - \left[\frac{\partial \rho}{\partial z_j} \Big/ \frac{\partial \rho}{\partial z_{n+1}}\right] \frac{\partial}{\partial z_{n+1}} \ , \ j=1, \ldots, n \ .$$

(We recall that $\frac{\partial}{\partial z_j} = \frac{1}{2}\left[\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right]$, $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left[\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right]$.)

Clearly, $L_1, \ldots, L_n, \bar{L}_1, \ldots, \bar{L}_n$ are linearly independent, i.e. (6) holds.

In the CR case the solutions of eqns (1) are called CR functions (or CR distributions).

The prototype of CR structures is that defined on $\mathbb{R}^3$ (where we call the coordinates x,y,s) by the *Lewy* vector field,

$$L = \frac{\partial}{\partial \bar{z}} - iz \frac{\partial}{\partial s} . \tag{10}$$

We can map $\mathbb{R}^3$ onto the hypersurface S of $\mathbb{C}^2$ (coordinates: $z = x + iy$, $w = s + it$) defined by the equation

$$t = |z|^2 . \tag{11}$$

Then L is transformed into the tangent Cauchy-Riemann operator

$$\tilde{L} = \frac{\partial}{\partial \bar{z}} - 2iz \frac{\partial}{\partial \bar{w}} . \tag{12}$$

There are formally integrable structures on manifolds that are neither elliptic nor CR . A simple example is the one defined on $\mathbb{R}^2$ (coordinates: x,t) by the *Mizohata* vector field,

$$L_0 = \frac{\partial}{\partial t} - it \frac{\partial}{\partial x} . \tag{13}$$

It defines the standard complex structure in the upper half-plane $t>0$, the opposite one in the lower half-plane $t < 0$; and $L_0 = \bar{L}_0$ when $t = 0$. The Mizohata structure on $\mathbb{R}^2$ is somehow related to the Lewy structure on $\mathbb{R}^3$.

Notice that, in the last two examples, the base manifold is real-analytic, and the coefficients of the generating vector fields are also analytic. We may deal with formally integrable structures on an analytic manifold Ω , which are themselves analytic: the structure bundle $\mathcal{V}$ is then an analytic subbundle of $\mathbb{C}T\Omega$ (satisfying (3')).

§2. LOCALLY INTEGRABLE STRUCTURE

The definition of local integrability is best given in terms of the vector subbundle T' of the complex *co*tangent bundle $\mathbb{C}T^*\Omega$ which is the *orthogonal* of $\mathcal{V}$ for the duality between tangent and cotangent vectors (at one and the same point of the base Ω). The property that $\mathcal{V}$ is involutive (i.e., (3) or (3')) is equivalent to the property that T' is *closed*, which means that, given any smooth section φ of T', in the neighbourhood of any point of Ω we can write

(14) $$d\varphi = \psi_1 \wedge \varphi_1 + \dots + \psi_m \wedge \varphi_m ,$$

with $\varphi_1,\dots,\varphi_m$ smooth local sections of T' and $\psi_1,\dots,\psi_m$ smooth differential forms (one-forms). Note also that the fibre-dimension (i.e., the rank) of the vector bundle T' is equal to N-n , recalling that the fibre-dimension of $\mathcal{V}$ is equal to n. It is possible, and convenient, to choose the sections $\varphi_1,\dots,\varphi_m$ in (14) to be linearly independent, and write $N = m + n$.

DEFINITION. The formally integrable structure $\mathcal{V}$ (or T') on Ω is said to be *locally integrable* if T' is generated locally by *exact* smooth one-forms.

To say that $\mathcal{V}$ is locally integrable is to say that an arbitrary point $0 \in \Omega$ has an open neighbourhood U in which there are functions $Z_j \in C^\infty(U), j = 1,\ldots,m,$ such that

(15) $LZ_j = 0$ whatever the smooth section of $\mathcal{V}, L$;

(16) $dZ_1,\ldots, dZ_m$ are linearly independent at every point of U

To the question "When is the structure $\mathcal{V}$ locally integrable?" no general answer is known; only in special cases has it been answered.

First of all, in the *essentially real* case: this is the Frobenius theorem, for it asserts that local coordinates $x_1,\ldots,x_N$ can be found, such that the vector fields $\frac{\partial}{\partial x_{m+1}},\ldots,\frac{\partial}{\partial x_N}$ (recall that $n = N - m$) span $\mathcal{V}$ over the domain of the coordinates, U . It means that the functions $Z_j = x_j$, $j = 1,\ldots,m$, satisfy (15) and (16).

This observation also takes care of the *analytic* structures, for one can complexify the base manifold Ω and holomorphically extend the vector bundle $\mathcal{V}$. It then suffices to apply the holomorphic version of the Frobenius theorem and restrict the resulting coordinates to the real domain: Thus *every formally integrable analytic structure* (on an analytic manifold) *is locally integrable, and the functions* Z_j *in* (15) - (16) *can be taken to be analytic.*

Returning to the case of C^∞ structures the first nontrivial statement in the literature is the Newlander-Nirenberg theorem, which asserts that *every complex structure is locally integrable* (in the old terminology, every almost-complex structure is a complex structure; see

Newlander-Nirenberg [12]). An easy consequence is the result that *every elliptic structure is locally integrable* (Treves [16]).

The first example of a formally integrable structure which is *not* locally integrable was found by Nirenberg ([13]) in 1972. The defining vector field is a perturbation of the Mizohata operator, (13):

(18) $$L_0^{\#} = \frac{\partial}{\partial t} - it(1+\varphi(x,t))\frac{\partial}{\partial x} ,$$

where φ is a certain C^{∞} function in $\mathbb{R}^2$, $\varphi \equiv 0$ for $t < 0$, and therefore vanishes to infinite order at $t = 0$. In 1973, Nirenberg ([14]) constructed a similar (but more complicated) perturbation of the Lewy vector field, (10), which is not locally integrable. As a matter of fact the Nirenberg operators have the stronger property that all the solutions of the homogeneous equations $Lh = 0$, say in the whole space ($\mathbb{R}^2$ or $\mathbb{R}^3$), are constant. Such examples have more recently been extended to *systems* of n vector fields defining CR structures on $\mathbb{R}^{2n+1}$ whose Levi form is nondegenerate and has signature $(n-1,1)$ (see Jacobowitz-Treves [4],[5],[6]). The difficulty for systems is that one must only deal with perturbations that respect the formal integrability condition (3).

On the positive side, in addition to the cases mentioned above (essentially real, elliptic, analytic), there is a remarkable result of Kuranishi [8] concerning CR structures on $\mathbb{R}^{2n+1}$ defined by n vector fields, whose Levi form has all its eigenvalues >0 (or all <0). If $n \geq 4$ the structure is locally integrable.

One of the advantages of locally integrable structures is the *approximation formula* for solutions of the homogeneous equation, which we now describe. We reason in some open neighborhood of 0, U, in which we assume that there exist functions Z_j satisfying (15)-(16). After a $\mathbb{C}$-linear substitution we may assume that

(19) $d(\mathcal{R}e Z_1),\ldots,d(\mathcal{R}e Z_m)$ are linearly independent at every point of U,

and take $x_j = \mathcal{R}e Z_j$, $j = 1,\ldots,m$, as coordinates. We complete this set of coordinates by adjoining to them n additional ones, $t_1,\ldots,t_n$, and thus write

$$Z_j = x_j + i\Phi_j(x,t)\ , \quad j = 1,\ldots,m\ . \tag{20}$$

We can also achieve that $\Phi_j|_0 = 0$, and that all first partial derivatives of every Φ_j with respect to $x_1,\ldots,x_m$, vanish at the origin. Note that the "tangent" structure bundle $\mathcal{V}$ is then spanned, over U, by vector fields,

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} \lambda_{jk}(x,t)\frac{\partial}{\partial x_k}\ , \quad j = 1,\ldots,n\ . \tag{21}$$

One can then easily prove (see Baouendi-Treves [2], Treves [16]) that there is a an open neighborhood $U_0 \subset U$ of 0 such that, if h is any solution of the homogeneous equations (1) in U, then, in the smaller neighborhood U_0,

$$h(x,t) = \tag{22}$$

$$\lim_{\nu\to+\infty} \left[\frac{\nu}{\pi}\right]^{m/2} \int \exp\left\{-\nu \sum_{j=1}^{m} [Z_j(x,t)-Z_j(y,0)]^2\right\}\chi(y,0)h(y,0)dZ(y,0) ,$$

where $\chi \in C_0^\infty(U)$, $\chi \equiv 1$ in a suitably large neighborhood of the closure of U_0 , and

$$dZ(y,0) = dZ_1(y,0) \wedge \ldots \wedge dZ_m(y,0) = \det\left[\frac{\partial Z}{\partial y}(y,0)\right]dy .$$

The limit in (22) can be understood in a variety of ways: in the distribution sense, if h is a distribution solution (then the integral at the right is a duality bracket); in the sense of $C^k(U_0)$ if h is a C^k solution ($0\leq k\leq+\infty$); in the sense of Sobolev spaces if h belongs to one of these. Etc.

Recently (spring 1985) G.Métivler [11] was able to extend (22) to systems of *nonlinear* first-order PDE (in involution), under the proviso that the solution h be of class C^2. The formula (22) has the following consequences;

a) **APPROXIMATION**: every solution h of Eqns (1) is the limit, in a suitable neighborhood of the origin, and in the appropriate distribution space, of a sequence of polynomials $P_\nu(Z)$ with complex coefficients in $Z(x,t)$ (we write $Z = (Z_1,\ldots,Z_m)$).

In particular, a) implies that any CR function (or distribution) on a CR submanifold Σ of $\mathbb{C}^m$ is locally (in Σ) the limit (uniform, C^k, distribution, etc.), of a sequence of holomorphic polynomials.

(b) **LOCAL CONSTANCY ON FIBRES:** call "fibres" of Z in any subset S of U the intersection of S with the pre-images of points under the map $Z:U \to \mathbb{C}^m$ (if Z were valued in $\mathbb{R}$ the fibres would be its level sets). A consequence of Th.1 is that every continuous solution of (1) in U is constant on the fibres of Z in U'. Among other things this shows that the germs of fibres of Z at 0 are independent of the choice of Z; they are invariants of the structure T'. That they are an essential feature of the structure T' is evidenced by the Theorem at the end of §3.

The fibres of Z can be highly singular sets, and certainly need not be connected: for instance, the fibres of $Z = x + it^2/2$ in any neighborhood of the origin in $\mathbb{R}^2$ consist of the points $(x,\pm t)$. Z defines the Mizohata structure on $\mathbb{R}^2$ (see [13]). According to (2) all the solutions of the Mizhohata equation, $h_t - ith_x = 0$, are even with respect to t.

The local constancy on fibres can be rephrased as follows: every continuous solution h in U is, in U' (possibly contracted about 0), of the form $\tilde{h}\circ Z$, with $\tilde{h}$ a continuous function in $Z(U')$.

c) **UNIQUENESS IN THE CAUCHY PROBLEM:** Formula (22) imlpies that, if $h \equiv 0$ on the submanifold $t = 0$ of U, then $h \equiv 0$ in U'.

Recall the examples Cohen [3] of smooth vector fields $L = \partial/\partial t - ib(t,x)\partial/\partial x$ in $\mathbb{R}^2$ such that there are C^∞ solutions of $Lh = 0$ (in the whole plane) whose support is exactly the half-plane $t \geq 0$. In view of Formula (22) Cohen's vector fields L do not define a locally integrable structure on $\mathbb{R}^2$.

In order to give an invariant meaning to the uniqueness in c) let us adopt a global viewpoint. Thus Ω is once again a C^∞ manifold. Let us say that a smooth submanifold X of Ω is *maximally real* if dim X = m and if the pull-back of the structure bundle T' to X is equal to $\mathbb{C}T^*X$ or, equivalently, if the natural injection of $\mathbb{C}TX$ into $\mathbb{C}T\Omega|_X$ is transversal to $\mathcal{V}$. Suppose then that the origin lies on X. It is easily seen that the coordinates $t_1,\ldots,t_n$ in the open neighborhood U (see above) can be chosen so that $X \cap U$ is exactly defined by the equations $t = 0$. In passing let us point out that the map $Z : U \to \mathbb{C}^m$ induces a diffeomorphism of $X \cap U$ onto a totally real submanifold of $\mathbb{C}^m$ of (maximum) dimension m. A reformulation of c) is then that, if the trace on X of a distribution solution h (trace which is always well-defined) vanishes identically in X, then h itself vanishes identically in some open neighborhood of X. This property enables us to "circumscribe" the support of any distribution solution: it is a union of orbits of the family of real vector fields $\mathcal{Re}\, L$ $\mathcal{Im}\, L$ when L ranges over all smooth sections of U (see Treves [16], Ch. II, Th. 2.5).

§3. INHOMOGENEOUS EQUATIONS

Assume that the vector fields L_j are given by (21). In this case the commutators $[L_j,L_k]$ only involve partial derivatives with respect to $x_1,\ldots,x_m$. They can be linear combinations of $L_1,\ldots,L_n$, if and only if they vanish:

$$[L_j,L_k] = 0 \ , \ j,k = 1,\ldots,n \ . \tag{23}$$

If then we want to solve the inhomogeneous equations (2), the right-hand sides f_j must evidently satisfy the compatibility conditions:

(24) $$L_k f_j = L_j f_k \ , \ j,k = 1,\dots,n \ .$$

The question we briefly discuss here is that of the possibility of solving the equations (2) for any smooth right-hand side $f = (f_1,\dots,f_n)$ satisfying (24). We can regard f as a one-form in the neighborhood U of the origin. We might be content with solving (2) in a smaller neighborhood of the origin. The Poincaré lemma for one-forms tells us that this is always possible when the structure T' is essentially real (we may then select the vector fields L_j to be real, i.e., $L_j = \partial/\partial t^j$, $j = 1,\dots,n$). The Dolbeault lemma entails the same for elliptic structures.

As with local integrability the first counter-examples to local solvability arise with structures that are neither real nor elliptic. It suffices to deal with a single vector field L (i.e., the fibres of U have dimension one). Then the compatibility conditions (24) are void. The celebrated example of Hans Lewy [10] is that of the hyperquadric $t = x^2 + y^2$ in $\mathbb{C}^2$ (where the variables are $z = x + iy$, $w = s + it$) equipped with its natural CR structure. The Lewy vector field is $L = \partial/\partial\bar{z} - iz\partial/\partial s$. It is nowhere locally solvable. But already the Mizohata vector field in $\mathbb{R}^2$ (where the coordinates are t,x) , $L_0 = \partial/\partial t - it\partial/\partial x$ provides a counter-example: it is not locally solvable in the neighborhood of any point $(0,x)$.

Actually, in the case of a single vector field $(n = 1)$, the complete answer is known. It is a particular case of the local solvability theory for single linear PDE. Suppose the vector field L has the form (21); actually one can change variables and "straighten"

$\mathcal{R}e L$ (in general, this is not feasible when dealing with systems $L_1,\dots,L_n$) :

$$L = \frac{\partial}{\partial t} + i \sum_{k=1}^{m} b_k(x,t)\frac{\partial}{\partial x_k} \tag{25}$$

With L written in this form, the local solvability of (2) is equivalent to the existence of an open ball $U \subset \mathbb{R}^m$ and an open interval $I \subset \mathbb{R}^1$, both centered at 0, such that the following holds:

(P) $\forall\, x \in U$, $\forall\, \xi \in \mathbb{R}^m$, the function

$$I \ni t \mapsto \sum_{k=1}^{m} b_k(x,t)\xi_k$$

does not change sign.

This property (P) can be restated in completely invariant fashion, but we shall not do so here (see Nirenberg-Treves [15]). Let us only restate it as follows:

(P) There is a unit vector $\vec{v}(x)$ in U such that

$$\vec{b}(t,x) = |\vec{b}(t,x)|\vec{v}(x) \quad \text{in} \quad U \times I .$$

We have used the notation $\vec{b} = (b_1,\dots,b_m)(\in\mathbb{R}^m)$. Writing $\vec{v} = (v_1,\dots,v_m)$, we see that

$$L = \frac{\partial}{\partial t} + i|\vec{b}(t,x)| \sum_{k=1}^{m} v_k(x)\frac{\partial}{\partial x_k} .$$

The vector $\vec{v}$ is well-defined and smooth in the region

$$U_0 = \left\{x \in U ; \exists\, t \in I , \vec{b}(t,x) \neq 0\right\} .$$

The region $U_0 \times I$ is foliated by the integral manifolds of the pair of vector fields $\frac{\partial}{\partial t}$, $\sum_{k=1}^{m} v_k(x)\frac{\partial}{\partial x_k}$. The region $(U \backslash U_0) \times I$ (in which $\vec{b}(x,t) \equiv 0$) is foliated by the integral manifolds of $\frac{\partial}{\partial t}$. In $U_0 \times I$ the leaves are two-dimensional; in $(U \backslash U_0) \times I$ they are one dimensional. In both subsets L is tangent to the leaves. One method for solving the inhomogeneous equation

$$Lu = f \tag{26}$$

is to solve it on each leaf individually, in such a way that the solution u varies smoothly with the leaf. This is possible (see Treves [19]).

It follows from a theorem in Hörmander [7] that when condition (P) holds, possibly after contracting U and I, there are m solutions Z_j of $LZ_j = 0$ in $U \times I$ $(j = 1,\ldots,m)$ whose differentials are linearly independent, In other words the local solvability of the vector field L entails the local integrability of the structure (on $U \times I \subset \mathbb{R}^{m+1}$) it defines. But of course the structure can be locally integrable without there being local solvability of the inhomgeneous equation, for instance when the coefficients are real-analytic but do not satisfy condition (P). The simplest examples of this are the Lewy operator (10) on $\mathbb{R}^3$, and the Mizohata operator (13), in an open neighborhood of the origin in $\mathbb{R}^2$.

For smooth hypersurfaces in $\mathbb{C}^{n+1}$ whose Levi form is nondegenerate a complete answer (also for p-forms with $p>1$) is provided in Andreotti-Hill [1]. The answer depends on the signature of the Levi form. For one-forms the only case of nonsolvability occurs when the Levi form has $n-1$ eigenvalues of one sign and one of the opposite sign. Exactly the same result is valid for all *analytic* structures whose Levi form is nondegenerate at every point of the characteristic set, as shown in Treves [17].

Beyond that only special and fragmentary results are known. Let us mention such a result that sheds some light on the role of the fibres of the mapping Z, both from the viewpoint of local integrability and from that of local solvability:

Consider the structure on $\Omega \subset \mathbb{R}^{n+1}$ defined by a single analytic function Z (such that dZ is nowhere zero). In other words the fibre dimension of T' is one, and that of $\mathcal{V}$ is n. We shall consider local generators of $\mathcal{V}$ of the following kind (and thus commuting pairwise):

$$L_j = \frac{\partial}{\partial t_j} + \lambda_j(t,x)\frac{\partial}{\partial x}\ ,\quad j = 1,\dots,n\ , \tag{27}$$

with analytic coefficients λ_j.

Let us use the following terminology:

We say that the equations (2) are *locally solvable at* x_0 if, given any open neighborhood U of x_0, there is an open neighborhood $U' \subset U$ of x_0 such that, to any smooth one-form f in U satisfying the compatibility conditions (24), there is a C^∞ function u verifying (2) in U'.

We say that *Condition* (P) *holds at a point* x_0 of Ω if there is a basis of neighborhoods of x_0 in each of which the fibres of Z are connected.

It can be shown that Condition (P) as defined is the same as the property (P) above, in the local solvability theory for a single vector field $L = \partial/\partial t + \lambda(t,x)\partial/\partial x$ (with λ analytic).

THEOREM. If Condition (P) holds at every point of Ω the equations (2) are locally solvable at every point of Ω.

If Condition (P) does not hold at a point x_0 of Ω the equations (2) are not locally solvable at that point.

Furthermore there exist C^∞ vector fields $L_j^\#$ $(j = 1,\dots,n)$ in an open neighborhood $U \subset \Omega$ of x_0 that have the following properties:

(28) for each j, $L_j - L_j^\#$ vanishes to infinite order at the point x_0;

(29) the vector fields $L_j^\#$ commute pairwise;

(30) the structure on U defined by the $L_j^{\#}$ (which is formally integrable by virtue of (29) is not locally integrable.

For a proof see Treves [18].

REFERENCES

[1] Andreotti, A. & Hill, C.D., *E.E. Levi convexity and the Hans Lewy problem*, I and II, Ann. Sc. Norm. Sup. Pisa, 26 (1972), 325-363 and 747-806.

[2] Baouendi, M.S. & Treves, F., *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. of Math., 113 (1981), 387-421.

[3] Cohen, P., *The non-uniqueness of the Cauchy problem*, O.N.R. Techn. Report 93, Stanford 1960.

[4] Jacobowitz, H. & Treves, F., *Non-realizable CR structures*, Inventiones Math. 66 (1982), 231-249.

[5] Jacobowitz, H. & Treves, F., *Nowhere solvable homogeneous partial differential equations*, Bull. A.M.S., Vol. 8 (1983), 467-469.

[6] Jacobowitz, H. & Treves, F., *Aberrant CR structures*, Hokkaido J. Math., Vol. 12, in honor of T.Sirota (1983), 268-284.

[7] Hörmander, L., *Propagation of singularities and semiglobal existence theorems for (pseudo-) differential operators of principal type*. Ann. of Math. 108 (1978), 569-609.

[8] Kuranishi, M., *Strongly pseudoconvex CR structures over small balls*, Ann. of Math., Part I: 115 (1982) 451-500, Part II: 116 (1982) 1-64, Part III: 117 (1983).

[9] Lewy, H., *On the local character of the solution of a typical differential equation in three variables and a related problem for regular functions of two complex variables*, Ann. of Math. 64 (1956), 514-522.

[10] Lewy, H., *An example of a smooth linear partial differential equation without solutions*, Ann. of Math. 66, 155-158.

[11] Métivler, G., *Uniqueness and approximations of solutions of first order nonlinear equations*, to appear.

[12] Newlander, A & Nirenberg, L., *Complex analytic coordinates in almost complex manifolds*, Ann. Math., 65 (1957), 391-404.

[13] Nirenberg, L., *Lectures on linear partial differential equations*, Amer. Math. Soc. Region. Conf. Series in Math., No. 17 (1973).

[14] Nirenberg, L., *On a question of Hans Lewy*, Russian Math. Surveys 29 (1974), 251-262.

[15] Nirenberg, L. & Treves, F., *Solvability of a first-order linear partial differential equation*, Comm. Pure Applied Math. 16 (1963), 331-351.

[16] Treves, F., *Approximation and representation of functions and distributions annihilated by a system of complex vector fields*, Centre Math. Ecole Polytechnique, Palaiseau France 1981.

[17] Treves, F., *A remark on the Poincaré lemma in analytic complexes with nondegenerate Levi form*, Comm. in P.D.E., Vol. 7 (1982), 1467-1482.

[18] Treves, F., *On the local integrability and local solvability of systems of vector fields*, Acta Math. (1983).

[19] Treves, F., *Integral representations of solutions of first-order linear partial differential equations*, I.Ann. Scuola Norm. Pisa 3 (1976),1-36.

Department of Mathematics
Rutgers University
Hill Center for Mathematical Sciences
NEW BRUNSWICK, New Jersey USA 08903