# DEFORMING RIEMANNIAN METRICS ON THE 2-SPHERE

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In 1982, Hamilton [Ha] proved the following:

<u>Theorem</u> Let X be a compact 3-dimensional Riemannian manifold of positive Ricci curvature. The evolution equation $\frac{\partial}{\partial t} g_{ij} = \frac{2}{3} r g_{ij} - 2R_{ij}$, where $r = \int_X R d\mu_X / \int_X d\mu_X$, has a unique solution for all t and it converges as $t \to \infty$ to a metric of constant positive curvature. Furthermore, any isometries of X are preserved as the metric evolves.

The aim of this paper is to prove a 2-dimensional version of this theorem. We have also obtained analogous results for Kähler and Hermitian manifolds by applying the same method with Huisken's higher dimensional version of Hamilton's theorem [Hu].

We start with a compact, oriented Riemannian surface of positive Gaussian curvature (already this is enough to show that M is diffeomorphic to $S^2$ by the Gauss-Bonnet theorem and the classification of compact surfaces). We then show that there is a principal $S^1$ bundle over M with a metric of positive Ricci curvature such that the projection map is a Riemannian submersion. We allow the metric on this bundle to evolve to a metric of constant curvature; the metric on M then evolves to a metric of constant curvature also.

Let P be a principal $S^1$ bundle over M and let $\pi$ be the projection map. Let $\omega$ be the connection form and $\Omega$ the curvature form of a connection in the bundle P. $\Omega$ is a horizontal, invariant 2-form (because $S^1$ is abelian) so $\Omega = \pi^*(\gamma)$ for some 2-form $\gamma = g d\mu_M$ on M where $d\mu_M$ is the volume form on M and g is a smooth function on M.

Let f be a smooth positive function on M. As in [K], define an invariant metric on P via $\langle u, v\rangle_P = \langle \pi_* u, \pi_* v\rangle_M + \pi^*(f^2)\omega(u)\omega(v)$. Note that any invariant metric on P may be constructed in this way; in fact we can recover the connection by defining the horizontal space to be the orthogonal complement of the fundamental vector field V, the metric on M via $\langle u, v\rangle_M = \langle u^*, v^*\rangle_P$ where $u^*$ and $v^*$ are the horizontal lifts, with respect to the connection just defined, of u and v respectively and f via $f^2 = \langle V, V\rangle_P$.

Let $p \in P$, $m = \pi(p)$ and let $X_1$, $X_2$ be an orthonormal basis for $T_m(M)$. Let $Y_1$ and $Y_2$ be the horizontal lifts at p of $X_1$ and $X_2$ respectively and let $Y_0 = \frac{1}{f} V$, so that $Y_0$, $Y_1$, $Y_2$ is an orthonormal basis for $T_p(P)$.

A straightforward but lengthy calculation shows that the Ricci curvature of P with respect to the basis $Y_0$, $Y_1$, $Y_2$ is given by:

$$\frac{1}{2}\pi^* \begin{bmatrix} f^2g^2 - \frac{2}{f}\Delta f & fg_{;2} + 3f_{;2}g & -fg_{;1} - 3f_{;1}g \\ fg_{;2} + 3f_{;2}g & 2K - f^2g^2 - \frac{2}{f} f_{;11} & -\frac{2}{f} f_{;12} \\ -fg_{;1} - 3f_{;1}g & -\frac{2}{f} f_{;21} & 2K - f^2g^2 - \frac{2}{f} f_{;22} \end{bmatrix}$$

where ; denotes covariant differentiation in M with respect to the basis $X_1$, $X_2$ and K denotes the Gaussian curvature of M.

For any harmonic 2-form $\gamma$ on M which represents an element of $H^2(M; \mathbb{Z})$, there exists a principal $S^1$ bundle over M and a connection in this bundle such that the curvature form is $\pi^*(\gamma)$ (see [K], proposition 9). Thus there exists a principal $S^1$ bundle P over M with g a positive constant function chosen so that $\gamma = g d\mu_M \in H^2(M; \mathbb{Z})$.

Let $\delta$ be a lower bound for the Gaussian curvature, so $0 < \delta \leq K$. Choose f to be a constant function such that $0 < f < \frac{1}{g}\sqrt{2\delta}$, so that $0 < f^2g^2 < 2\delta \leq 2K$. With this choice the Ricci curvature of P with respect to $Y_0$, $Y_1$, $Y_2$ is given by:

$$\frac{1}{2}\pi^* \begin{bmatrix} f^2g^2 & 0 & 0 \\ 0 & 2K - f^2g^2 & 0 \\ 0 & 0 & 2K - f^2g^2 \end{bmatrix}$$

which is obviously positive definite.

We now let the metric on P evolve, as in Hamilton's theorem, according to the equation $\frac{\partial}{\partial t} g_{ij} = \frac{2}{3} r g_{ij} - 2R_{ij}$.

As the initial metric is invariant under the $S^1$ action, it remains so for all time and hence it induces a metric on M, a connection and a function f, all of which will evolve as the metric on P does.

Another long but straightforward calculation shows that the evolution equation for the metric on M is:

$$\frac{\partial}{\partial t} g_{ij} = \left(\frac{2}{3} r - 2K + f^2g^2\right) g_{ij} + \frac{2}{f} f_{;ij}$$

and for f is:

$$\frac{\partial}{\partial t} f = \Delta f + (\frac{1}{3} r - \frac{1}{2} f^2 g^2) f$$

where $r = \int_M f \, (2K - \frac{1}{2} f^2 g^2 - \frac{2}{f} \Delta f) d\mu_M / \int_M f d\mu_M$.

The evolution equation for g is more difficult to calculate, however the scalar curvature R of P is $S^1$ invariant and $R = \pi^*(2K - \frac{1}{2} f^2 g^2 - \frac{2}{f} \Delta f)$, so $f^2 g^2 = 4K - 2\tilde{R} - \frac{4}{f} \Delta f$ (where $\tilde{R}$ is the function on M for which $R = \pi^*(\tilde{R})$).

Hamilton [Ha] has already calculated the evolution equation for R as: $\frac{\partial}{\partial t} R = \Delta R - \frac{2}{3} rR + 2S$, where $S = g^{ik} g^{jl} R_{ij} R_{kl}$.

From previous calculations of the Ricci curvature, we have

$$\tilde{S} = \frac{1}{4} (f^2 g^2 - \frac{2}{f} \Delta f)^2 + \frac{1}{2} (fg_{;2} + 3f_{;2} g)^2 + \frac{1}{2} (fg_{;1} + 3f_{;1} g)^2$$

$$+ \frac{1}{4} (2K - f^2 g^2 - \frac{2}{f} f_{;11})^2 + \frac{1}{2} (\frac{2}{f} f_{;12})^2 + \frac{1}{4} (2K - f^2 g^2 - \frac{2}{f} f_{;22})^2$$

which may be written as a function of $\tilde{R}$, K and f although it is unpleasant.

From this we may derive the evolution equation for $\tilde{R}$ as $\frac{\partial}{\partial t} \tilde{R} = \Delta \tilde{R} + \frac{1}{f} \langle \nabla f, \nabla \tilde{R} \rangle - \frac{2}{3} r\tilde{R} + 2\tilde{S}$, where the extra term is because the Laplacian is now taken in M.

Thus we have the following:

<u>Theorem</u> Let M be a compact, oriented surface of positive Gaussian curvature. The system of equations:

$$\frac{\partial}{\partial t} g_{ij} = (\frac{2}{3} r + 2K - 2\tilde{R} - \frac{4}{f} \Delta f) g_{ij} + \frac{2}{f} f_{;ij}$$

$$\frac{\partial}{\partial t} f = 3\Delta f + (\frac{1}{3} r - 2K + \tilde{R}) f$$

$$\frac{\partial}{\partial t} \tilde{R} = \Delta\tilde{R} + \frac{1}{f} \langle \nabla f, \nabla\tilde{R} \rangle - \frac{2}{3} r\tilde{R} + 2\tilde{S}$$

where $\tilde{S}$ is a function of $\tilde{R}$, K and f, $r = \int_M f\tilde{R}\, d\mu_M / \int_M f\, d\mu_M$ and initially $g_{ij}$ is the metric, f is the constant function chosen before and $\tilde{R} = 2K - \frac{1}{2} f^2 g^2$ with g the constant function chosen before, has a unique solution for all t and $g_{ij}$ converges as $t \to \infty$ to a metric of constant positive curvature on M while f and $\tilde{R}$ each converge to constant functions.

It is possible to extend this theorem to allow the Gaussian curvature of M to have isolated zeros, the only added complication being that we can no longer choose f to be constant.

## References

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