REMARKS ON MULTIPLE SOLUTIONS OF NONLINEAR EQUATIONS

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In this lecture, we discuss some of our recent work on the number of solutions of weakly nonlinear elliptic partial differential equations. In particular, we discuss cases where the answers were surprising (at least to me). In §1, we have rather fewer solutions than one might expect while in §2, we have rather more solutions than one might expect. The discussion in §1 is rather brief since a more complete discussion appears in [4].

§1 TOO FEW SOLUTIONS

In this section, we discuss the following problem.

$$(1) \qquad \begin{aligned} -\Delta u &= f(x,u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Here Ω is a smooth bounded domain in R^n, $f:\Omega\times R\to R$ is C^1, $f(x,0) = 0$ for $x \in \Omega$ and $y^{-1}f(x,y) \to a_2(x)$ as $|y| \to \infty$ uniformly in x, where $a_2(x)>0$ in Ω. Let $a_1(x) = f_2^1(x,0)$ and assume that $a_1(x)>0$ in Ω. Finally assume that the two eigenvalue problems

$$\begin{aligned} -\Delta h-a_i h &= \lambda h \text{ in } \Omega, \\ h &= 0 \text{ on } \partial\Omega \end{aligned}$$

do not have zero as an eigenvalue (where i=1,2). Let m_1, m_2 denote the number of negative eigenvalues counting multiplicity of these two eigenvalue problems respectively. Then Clark [1] proved that (1) has at least $2|m_1-m_2|$ distinct non-trivial solutions <u>provided that</u> f <u>is odd in</u> y. It can also be shown that this result holds without the oddness assumption if n=1 (that is, in the ordinary differential equation case). Intuitively, if we cross many eigenvalues of the linearization as we move from 0 to ∞, then we have many solutions. Recently, the author has constructed examples where $|m_1-m_2| = n$ but there are only two non-trivial solutions. Thus, although we cross many eigenvalues, we only have 2 non-trivial solutions. In particular, we see that Clark's result is not true if we delete the oddness assumption.

The construction of the counterexample depends upon the following abstract result. Assume that E and H are Banach spaces and $F:E\to H$ is C^1 and Fredholm of index zero such that $F(m)=0$ for $m \in M$, where M is a compact C^1-submanifold of E. In addition assume that $N(F'(m))$, the kernel of $F'(m)$, has the same dimension as M for $m \in M$. If $G:E\to H$ is C^1 and $P(m)$ is a projection onto $N(F'(m))$ depending C^1 on m, then it can be shown that the zeros of $F(x)+\varepsilon G(x)$ near M for small ε are largely determined by the zeros of $S(m) = P(m)G(m)$ on M. In particular, if S has only non-degenerate zeros in an appropriate sense, then the number of zeros of $F(x)+\varepsilon G(x)$ near M for fixed small ε is equal to the number of zeros of S on M. This result is useful because S is reasonable to compute in simple cases.

The idea to construct the counterexample is as follows. We choose Ω the unit ball in R^n, $\bar{\lambda}$ an eigenvalue of $-\Delta$ (for Dirichlet boundary conditions) of multiplicity n and then choose $\bar{f}$ independent of x and sublinear such that a_1 is slightly less then $\bar{\lambda}$ while a_2 is close to but less than the next eigenvalue of $-\Delta$. It then turns out that $-\Delta u-\bar{f}(u)$ satisfies the assumptions of the previous paragraph on F where the set of solutions consists of $\{0\}$ and a manifold M of non-zero solutions. Note that the manifold M of solutions comes from the symmetries in the problem. Moreover, M turns out to be, up to a scale factor, close to the unit sphere in the eigenspace for $\bar{\lambda}$. We then choose a suitable $G=g(x,u(x))$ such that S has only 2 zeros and the result follows. This requires some work. Here our knowledge of M helps us to understand S. In practice, it is easier technically to do the construction slightly differently. Details appear in [4].

It seems likely that there is a counterexample with f independent of x but we have been unable to construct one. The same idea of perturbing symmetric situations is also used to construct several other counterexamples in [4].

§2 TOO MANY SOLUTIONS

In this section, we show that some problems have rather more solutions than one might hope. Our basic construction is a variant of one in Schaeffer [11] and Hale and Vegas [9]. (We obtain a little more information than in [11] while our construction seems easier to use and to hold more generally than the one in [9]). Our results emphasize that it is the geometry of the underlying domain Ω and not only its topology which matters for determining the number of solutions.

Firstly, assume $n \geq 2$, $1<p<(n+2)(n-2)^{-1}$ ($1<p<\infty$ if $n=2$) and $k \in Z^+$. We construct a smooth bounded domain Ω in R^n such that Ω is topologically a ball in R^n but the equation

$$\begin{aligned} -\Delta u &= u^p \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

has at least k non-trivial positive solutions. Moreover, each of these solutions is non-degenerate, that is, the linearization is invertible. This last property is useful when one wants to study other problems (as we will see later) and it implies that our large number of solutions holds for all domains close to Ω in a suitable sense.

To construct Ω, we will use Ω_ε where ε is small. First let $\tilde{\Omega}_\varepsilon$ consist of $B_\varepsilon(0) \cup B_{\frac{1}{2}}(me_1)$ (where m is reasonably large and fixed) *together with* a "straight tube" of radius ε^2 joining the two balls. Here $B_s(t)$ is the ball with centre t and radius s. We can make $\tilde{\Omega}_\varepsilon$ smooth by rounding off the edges where the tube meets the two balls. Let T denote the rotation through $2\pi k^{-1}$ in the e_1-e_2 plane. (Thus T fixes $e_3,\dots,e_n$.) L fine $\Omega_\varepsilon = \bigcup_{0 \leq i \leq k-1} T^i \tilde{\Omega}_\varepsilon$. It is easy to see that, provided m is large enough and ε is small, Ω_ε is a smooth manifold with boundary diffeomorphic to the unit ball and Ω_ε is T invariant. Note also that, as $\varepsilon \to 0$, Ω_ε tends in some generalized sense to $\Omega_0 = \bigcup_{0 \leq i \leq k-1} T^i B_{\frac{1}{2}}(me_1)$.

To construct our solutions, we maximize $J_\varepsilon(u) = \int_{\Omega_\varepsilon} u^{p+1} dx$ over those $u \in \dot{W}^{1,2}(\Omega_\varepsilon)$ subject $\int_{\Omega_\varepsilon} |\nabla u|^2 dx = 1$. It is easy to see that there is a u_ε

where J_ε attains its maximum, that $u_\varepsilon(x)\geq 0$ in Ω_ε and that $J_\varepsilon(u_\varepsilon)\to J_0(u_0)$ as $\varepsilon \to 0$. (Here J_0 is the corresponding functional when Ω_ε is replaced by Ω_0 and u_0 is the corresponding maximizer). To prove the last claim we note that, since $\Omega_0 \subseteq \Omega_\varepsilon$, $J_\varepsilon(u_\varepsilon)\geq J_0(u_0)$. (Here we are using that $\dot{W}^{1,2}(\Omega_0) = \{u\in\dot{W}^{1,2}(\Omega_\varepsilon): u = 0 \text{ outside } \Omega_0\}$.) To prove the inequality the other way, we note that a subsequence of u_ε must converge weakly (in fact strongly) in $\dot{W}^{1,2}(\tilde{B})$ to $\tilde{u}_0$, where $\tilde{u}_0 \in \dot{W}^{1,2}(\Omega_0)$, $\tilde{u}_0$ maximizes $J_0(u_0)$, and $\int_{\Omega_0} |\nabla\tilde{u}_0|^2dx=1$. Here $\tilde{B}$ is a large ball containing all the Ω_ε's. Now, u_0 must be a non-negative solution of $-\Delta u = \lambda_0 u^p$ in Ω_0, where $\lambda_0 > 0$. Thus, on each component of Ω_0, u_0 must vanish identically or be the unique positive solution of (2) (for $\Omega = B_{\frac{1}{2}}(me_1)$) up to translation and scaling. Here we are using a result in [7] to ensure uniqueness. Of these possible choices of u_0, it is not difficult to show the one that maximizes $J_0(u_0)$ is the one where u_0 vanishes except on one component of Ω_0. Since

(3) $$-\Delta u_\varepsilon = \lambda_\varepsilon {u_\varepsilon}^p$$

where $\lambda_\varepsilon > 0$, we can use a simple scaling to obtain a positive solution of (2) (for $\Omega = \Omega_\varepsilon$) mostly concentrated near a single component of Ω_0. If we use the symmetry, we then obtain k different positive solutions of (2) (each concentrated near a different component of Ω_0).

To prove the non-degeneracy, we first note that it suffices to prove u_ε is a non-degenerate solution of (3) for small ε. By scalar multiplying (3) by u_ε, we see that λ_ε is uniformly bounded above. By using the independence of the constants in the Sobolev inequalities and the L^p regularity theory upon the domain for domains contained in $\tilde{B}$ (cp [8]), we deduce that the u_ε are uniformly bounded in L^∞. If $-\Delta h_\varepsilon = \lambda_\varepsilon p\, u_\varepsilon^{p-1}h_\varepsilon$ in Ω_ε, $h_\varepsilon = 0$ on $\partial\Omega_\varepsilon$ and $\|h_\varepsilon\|_2 = 1$, we see similarly that h_ε are uniformly bounded in $\dot{W}^{1,2}(\Omega_\varepsilon)$ and thus (as before, working in $\dot{W}^{1,2}(\tilde{B})$), we deduce that a subsequence of h_ε converges weakly in $\dot{W}^{1,2}(\tilde{B})$ to $h_0 \in \dot{W}^{1,2}(\Omega_0)$ where $\|h_0\|_2 = 1$ and $-\Delta h_0 = \lambda_0 p {u_0}^{p-1} h_0$. This is impossible by the non-degeneracy of the solution of (2) on a ball (which follows from [3] and [7]). Thus the non-degeneracy holds for small positive ε.

We conjecture that (2) has a unique positive solution if Ω is convex.

The above example has additional interest because, it can be used to obtain multiple solutions for some related equations. Firstly, if f is C^1, if $f(0)=0$, if $1<p<(n+2)(n-2)^{-1}$ and if $y^{1-p}f'(y)\to a>0$ as $y\to 0$, then it follows easily from our work above and ideas in [6] that the equation

$$-\Delta u = \lambda f(u) \text{ in } \Omega_\varepsilon,$$
$$u = 0 \quad \text{on } \partial\Omega_\varepsilon$$

has at least k small positive solutions for λ large.

Secondly, but a little less obviously, it follows also that there exist $a,b,c,e,f,g > 0$ for which the problem

$$(4) \qquad \begin{aligned} -\Delta u &= u(a-bu-cv) \\ -\Delta v &= v(e-fu-gv) \quad \text{in } \Omega_\varepsilon, \\ u &= v = 0 \qquad \text{on } \partial\Omega_\varepsilon \end{aligned}$$

has at least k strictly positive solutions provided $n \le 5$. Here a solution is strictly positive if both components are strictly positive in Ω_ε. Note that this system occurs in animal population problems. To prove this result, we replace $-\Delta v$ by $-d\Delta v$ in the second equation where $d > 0$. (This of course corresponds to changing e,f,g in the second equation). By the theory in [5], there is an explicit function $h:R\to R$ such that the number of strictly positive solutions of (4) for small d is at least the number of isolated positive solutions of non-zero index of

$$(5) \qquad \begin{aligned} -\Delta u &= h(u) \quad \text{in } \Omega_\varepsilon, \\ u &= 0 \qquad \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

For a suitable choice of a,b,c,e,f,g, it turns out from the formula for h in [5] that $h(y) = ry^2$ for $0\le y\le s$ where r may be large without s being small. It now follows easily from our earlier result that (5) may have many isolated solutions of non-zero index. Hence the result follows. Note that there is a trivial method of obtaining a line of solutions of (4). We simply choose $a=e$, $b=f$, $c=g$. However our examples, which occur for very different values of the parameters, have the properties that there are at least k

components of strictly positive solutions, that the other solutions $(\bar{u},0)$ and $(0,\bar{v})$ are non-degenerate and that there are at least k solutions for any nearby set of parameter values. Thus, the large number of solutions is a "stable" property. This contrasts with the other example of non-uniqueness mentioned above. The above solutions we have constructed are all unstable (for the natural corresponding parabolic problem). It turns out that there must also be an asymptotically stable strictly positive solution. This is a special case of a rather more general result. Note that, in [10], an example is constructed where there are several stable solutions for Neumann boundary conditions and for quite different parameter values.

Lastly, the same ideas can be used for the Gelfand equation

$$(6)\qquad \begin{aligned} -\Delta u &= \lambda e^{u} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

if $n=2$. More precisely, we find that for all sufficiently small ε there is a $\lambda=\lambda(\varepsilon)>0$ such that (6) (for $\Omega = \Omega_\varepsilon$) has at least k solutions each of which is non-degenerate. The proof of this is similar in outline to the proof of the corresponding result for (2) though it is rather more technical. We briefly sketch the ideas. The idea is to maximize $J_\varepsilon(u) = \int_{\Omega_\varepsilon}(e^u-1)dx$ subject to the constraint $\int_{\Omega_\varepsilon}|\nabla u|^2dx = R$ where R is large. Now for $\Omega = B_{\frac{1}{2}}(0)$, it is known that there is a $\bar{\lambda} > 0$ such that (6) is uniquely solvable for $\lambda = \bar{\lambda}$, has no solution for $\lambda > \bar{\lambda}$ and has exactly two solutions $u_1(\lambda)$, $u_2(\lambda)$ for $0<\lambda<\bar{\lambda}$, where $u_1(\lambda)$ is the minimal solution and both $u_1(\lambda)$ and $u_2(\lambda)$ are non-degenerate. (This follows from [2] and [3]). Now one shows that, for large R, $J_0(u) = \int_{\Omega_0}(e^u-1)dx$ is maximized (subject to the constraint) by u_0 where u_0 is a translate of the minimal solution $u_1(\lambda_0)$ on (k-1) components of Ω_0 and is a translate of $u_2(\lambda_0)$ on the other.

The above result together with some of the ideas in [2] can be easily used to show that there is an $\alpha(\beta,\gamma) > 0$ such that

$$-\Delta u = \alpha(1+\beta-u)\exp(-\gamma u^{-1}) \text{ in } \Omega_\varepsilon,$$
$$u = 1 \text{ on } \partial\Omega_\varepsilon$$

has at least k solutions if $\beta > 0$ and γ is large. This equation occurs in catalysis theory (cp [2]).

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