THE GAUSS MAP OF A SUBMERSION

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1. MOTIVATION

Let $\phi: (M, g) \to (N, h)$ be a mapping of Riemannian manifolds. Then, following Eells and Sampson [11], we let τ_ϕ denote the tension field of the map ϕ. Thus τ_ϕ is a section of the pull-back bundle $\phi^{-1}TN$, and ϕ is harmonic if and only if $\tau_\phi = 0$ [9].

The fundamental problem of harmonic mappings is, given a particular homotopy class of mappings between Riemannian manifolds; does it contain a harmonic representative. In the case when M and N are surfaces a complete picture is known. For example, consider the homotopy classes of maps from the torus (T^2, g) to the sphere (S^2, h), with any metrics. All classes with degree $|d| \geq 2$ have harmonic representatives. The classes with $d = \pm 1$ have no harmonic representatives [9]. In the case when M and N have arbitrary dimensions we have a much less complete picture.

A natural case to study is when M and N are spheres. Many examples of harmonic mappings were constructed by Smith [17] by assuming certain symmetry. He showed in particular that all classes of $\pi_n(S^n)$ have a harmonic representative for $n \leq 7$. In [1] this work was carried further and it was shown by allowing ellipsoidal deformations of the metrics that all classes of $\pi_n(S^n)$ for any n have a harmonic representative at least in some metric.

Another case considered by Smith were the classes $\pi_3(S^2) \cong Z$. The homotopy classes of maps from S^3 to S^2 are parametrized by their Hopf invariant. Topologically this number can be interpreted as the winding number of the fibres. In the case of Hopf invariant equal to a square d^2, the methods of Smith yield a harmonic representative. However, in the case of Hopf invariant $k\ell$, $k \neq \ell$, it is unknown whether the class contains a harmonic representative or not. It becomes interesting then, to look for an object which relates the harmonicity of a map to properties of the fibres. This motivated the development of the Gauss map of a submersion in [2].

If $\phi: M \to N$ is a submersion between oriented manifolds, then the Gauss map $\gamma: M \to G_{m-n}(M)$ is the mapping which associates to each point x of M, the tangent plane to the fibre passing through x. Here, $G_{m-n}(M)$ denotes the Grassmann bundle over M, whose fibre at each point $x \in M$ is the Grassmannian of oriented $(m - n)$-planes in T_xM ($m = \dim M$, $n = \dim N$). In the case when M is a domain in $\mathbb{R}^m$ we obtain a mapping γ into the usual Grassmannian manifold by identifying $T_xM \cong \mathbb{R}^m$. There is a similar interpretation when M is a domain in the Euclidean sphere S^m. Such mappings have also been considered by Gluck, Warner and Ziller [14] in classifying fibrations of spheres by parallel great spheres.

In this article we highlight the remarkable analogies between the Gauss map of a submersion and the usual Gauss map associated to an immersion into Euclidean space. We derive analogous results to those of Chern [6] and Ruh-Vilms [16] for the Gauss map of an immersion. The case of 2-dimensional range turns out to be special for a submersion.

By way of application we classify a class of harmonic morphisms from domains in Euclidean space onto a Riemann surface in terms of holomorphic mappings. This latter work is joint collaboration with J.C. Wood.

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2. THE GAUSS MAP OF AN IMMERSION.

Let $\phi: M \to \mathbb{R}^n$ be a mapping of an oriented Riemannian manifold into Euclidean space and let m denote the dimension of M. Assume that ϕ is a conformal immersion if m = 2 and an isometric immersion if $m \geq 3$.
Let $\gamma: M \to G_m(\mathbb{R}^n)$ be the Gauss map of ϕ, where $G_m(\mathbb{R}^n)$ denotes the Grassmannian of oriented m-planes in $\mathbb{R}^n$. Thus to each point $x \in M$, $\gamma(x)$ is the plane $d\phi_x(T_xM)$ translated to the origin in $\mathbb{R}^n$ together with the orientation induced from M.

The derivative $d\gamma: TM \to TG_m(\mathbb{R}^n)$. For $p \in G_m(\mathbb{R}^n)$, let H_p denote the plane determined by p and V_p its orthogonal complement in $\mathbb{R}^n$, $V_p = H_p^{\perp}$. Then the identification $T_pG_m(\mathbb{R}^n) \cong H_p^* \otimes V_p$ is well known [16]. Thus $d\gamma$ can be regarded as a section of the bundle $\otimes^2 T^*M \otimes NM$ (NM denotes the normal bundle of M in $\mathbb{R}^n$), as is the second fundamental form $\nabla d\phi$ of the immersion ϕ and in fact the two sections can be identified. We therefore have the well known result of Ruh-Vilms' [16]:

$$\tau_\gamma = \text{trace } \nabla d\gamma = \text{trace } \nabla\nabla d\phi = \nabla \text{ trace } \nabla d\phi = \nabla\tau_\phi.$$

That is:

(2.1) γ <u>is harmonic if and only if the immersion</u> ϕ <u>has parallel mean curvature</u>.

In the case when $\phi: M \to S^n$, the Gauss map $\gamma: M \to G_{m+1}(\mathbb{R}^{n+1})$ is obtained by embedding S^n in $\mathbb{R}^{n+1}$ in the usual way.
Then $\gamma(x) = \phi(x) \wedge d\phi_x(T_xM)$. The above result now becomes:

(2.2) γ <u>is harmonic if and only if</u> ϕ <u>is harmonic</u>.

The case when dim $M = 2$ is special. For then the Grassmannian $G_2(\mathbb{R}^n)$ is biholomorphically equivalent to the complex quadric hypersurface Q_{n-2} in $\mathbb{C}P^{n-1}$, where $Q_{n-2} = \{[z_1, \ldots, z_n]; \sum z_i^2 = 0\}$. The identification is given by sending the plane $X \wedge Y$ to $[X - iY] \in Q_{n-2}$, where X, Y are orthonormal vectors. If we pick an isothermal coordinate $z = x + iy$ locally on M, then the Gauss map $\gamma: M \to Q_{n-2}$ is given explicitly by $\gamma = [\frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}]$ and we immediately obtain the result of Chern [6]:

(2.3) <u>The conformal immersion</u> ϕ <u>is harmonic if and only if the Gauss map</u> γ <u>is</u> - <u>holomorphic</u>.

3. THE GAUSS MAP OF A SUBMERSION.

Let $\phi: M \to N$ be a submersion for a domain M in Euclidean space $\mathbb{R}^m$ onto an oriented Riemannian manifold N. For $x \in M$, let $V_xM = \ker d\phi(x)$, and $HxM = \tilde{VxM}$. Define the Gauss map $\gamma: M \to G_{m-n}(\mathbb{R}^m)$ by $\gamma(x) = V_xM$ translated to the origin in $\mathbb{R}^m$ together with an orientation induced from those on M and N ($n = \dim N$). Then $d\gamma$ is a section of the bundle $T^*M \otimes V^*M \otimes HM$ over M, whereas $\nabla d\phi$ is a section of the bundle $T^*M \otimes T^*M \otimes \phi^{-1}TN$ over M. Let $d\phi \circ d\gamma$ denote the section of the bundle $T^*M \otimes V^*M \otimes \phi^{-1}TN$ obtained by composition on the last factor. Thus $d\phi \circ d\gamma(X, V) = d\phi[d\gamma(X, V)]$ for each $x \in TM$, $V \in VM$.

(3.1) THEOREM: $d\phi \circ d\gamma = -\nabla d\phi|_{TM \otimes VM}$.

Proof: Let $(X_a)_{a=1,\ldots,m} = (X_i, X_r)^{r=n+1,\ldots,m}_{i=1,\ldots,n}$ represent an orthonormal frame field about a point $x \in M$ adapted to the horizontal and vertical spaces, so that the X_i span HM and the X_r span VM. Then for each $x \in M$,

$$\gamma(x) = X_{n+1} \wedge \ldots \wedge X_m.$$

The canonical Riemannian structure on $G_{m-n}(\mathbb{R}^m)$ is defined by requiring that $X_r^* \otimes X_i$ is an orthonormal basis of $T_{\gamma(x)}G_{m-n}(\mathbb{R}^m)$. Then $X_r^* \otimes X_i$ can be identified with the plane $E_r^i = X_{n+1} \wedge \ldots \wedge X_{r-1} \wedge X_i \wedge X_{r+1} \wedge \ldots \wedge X_m$. Writing $\langle \nabla_{X_a} X_r, X_j \rangle = h_r^{aj}$ so that the horizontal projection $\mathcal{H}\nabla_{X_a} X_r = \sum_j h_r^{aj} X_j$; we obtain

$$d\gamma(X_a) = \sum_{r=n+1}^{m} X_{n+1} \wedge \ldots \wedge X_{r-1} \wedge \nabla_{X_a} X_r \wedge X_{r+1} \wedge \ldots \wedge X_m$$

$$= \sum_{r=n+1}^{m} \sum_{j=1}^{n} h_r^{aj} E_r^j$$

$$= \sum_r \sum_j h_r^{aj} X_r^* \otimes X_j.$$

Thus $d\gamma(X_a)(X_r) = \sum_j h_r^{aj} X_j = \mathcal{H}(\nabla_{X_a} X_r)$. So that $d\phi[d\gamma(X_a)X_r)] = d\phi(\nabla_{X_a} X_r)$.

Also
$$\nabla d\phi(X_a, X_r) = -d\phi(\nabla_{X_a} X_r) + \nabla_{X_a} d\phi(X_r)$$

$$= -d\phi(\nabla_{X_a} X_r). \qquad //$$

Given a bundle E over M, let $\mathcal{C}(E)$ denote the space of smooth sections of E. We use the same ∇ to denote the Levi-Civita connection on the various bundles. Then trace $\nabla(d\phi \circ d\gamma) \in \mathcal{C}(V^*M \otimes \phi^{-1}TN)$. Consider also ∇ trace $\nabla d\phi \in \mathcal{C}(T^*M \otimes \phi^{-1}TN)$ and let ∇^V trace $\nabla d\phi$ denote the restriction to the bundle $V^*M \otimes \phi^{-1}TN$.

(3.2) THEOREM. Trace $\nabla(d\phi \circ d\gamma) = -\nabla^V$ trace $\nabla d\phi$. _Furthermore if_ M _is the flat torus_ T^m _and hence compact, then_ trace $\nabla(d\phi . d\gamma) = 0$ _if and only if_ $d\phi \circ d\gamma$ _is a harmonic_ $V^*M \otimes \phi^{-1}TN$ _valued_ 1-_form_.

Proof: Consider a point $x \in M$ and choose an orthonormal frame field $(e_a)_{a=1,\ldots,m}$ about x with the property that $\nabla_{e_a} e_b = 0$ at x, and such that $(e_a)_{a=1,\ldots,m} = (e_i, e_r)_{i=1,\ldots,n}^{r=n+1,\ldots,m}$ is adapted to HM and VM at x. Then, evaluating at the point x and summing over repeated indices,

$$\begin{aligned} \text{trace } \nabla(d\phi \circ d\gamma)(e_r) &= [\nabla_{e_a}(d\phi \circ d\gamma)(e_a)](e_r) \\ &= -\nabla_{e_a}[\nabla d\phi(e_a, e_r)] \\ &= d\phi(\nabla_{e_a}\nabla_{e_r} e_a) - \nabla_{e_a}\nabla_{e_r} d\phi(e_a). \end{aligned}$$

Also

$$\begin{aligned} \nabla_{e_r} \text{ trace } \nabla d\phi &= \nabla_{e_r}[\nabla d\phi(e_a, e_a)] \\ &= -d\phi(\nabla_{e_r}\nabla_{e_a} e_a) + \nabla_{e_r}\nabla_{e_a} d\phi(e_a). \end{aligned}$$

Thus $$-\text{trace } \nabla(d\phi\circ d\gamma)(e_r) - \nabla_{e_r} \text{trace } \nabla d\phi$$

$$= -d\phi[R^M(e_a, e_r)e_a + \nabla_{[e_a, e_r]}e_a]$$

$$+ R^N(d\phi(e_a), d\phi(e_r))d\phi(e_a) + \nabla_{[e_a,e_r]}d\phi(e_a),$$

which vanishes at x since $R^M = 0$.

To show that $d\phi\circ d\gamma$ is a harmonic 1-form, we must show $\Delta(d\phi\circ d\gamma) = 0$, where $\Delta = dd^* + d^*d$, and d, d* denote the differential and codifferential operators acting on vector bundle valued 1-forms [8].

If trace $\nabla(d\phi\circ d\gamma) = 0$, then $d^*(d\phi\circ d\gamma) = 0$ follows immediately. Furthermore

$$d(d\phi\circ d\gamma)(e_a, e_b) = \nabla_{e_a}(d\phi\circ d\gamma)(e_b) - \nabla_{e_b}(d\phi\circ d\gamma)(e_a) + d\phi\circ d\gamma([e_a, e_b]),$$

which vanishes after identification of $d\phi\circ d\gamma$ with $-\nabla d\phi|_{TM\otimes VM}$. For

$$[\nabla_{e_a}(d\phi\circ d\gamma)(e_b) - \nabla_{e_b}(d\phi\circ d\gamma)(e_a)](e_r) = d\phi[R^M(e_a, e_b)e_r]$$

$$= 0,$$

since $R^M = 0$. //

Similar considerations apply when M is a domain in the Euclidean sphere S^m. Then for each $x \in M$, $\gamma(x)$ assigns the space spanned by ker $d\phi(x)$ together with the vector x. Thus $\gamma: M \to G_{m-n+1}(\mathbb{R}^{m+1})$. The above theorem now becomes [2]:

(3.3) THEOREM: trace $\nabla(d\phi \circ d\gamma) = 0$ <u>if and only if</u> $\tau_\phi = 0$.

For horizontally conformal maps, the condition $\nabla^V \tau_\phi = 0$ has an interpretation in terms of the mean curvature of the fibres. We see this in the next section.

4. HORIZONTALLY CONFORMAL MAPPINGS.

Let $\phi: (M, g) \to (N, h)$ be a mapping of Riemannian manifolds. Then ϕ is <u>horizontally conformal</u> if for each point $x \in M$ at which $d\phi(x) \neq 0$, $d\phi(x)|_{H_x M}$ is conformal and surjective. Thus, there exists a positive number $\lambda(x)$ such that $\|d\phi_x(X)\|^2 = \lambda(x)^2 \|X\|^2$ for each $X \in H_x M$. Setting $\lambda = 0$ on the critical set $C_\phi = \{x \in M;\ d\phi(x) = 0\}$ we obtain a continuous function $\lambda: M \to \mathbb{R}$ called the <u>dilation</u> of ϕ. The function λ^2 is smooth.

Let $S_\phi = \frac{1}{2}\|d\phi\|^2 g - \phi^* h$ denote the stress-energy tensor of ϕ. Then [4]

(4.1) $$\nabla^* S_\phi(X) = -h(\tau_\phi, d\phi(X))$$

for each $x \in M$ and for each $X \in T_x M$.

If ϕ is horizontally conformal, then $\|d\phi\|^2 = n\lambda^2$, and for each horizontal vector X

(4.2) $\nabla^* S_\phi(X) = \frac{1}{2}(n - 2)X(\lambda^2) + \lambda^2$ (mean curvature of fibre in X direction)
(see [4] for a proof of this fact).

Let μ denote the mean curvature of the fibres, so that μ is the horizontal projection $\mathcal{H}(\nabla_{X_r} X_r)$. Say that the fibres have basic mean curvature if $d\phi(\mu) = C \circ \phi$ for some vector field $C \in \mathcal{C}(TN)$. From (4.1) and (4.2) we obtain the following

(4.3) THEOREM: If either (i) $n = 2$, or (ii) $n \geq 3$ and $\nabla\lambda^2$ is vertical, then

(a) ϕ is harmonic if and only if the fibres of ϕ are minimal.

(b) $\tau_\phi = B \circ \phi$ for some vector field $B \in \mathcal{C}(TN)$ if and only if the fibres of ϕ have basic mean curvature.

With reference to the last section, we observe that provided the fibres are connected, $\nabla^V \tau_\phi = 0$ if and only if $\tau_\phi = B \circ \phi$ for some $B \in \mathcal{C}(TN)$ (if the fibres are not connected then the above applies to each connected component). If $M \subset \mathbb{R}^m$ we therefore obtain the analogue of the theorem of Ruh-Vilms:

(4.4) THEOREM: If $\phi: M \to N$ is a horizontally conformal submersion, such that either (i) $n = 2$, or (ii) $n \geq 3$ and $\nabla\lambda^2$ vertical, and $\gamma: M \to G_{m-n}(\mathbb{R}^m)$ is the associated Gauss mapping, then trace $\nabla(d\phi \cdot d\gamma) = 0$ if and only if the components of the fibres of ϕ have basic mean curvature.

(4.5) REMARK: We observe that for a submersion with $n \geq 3$, $\nabla\lambda^2$ being vertical is the analogue of a homothetic immersion with $m \geq 3$.

In general basic mean curvature is not equivalent to parallel mean curvature. Strong conditions are required to ensure equivalence [2]. In fact if (i) $\lambda^2 = A \circ \phi$, for some $A \in \mathcal{C}(N)$, and (ii) the horizontal distribution HM is integrable, then the fibres of ϕ have parallel mean curvature if and only if they have basic mean curvature. Conversely, if the fibres of ϕ have both basic mean curvature and parallel mean curvature, then condition (i) above holds.

In the next section we seek the analogue of Chern's result for the case of 2-dimensional range.

5. SUBMERSIONS ONTO A SURFACE

Let $\phi: M \to N$ be a submersion from a domain M in $\mathbb{R}^m$ onto an oriented surface N. Then the Gauss map $\gamma: M \to G_{m-2}(\mathbb{R}^m)$. The Grassmannian $G_{m-2}(\mathbb{R}^m)$ can be identified with $G_2(\mathbb{R}^m)$ by sending each (m -2)-plane in $\mathbb{R}^m$ to its orthogonal complement. As we have already seen $G_2(\mathbb{R}^m)$ has a natural complex structure induced by identifying it with the complex quadric hypersurface $Q_{m-2} \subset \mathbb{C}P^{m-1}$. We thus have induced a natural complex structure J on $G_{m-2}(\mathbb{R}^m)$ which alternatively can be described as follows.

Let $p \in G_{m-2}(\mathbb{R}^m)$, $a \in T_pG_{m-2}(\mathbb{R}^m)$. Then $a: V_p \to H_p$. Define Ja by

$$(Ja)(v) = J^H(a(v)),$$

for each $v \in V_p$, where J^H is rotation through $\pi/2$ in the horizontal space H_p (this is well defined in terms of the orientation on N). We say that γ is <u>horizontally holomorphic</u> if $d\gamma(J^H X) = Jd\gamma(X)$ for each $X \in H_xM$, $x \in M$.

(5.1) THEOREM: <u>If</u> ϕ <u>is horizontally conformal, then the Gauss map</u> γ <u>is horizontally</u> $\pm$ <u>holomorphic</u>. The result follows from the following two Lemmas.

(5.2) LEMMA: The Gauss map γ is horizontally holomorphic if and only if

$$\langle \nabla_{X_1} X_1 - \nabla_{X_2} X_2, V\rangle = 0$$

$$\langle \nabla_{X_1} X_2 + \nabla_{X_2} X_1, V\rangle = 0$$

for each vertical vector field V, where (X_1, X_2) is an orthonormal horizontal frame field and $\langle,\rangle$ denotes the Euclidean inner product.

Proof: This follows directly after noting that $d\gamma(X_i)X_r = \nabla_{X_1} X_r$, for each $i = 1, 2$, $r = 3, \ldots, m$.

(5.3) LEMMA: If ϕ is horizontally conformal with dilation λ and X is a horizontal vector field, V a vertical vector field, then the horizontal projection $[X, V]$ is given by

$$[X, V] = -d(\log \lambda)(V)X.$$

Proof: Choose an orthonormal frame field $(X_a) = (X_i, X_r)_{i=1,2}^{r=3,\ldots,m}$ adapted to the horizontal and vertical spaces. Assume also that $d\phi(X_i) = \lambda\bar{X}_i$ locally about some point, for some orthonormal frame field $(\bar{X}_i)_{i=1,2}$ on N. That is X_i is the horizontal lift of $\bar{X}_i$ normalized to have unit length. Then

$$\begin{aligned}\nabla d\phi(X_i, X_r) &= -d\phi(\nabla_{X_i} X_r) + \nabla_{X_i} d\phi(X_r)\\ &= -d\phi(\nabla_{X_i} X_r).\end{aligned}$$

By the symmetry of $\nabla d\phi$ this also equals

$$\nabla d\phi(X_r, X_i) = -d\phi(\nabla_{X_r} X_i) + \nabla_{X_r} d\phi(X_i).$$

But

$$\begin{aligned}\nabla_{X_r} d\phi(X_i) &= \nabla_{X_r} \lambda\bar{X}_i \\ &= d\lambda(X_r)\bar{X}_i + \lambda\nabla_{X_r}\bar{X}_i \\ &= d\lambda(X_r)\bar{X}_i.\end{aligned}$$

Then $d\phi(\nabla_{X_i} X_r - \nabla_{X_r} X_i) = -d(\log\lambda)(X_r)d\phi(X_i)$ and the result follows. //

The Theorem now follows directly by verifying that Lemma (5.2) holds, using Lemma (5.3).

6. APPLICATION TO HARMONIC MORPHISMS WITH TOTALLY GEODESIC FIBRES.

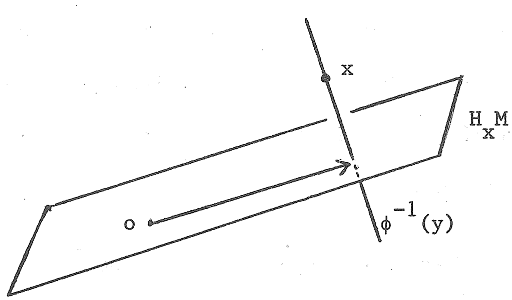
The point here is that since the fibres are totally geodesic they are parts of planes in $\mathbb{R}^m$. Thus, if the fibres are connected, the Gauss map γ is constant along them, thus reducing to a map from the base space N. From the results of the last section this map will be $\pm$ holomorphic, and we obtain our classification in terms of $\pm$ holomorphic mappings into Grassmannians. We omit proofs which can be found in [5].

Let $\phi: M \to N$ be a harmonic morphism from a domain M in Euclidean space $\mathbb{R}^m$ onto a Riemann surface N. Then the fibres of ϕ are minimal [4]. Suppose in addition the fibres are totally geodesic and connected, and that ϕ is a submersion everywhere. Let γ denote the Gauss map, but now assume that to each $x \in M$, $\gamma(x)$ associates the horizontal space $H_x M$ instead of the vertical space $V_x M$. Thus $\gamma: M \to G_2(\mathbb{R}^m) \cong Q_{m-2} \subset \mathbb{C}P^{m-1}$. Then we obtain the commutative diagram:

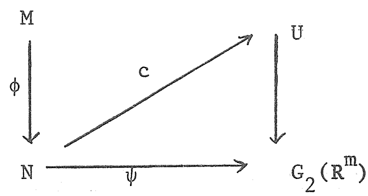
$$\begin{array}{ccc} M & & \\ \phi \downarrow & \searrow \gamma & \\ N & \xrightarrow{\psi} & G_2(\mathbb{R}^m) \end{array}$$

where $\psi(y)$ determines the fibre of ϕ over y, for each $y \in N$.

Consider now the position vector $c(y)$ of the fibre $\phi^{-1}(y)$. That is $c(y)$ is a vector in $\mathbb{R}^m$, perpendicular to the fibre $\phi^{-1}(y)$, which determines the position of $\phi^{-1}(y)$ in $\mathbb{R}^m$.



Then if $x \in \phi^{-1}(y)$, $c(y) \in H_x M$. Let U denote the universal bundle over $G_2(\mathbb{R}^m)$. Then the fibre of U over $p \in G_2(\mathbb{R}^m)$ is precisely the plane H_p determined by p. The vector field c now becomes a section of $\psi^{-1}U$:



If $\pi: U \to G_2(\mathbb{R}^m)$ is the canonical projection, then at each point $a \in U$, the tangent space T_aU decomposes: $T_aU \cong T_a^1U \oplus T_a^2U$, where $T_a^2U = \ker d\pi(a)$ and T_a^1U is the orthogonal complement. If H_p is the plane determined by $\pi(a) = p$, then $T_a^2U \cong H_p$. There is then a complex structure J^1 on T_a^1U determined by the complex structure J on the base $G_2(\mathbb{R}^m)$, and a complex structure J^2 on T_a^2U determined by rotation through $\pi/2$ in H_p. We thus obtain two distinct complex structures on U:

$$J^I = (J^1, J^2) \text{ and}$$

$$J^{II} = (J^1, -J^2).$$

Exploiting the fact that ψ is $\pm$ holomorphic, and deriving equations for c resulting from the horizontal conformality of ϕ we obtain [5]:

(6.1) THEOREM: The section $C: N \to U$ is $\pm$ holomorphic with respect to the complex structure J^{II} on U. Conversely, any such section yields a harmonic morphism from a domain $M \subset \mathbb{R}^m$.

We note the similarity between this result and the results of Eells-Wood [12] and Eells-Salamon [10] for classifying harmonic conformal immersions into various homogeneous spaces.

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