THE OBLIQUE DERIVATIVE PROBLEM FOR EQUATIONS OF MONGE-AMPERE TYPE IN TWO DIMENSIONS

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1. Introduction

In this paper we are concerned with the existence of convex classical solutions of the oblique derivative problem for equations of Monge-Ampère type,

$$\det D^2u = f(x,u,Du) \text{ in } \Omega \, , \tag{1.1}$$

$$D_\beta u = \varphi(x,u) \text{ on } \partial\Omega \, . \tag{1.2}$$

Here Ω is a uniformly convex domain in $\mathbb{R}^2$, f , φ are prescribed functions on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2$, $\partial\Omega \times \mathbb{R}$ respectively with f positive, and β is a unit vector field on $\partial\Omega$ satisfying

$$\beta\cdot\nu > 0 \, , \tag{1.3}$$

where ν is the inner unit normal to $\partial\Omega$. This problem, and in particular the case $\beta = \nu$, was recently studied for domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, by Lions, Trudinger and Urbas [5], who proved a priori estimates for the derivatives up to second order for convex solutions of (1.1), (1.2) under suitable regularity and structure hypotheses on Ω , f , φ and β . In particular, the second derivative estimate in [5] requires $\beta = \nu$, and it does not appear that the method used there can be modified to work for more general β . However, if the domain Ω is a ball and $f^{1/n}$ is convex with respect to the gradient variables, the argument given in [5], Section 4 can be used to obtain second derivative bounds for more general β .

Here we derive second derivative bounds and existence theorems for convex solutions of (1.1), (1.2) in two dimensions. This is still an open problem in higher dimensions, except for the special cases mentioned above. For the main existence theorem, which is stated below, we shall assume that Ω is a uniformly convex domain in $\mathbb{R}^2$ with boundary $\partial\Omega \in C^{2,1}$, $f \in C^{1,1}(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^2)$ is positive and nondecreasing in z for all $(x,z,p) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^2$, $\varphi \in C^{1,1}(\partial\Omega\times\mathbb{R})$ is nondecreasing in z with

$$(1.4) \qquad \varphi_z(x,z) \ge \gamma_0$$

for all $(x,z) \in \partial\Omega \times \mathbb{R}$, for some positive constant γ_0 , and $\beta \in C^{1,1}(\partial\Omega,\mathbb{R}^2)$ is a vector field satisfying

$$(1.5) \qquad \beta\cdot\nu \ge \mu_0 \ , \ |\beta| = 1 \quad \text{on} \quad \partial\Omega$$

and

$$(1.6) \qquad [-2\delta_i\beta_j(x) + \varphi_z(x,z)\delta_{ij}]\tau_i\tau_j \ge \mu_1$$

for all $(x,z) \in \partial\Omega \times \mathbb{R}$ and all directions τ tangential to $\partial\Omega$ at x , where μ_0 , μ_1 are positive constants and $\delta = (\delta_1,\delta_2)$ denotes the tangential gradient operator relative to $\partial\Omega$ given by

$$\delta_i = (\delta_{ij} - \nu_i\nu_j)D_j \ .$$

Without loss of generality we shall assume that β , φ and ν have been extended to be of class $C^{1,1}$ on $\bar{\Omega}$ with (1.4), (1.5), (1.6) holding near $\partial\Omega$. Furthermore, we assume the structural inequality

$$(1.7) \qquad f(x,N,p) \leq g(x)/h(p)$$

for all $(x,p) \in \Omega \times \mathbb{R}^2$, where N is a constant and $g \in L^1(\Omega)$, $h \in L^1_{loc}(\mathbb{R}^2)$ are positive functions satisfying

$$(1.8) \qquad \int_\Omega g < \int_{\mathbb{R}^2} h \ .$$

THEOREM 1.1 *Under the above hypotheses on the domain* Ω *and the functions* f , φ , β , *the boundary value problem* (1.1), (1.2) *has a unique convex solution* $u \in C^{2,\alpha}(\bar{\Omega})$ *for all* $\alpha < 1$.

If f , φ , β and $\partial\Omega$ are C^∞ , then the solution $u \in C^\infty(\bar{\Omega})$, by virtue of standard linear theory [2]. Two special cases included in Theorem 1.1 are the standard Monge Ampère equation

$$(1.9) \qquad \det D^2u = f(x) \ ,$$

and the equation of prescribed Gauss curvature

$$(1.10) \qquad \det D^2u = K(x)(1+|Du|^2)^2 \ ,$$

for which the conditions (1.7), (1.8) take the form

$$\int_\Omega K < \pi . \tag{1.11}$$

The condition (1.11) is also necessary for the existence of a classical solution (see [1], [7]).

The proof of Theorem 1.1 depends on the method of continuity (see [2], Theorem 17.28) which requires the a priori estimation of solutions in the Hölder space $C^{2,\alpha}(\bar{\Omega})$ for some $\alpha > 0$. The conditions (1.4), (1.7), (1.8) and

$$\beta \cdot \nu \geq 0 \tag{1.12}$$

enable us to prove a bound

$$\sup_\Omega |u| \leq C , \tag{1.13}$$

(see [5], Theorem 2.1), while (1.5), (1.13) and the convexity of u imply a gradient bound

$$\sup_\Omega |Du| \leq C , \tag{1.14}$$

(see [5], Theorem 2.2).

Once the second derivatives are bounded, the equation is uniformly elliptic, so we can apply the theory developed in [3], [4] to deduce a second derivative Hölder estimate

$$[D^2 u]_{\alpha;\Omega} \leq C \tag{1.15}$$

for some $\alpha > 0$. As noted in [3], the estimate (1.15) can be proved much more easily in two dimensions than in higher dimensions, and in the two variable case (1.15) is in fact valid under our somewhat weaker regularity hypotheses.

The estimation of the second derivatives is carried out in the following section. Some parts of our argument are similar to that of [5], but for completeness we include all the details.

Unless otherwise stated, our notation follows the book [2].

2. Second Derivative Bounds

As a preliminary to our main result, we first consider the special case of (1.1), (1.2) when $f(x,z,p) = f(x)$.

THEOREM 2.1 *Let* Ω *be a* $C^{2,1}$ *uniformly convex domain in* $\mathbb{R}^2$ *and* $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ *a convex solution of*

$$\det D^2u = f(x) \quad \text{in} \quad \Omega , \tag{2.1}$$

$$D_\beta u = \varphi(x,u) \quad \text{on} \quad \partial\Omega , \tag{2.2}$$

where $f \in C^{1,1}(\bar{\Omega})$ *is positive,* $\varphi \in C^{1,1}(\partial\Omega \times \mathbb{R})$ *is nondecreasing in* z , *and* $\beta \in C^{1,1}(\partial\Omega , \mathbb{R}^2)$ *is a vector field satisfying (1.5) and (1.6) . Then we have*

$$\sup_\Omega |D^2u| \le C , \tag{2.3}$$

where C *depends only on* μ_0 , μ_1 , $|u|_{1;\Omega}$, $|\log f|_{1,1;\Omega}$, $|\varphi|_{1,1;\partial\Omega\times(-M,M)}$, $|\beta|_{1,1;\partial\Omega}$ *and* Ω , *where* $M = \sup_{\Omega}|u|$.

PROOF We start with some preliminary identities. Writing the equation (2.1) in the form

$$F(D^2u) = \log\det D^2u = g(x) \tag{2.4}$$

and using the notation

$$F^{ij}(r) = \frac{\partial F}{\partial r_{ij}}(r) \; , \; F^{ij,kl}(r) = \frac{\partial^2 F}{\partial r_{ij}\partial r_{kl}} \; ,$$

we have

$$\begin{aligned} F^{ij} &= u^{ij} \\ F^{ij,kl} &= -F^{ik}F^{jl} = -u^{ik}u^{jl} \; , \end{aligned} \tag{2.5}$$

where $[u^{ij}]$ is the inverse matrix of D^2u . Since u is convex, F is a concave function of D^2u . Furthermore, we have $D_{\gamma\gamma}u \geq 0$ for any direction γ , so it is sufficient to bound $D_{\gamma\gamma}u$ from above. To do this we consider the function v on $\bar{\Omega} \times S^1$ defined by

$$v(x,\gamma) = D_{\gamma\gamma}u(x) + K|x|^2 \; ,$$

where K is a positive constant to be chosen. Using the concavity of F, we compute

$$
\begin{aligned}
F^{ij}D_{ij}v &= F^{ij}D_{ij\gamma\gamma}u + 2KF^{ij}\delta_{ij} \\
&= -F^{ij,kl}D_{ij\gamma}uD_{kl\gamma}u + D_{\gamma\gamma}g + 2K\mathcal{T} \\
&\geq 2K\mathcal{T} - C ,
\end{aligned}
\tag{2.6}
$$

where

$$\mathcal{T} = \operatorname{trace}\,[F^{ij}] \geq 2f^{-1/2} = 2e^{-g/2} . \tag{2.7}$$

Fixing K sufficiently large we therefore obtain $F^{ij}D_{ij}v \geq 0$ in Ω, so by the maximum principle, v attains its maximum on $\partial\Omega$.

We now proceed to estimate the second derivatives on $\partial\Omega$. Computing the tangential gradient of the boundary condition (2.2) on $\partial\Omega$, we obtain

$$D_k u\ \delta\beta_k + \beta_k\ \delta D_k u = \delta\varphi ,$$

so that if τ is any direction tangential to $\partial\Omega$ at any point $y \in \partial\Omega$, we have

$$D_{\tau\beta}u(y) = \tau_i\delta_i\varphi - \tau_i(\delta_i\beta_k)D_k u , \tag{2.8}$$

and hence,

$$|D_{\tau\beta}u(y)| \leq C . \tag{2.9}$$

Next we bound $D_{\nu\beta}u$ on $\partial\Omega$. Since we shall also need this for the general case $f = f(x,z,p)$, we carry out the argument for the boundary value problem (1.1), (1.2). Taking the logarithm of the equation (1.1), and differentiating in the k-th coordinate direction, we obtain

$$F^{ij}D_{ijk}u = g_{x_k} + g_z D_k u + g_{p_i} D_{ik}u \ . \tag{2.10}$$

Using (2.5) and (2.10) we therefore obtain, for

$$h = \beta_k D_k u - \varphi(x,u) \ ,$$

$$\begin{aligned} F^{ij}D_{ij}h = {} & \beta_k g_{x_k} + g_z \beta_k D_k u + g_{p_i} D_i h \\ & - g_{p_i} D_k u \, D_i \, \beta_k + g_{p_i} D_i \varphi \\ & + 2D_i \beta_i + F^{ij}(D_{ij}\beta_k) D_k u - F^{ij} D_{ij}\varphi \ , \end{aligned} \tag{2.11}$$

so that

$$|F^{ij}D_{ij}h - g_{p_i} D_i h| \leq C(1+\mathcal{T}) \leq C\mathcal{T} \ . \tag{2.12}$$

Consequently, by [2], Corollary 14.5, we obtain

$$\sup_{\partial\Omega} |D_\nu h| \leq C \ ,$$

and therefore

$$\sup_{\partial\Omega} |D_{\nu\beta}u| \le C \; . \tag{2.13}$$

Combining the estimates (2.9) and (2.13) we obtain

$$\sup_{\substack{x\in\partial\Omega \\ \gamma\in S^1}} |D_{\gamma\beta}u(x)| \le C \; . \tag{2.14}$$

We note that (2.14) holds in the more general case $f = f(x,z,p)$.

To complete the estimation of the second derivatives we need to use the fact that v attains its maximum on $\partial\Omega$. Let us therefore assume that v attains its maximum at a point $x_0 \in \partial\Omega$ and a direction $\xi \in S^1$. Let $\eta \in S^1$ be a direction normal to ξ. We may suppose that at x_0 we have

$$\xi\cdot\nu \ge 0 \quad \text{and} \quad \eta\cdot\nu \ge 0 \; .$$

Furthermore, if we now take ξ, η as the coordinate directions, the Hessian D^2u is diagonal at x_0 with maximum eigenvalue $D_{\xi\xi}u$, and the equation (2.1) takes the form

$$D_{\xi\xi}u\, D_{\eta\eta}u - (D_{\xi\eta}u)^2 = f \; . \tag{2.15}$$

Let a, b be constants, $a^2 + b^2 = 1$, such that at x_0,

$$\nu = b\xi + a\eta \tag{2.16}$$

and let

(2.17) $$\tau = a\xi - b\eta ,$$

so that τ is tangential to $\partial\Omega$ at x_0. Since $\xi\cdot\nu$, $\eta\cdot\nu \geq 0$ at x_0, we see that $a,b \geq 0$. Let c,d be constants with $c^2 + d^2 = 1$, so that at x_0,

(2.18) $$\beta = c\xi + d\eta .$$

We then have, since $D_{\xi\eta}u\,(x_0) = 0$,

$$\begin{aligned} D_{\beta\beta}u(x_0) &= c^2 D_{\xi\xi}u(x_0) + d^2 D_{\eta\eta}u(x_0) \\ &\geq c^2 D_{\xi\xi}u(x_0) , \end{aligned}$$

so by (2.14), $D_{\xi\xi}u(x_0)$ is bounded provided c^2 is bounded away from zero. We therefore need only consider the case that $|c|$ is small, say $|c| \leq c_0$ for a suitably chosen positive constant c_0.

Since $v(x,\xi)$ attains its maximum at x_0, we have

(2.19) $$D_{\xi\xi\beta}u(x_0) \leq C$$

and

(2.20) $$D_{\xi\xi\xi}u(x_0) \leq C .$$

The main step now is to show that these two inequalities imply

(2.21) $$D_{\tau\tau\beta}u(x_0) \leq C ,$$

provided $D_{\xi\xi}u(x_0)$ is sufficiently large, and $|c| = |\xi\cdot\beta(x_0)|$ is sufficiently small. This is where we use the two dimensionality. The required bound for $D_{\xi\xi}u(x_0)$ then follows from (2.21).

To prove (2.21) we first differentiate (2.1) in the directions ξ , η . Noting that $D_{\xi\eta}u(x_0) = 0$, we obtain at x_0 ,

$$D_{\xi\eta\eta}u = \frac{D_\xi f}{D_{\xi\xi}u} - \frac{f}{(D_{\xi\xi}u)^2} D_{\xi\xi\xi}u \tag{2.22}$$

and

$$D_{\eta\eta\eta}u = \frac{D_\eta f}{D_{\xi\xi}u} - \frac{f}{(D_{\xi\xi}u)^2} D_{\xi\xi\eta}u \ . \tag{2.23}$$

Furthermore, at x_0 we also have

$$D_{\tau\tau\beta}u = a^2 D_{\xi\xi\beta}u - 2abD_{\xi\eta\beta}u + b^2 D_{\eta\eta\beta}u \ .$$

Making use of (2.16), (2.17), (2.18), (2.22) and (2.23), we then obtain at x_0 ,

$$(2.24)\qquad \begin{aligned} D_{\tau\tau\beta}u &= a^2 D_{\xi\xi\beta}u - 2abcD_{\xi\xi\eta}u - 2abdD_{\xi\eta\eta}u \\ &\quad + b^2 cD_{\xi\eta\eta}u + b^2 dD_{\eta\eta\eta}u \\ &= a^2 D_{\xi\xi\beta}u - 2abcD_{\xi\xi\eta}u \\ &\quad - (2abd-b^2c)\left[\frac{D_\xi f}{D_{\xi\xi}u} - \frac{f}{(D_{\xi\xi}u)^2} D_{\xi\xi\xi}u\right] \\ &\quad + b^2 d\left[\frac{D_\eta f}{D_{\xi\xi}u} - \frac{f}{(D_{\xi\xi}u)^2} D_{\xi\xi\eta}u\right] \\ &= \left[a^2 - \frac{b^2 f}{(D_{\xi\xi}u)^2} - \frac{2abc}{d}\right] D_{\xi\xi\beta}u \\ &\quad + \left[\frac{2abc^2}{d} + 2abd \frac{f}{(D_{\xi\xi}u)^2}\right] D_{\xi\xi\xi}u \\ &\quad + \frac{1}{D_{\xi\xi}u}\left[b^2 cD_\xi f + b^2 dD_\eta f - 2abdD_\xi f\right] . \end{aligned}$$

Next, using (2.16), (2.18) and (1.5), we obtain

$$\beta \cdot \nu = bc + ad \geq \mu_0 ,$$

so that

$$ad \geq \mu_0 - bc \geq \mu_0 - |c| \geq \mu_0/2$$

provided

$$(2.25)\qquad |c| \leq \mu_0/2 .$$

Assuming henceforth that (2.25) is satisfied, we obtain, since $a^2, d^2 \leq 1$, and $a \geq 0$,

$$(2.26)\qquad \mu_0/2 \leq a,d \leq 1 ,$$

and therefore also

$$0 \le \frac{abc^2}{d} \le \mu_0/2 \tag{2.27}$$

and

$$0 \le abd \le 1 \ . \tag{2.28}$$

Next we have

$$a^2 - \frac{b^2 f}{(D_{\xi\xi}u)^2} - \frac{2abc}{d} \ge \frac{\mu_0^2}{4} - \frac{|f|_{0;\Omega}}{(D_{\xi\xi}u)^2} - \frac{4|c|}{\mu_0} \ ,$$

so that if we further assume

$$|c| \le c_0 = \mu_0^3/32 \tag{2.29}$$

and

$$(D_{\xi\xi}u(x_0))^2 \ge \frac{8|f|_{0;\Omega}}{\mu_0^2} \ , \tag{2.30}$$

then

$$0 \le a^2 - \frac{b^2 f}{(D_{\xi\xi}u)^2} - \frac{2abc}{d} \le 1 + \frac{\mu_0^2}{8} \ . \tag{2.31}$$

Using the estimates (2.19), (2.20), (2.27), (2.28), (2.30) and (2.31) in (2.24), we then obtain (2.21) as required.

Finally, we need to show that (2.21) implies a bound for $D_{\xi\xi}u(x_0)$. Computing the second tangential derivatives of the boundary condition (2.2), we obtain

$$(2.32) \quad D_k u \delta_i \delta_j \beta_k + \delta_i \beta_k \delta_j D_k u + \delta_j \beta_k \delta_i D_k u + \beta_k \delta_i \delta_j D_k u = \delta_i \delta_j \varphi \ ,$$

and hence at x_0 ,

$$(2.33) \qquad D_{\tau\tau\beta}u \geq -\, 2(\delta_i \beta_k) D_{jk} u \tau_i \tau_j + (\delta_i \nu_j) \tau_i \tau_j D_{\nu\beta} u + \varphi_z D_{ij} u \tau_i \tau_j - C \ .$$

Using (2.13), (2.21) and (1.6) in (2.33) we then obtain

$$(2.34) \qquad D_{\tau\tau}u(x_0) \leq C \ .$$

Now writing

$$\xi = \bar{a}\tau + \bar{b}\beta(x_0) \ ,$$

we have

$$|\bar{a}| \ , \ |\bar{b}| \leq C$$

by virtue of the obliqueness condition (1.5), and hence, using (2.14) and (2.34),

$$(2.35) \qquad D_{\xi\xi}u = \bar{a}^2 D_{\tau\tau}u + 2\bar{a}\bar{b}D_{\tau\beta}u + \bar{b}^2 D_{\beta\beta}u \le C$$

at x_0. We therefore deduce a bound for $\sup_{\Omega\times S^1} v$, from which we obtain (2.3). The proof of Theorem 2.1 is therefore complete.

Combining the estimates (1.13), (1.14), (1.15) and (2.3) we obtain a global estimate

$$(2.36) \qquad |u|_{2,\alpha;\Omega} \le C$$

for convex solutions $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ of the boundary value problem (2.1), (2.2), where $\alpha \in (0,1)$ and C depend on f, φ, β, Ω and γ_0, where

$$(2.37) \qquad \varphi_z \ge \gamma_0$$

for some positive constant γ_0. Using the method of continuity [2], Theorem 17.28, and a standard approximation argument to overcome the requirement $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ in Theorem 2.1, we can then infer the following special case of Theorem 1.1.

THEOREM 2.2 *Let Ω be a $C^{2,1}$ uniformly convex domain in $\mathbb{R}^2$, $f \in C^{1,1}(\bar{\Omega})$ a positive function, $\varphi \in C^{1,1}(\partial\Omega\times\mathbb{R})$ satisfy (1.4) and $\beta \in C^{1,1}(\partial\Omega,\mathbb{R}^2)$ satisfy (1.5) and (1.6). Then the classical boundary value problem (2.1), (2.2) has a unique convex solution $u \in C^{2,\alpha}(\bar{\Omega})$ for all $\alpha < 1$.*

The arbitrariness of the Hölder exponent α follows from standard linear theory [2]. We shall make use of Theorem 2.2 in the derivation of the second derivative bound in the general case.

THEOREM 2.3 *Let* Ω *be a* $C^{2,1}$ *uniformly convex domain in* $\mathbb{R}^2$ *and* $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ *a convex solution of the boundary value problem* (1.1), (1.2), *where* $f \in C^{1,1}(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^2)$ *is a positive function such that* $g = \log f$ *satisfies*

$$(2.38) \qquad |g(x,u,Du)| + |Dg(x,u,Du)| + |D^2g(x,u,Du)| \le \mu ,$$

$\varphi \in C^{1,1}(\partial\Omega\times\mathbb{R})$ *satisfies* (1.4), *and* $\beta \in C^{1,1}(\partial\Omega,\mathbb{R}^2)$ *satisfies* (1.5) *and* (1.6). *Then we have*

$$(2.39) \qquad \sup_{\Omega} |D^2u| \le C ,$$

where C *depends only on* μ , μ_0 , μ_1 , γ_0 , $|u|_{1;\Omega}$, $|\varphi|_{1,1;\partial\Omega\times(-M,M)}$, $|\beta|_{1,1;\partial\Omega}$ *and* Ω , *where* $M = \sup_{\Omega}|u|$.

PROOF Let $w \in C^{2,1}(\bar{\Omega})$ be a uniformly convex defining function for Ω , i.e., $w < 0$ in Ω , $w = 0$ on $\partial\Omega$, $|Dw| \ne 0$ on $\partial\Omega$, and

$$\theta I \le D^2w \le \theta I$$

for some positive constants θ , θ . From (2.38),

$$f(x,u,Du) \ge \delta_0 = e^{-\mu} .$$

Now, by Theorem 2.2, for each $\rho \in (0,1)$ there is a unique convex function $\psi = \psi_\rho \in C^2(\bar{\Omega})$ solving the boundary value problem

$$\text{(2.40)} \qquad \det D^2\psi = \delta_0/2 \quad \text{in} \quad \Omega ,$$

$$\text{(2.41)} \qquad D_\beta\psi = \varphi(x,\psi+\rho w) - \rho D_\beta w \quad \text{on} \quad \partial\Omega ,$$

and using Theorems 2.1 and 2.2 of [5], and Theorem 2.1, we obtain

$$\text{(2.42)} \qquad \sup_{\rho\in(0,1)} |\psi_\rho|_{2;\Omega} \le \Lambda$$

for some positive constant Λ. From (2.42) and the fact that $\delta_0 > 0$, we also have

$$\text{(2.43)} \qquad D^2\psi_\rho \ge \lambda I$$

for some positive constant λ, independent of $\rho \in (0,1)$. Letting $\bar{\psi} = \psi + \rho w$, we obtain

$$\det D^2\bar{\psi} \le \det D^2\psi + C(\rho\Lambda + \rho^2) ,$$

so fixing $\rho > 0$ sufficiently small, we have

$$\det D^2\bar{\psi} \le \delta_0 \quad \text{in} \quad \Omega .$$

By the mean value theorem, the function $u - \bar{\psi}$ satisfies an elliptic differential inequality

$$a^{ij}D_{ij}(u-\bar{\psi}) \ge 0 \quad \text{in} \quad \Omega ,$$

and by (1.2), (1.4) and (2.41)

$$D_\beta(u-\bar{\psi}) = \gamma(u-\bar{\psi}) \quad \text{on} \quad \partial\Omega$$

for some function $\gamma \geq \gamma_0$. We therefore deduce from the maximum principle that $u - \bar{\psi} \leq 0$ in Ω , and hence also that $D_\beta(u-\bar{\psi}) \leq 0$ on $\partial\Omega$. Thus

$$D_\beta(\psi-u) \geq -\rho D_\beta w \geq \sigma\rho \quad \text{on} \quad \partial\Omega \tag{2.44}$$

for some positive constant σ .

We now consider the function v on $\bar{\Omega} \times S^1$ given by

$$v(x,\gamma) = e^{\alpha(\psi-u)} D_{\gamma\gamma} u ,$$

where $\psi \in C^2(\bar{\Omega})$ is the unique convex solution of the boundary value problem (2.40), (2.41) with $\rho \in (0,1)$ fixed as above, and α is a positive constant to be chosen. As before, we need to bound v from above. Let us first assume that v attains its maximum at a point $x_0 \in \Omega$ and a direction $\xi \in S^1$. Differentiating $v = v(\cdot,\xi)$, we obtain

$$\frac{D_i v}{v} = \frac{D_{i\xi\xi} u}{D_{\xi\xi} u} + \alpha D_i(\psi-u) \tag{2.45}$$

and

$$(2.46)\qquad \frac{D_{ij}v}{v} - \frac{D_i v D_j v}{v^2} = \frac{D_{ij\xi\xi}u}{D_{\xi\xi}u} - \frac{D_{i\xi\xi}u\, D_{j\xi\xi}u}{(D_{\xi\xi}u)^2} + \alpha D_{ij}(\psi-u)\ .$$

Using (2.5) we therefore obtain

$$\begin{aligned}(2.47)\qquad e^{-\alpha(\psi-u)}F^{ij}D_{ij}v &\ge F^{ij}D_{ij\xi\xi}u - \frac{1}{D_{\xi\xi}u}F^{ij}D_{i\xi\xi}uD_{j\xi\xi}u \\ &\quad + \alpha D_{\xi\xi}uF^{ij}D_{ij}(\psi-u) \\ &\ge D_{\xi\xi}g + F^{ik}F^{jl}D_{ij\xi}uD_{kl\xi}u - \frac{1}{D_{\xi\xi}u}F^{ij}D_{i\xi\xi}uD_{j\xi\xi}u \\ &\quad + \alpha\lambda\mathcal{T}D_{\xi\xi}u - 2\alpha D_{\xi\xi}u\end{aligned}$$

by virtue of (2.43). Next, since $Dv(x_0) = 0$, we have at x_0

$$\begin{aligned}D_{\xi\xi}g &= g_{\xi\xi} + 2g_{\xi z}D_\xi u + 2g_{\xi Pi}D_{i\xi}u + g_{zz}(D_\xi u)^2 \\ &\quad + 2g_{zPi}D_\xi uD_{i\xi}u + g_{PiPj}D_{i\xi}uD_{j\xi}u \\ &\quad + g_z D_{\xi\xi}u + g_{Pi}D_{i\xi\xi}u \\ &\ge -C(1+|D^2u|^2|) - \alpha g_{Pi}D_i(\psi-u)D_{\xi\xi}u \\ &\ge -C(1+|D^2u|^2| - C\alpha D_{\xi\xi}u\ .\end{aligned}$$

Furthermore, by a rotation of coordinates we can assume that D^2u is in diagonal form at x_0 , with maximum eigenvalue $D_{\xi\xi}u$, so at x_0 we have

$$\frac{1}{D_{\xi\xi}u}F^{ij}D_{i\xi\xi}uD_{j\xi\xi}u \le F^{ik}F^{jl}D_{ij\xi}uD_{kl\xi}u\ .$$

Next, since we are in the two variable case, we have

$$(2.48)\qquad \mathcal{T} = \frac{\Delta u}{\det D^2u} \ge \frac{\Delta u}{e^{\mu}}$$

by virtue of (2.38). Using these estimates in (2.47), we therefore obtain at x_0,

$$0 \geq \alpha\lambda e^{-\mu}(D_{\xi\xi}u)^2 - C(1+|D^2u|^2) - C\alpha D_{\xi\xi}u ,$$

from which a bound for $D_{\xi\xi}u(x_0)$ follows by choosing α sufficiently large. An upper bound for $v(x_0)$ then follows.

We now consider the case that v attains its maximum at a point $x_0 \in \partial\Omega$ and a direction $\xi \in S^1$. Assuming as before that $\xi \cdot \nu \geq 0$ at x_0, we have

$$D_\xi v(x_0,\xi) \leq 0 , \quad D_\beta v(x_0,\xi) \leq 0 ,$$

from which we obtain

$$D_{\xi\xi\xi}u \leq -\alpha D_\xi(\psi-u)D_{\xi\xi}u \leq CD_{\xi\xi}u \tag{2.49}$$

and

$$D_{\xi\xi\beta}u \leq -\alpha D_\beta(\psi-u)D_{\xi\xi}u \leq -\alpha\sigma\rho D_{\xi\xi}u \tag{2.50}$$

at x_0, by virtue of (2.44). Since $D_{\xi\xi}u$ is the maximum eigenvalue of D^2u at x_0, for any direction γ, we have at x_0,

$$\left|\frac{D_\gamma f}{D_{\xi\xi}u}\right| = \frac{1}{D_{\xi\xi}u}\left|f_\gamma + f_z D_\gamma u + f_{p_i}D_{i\gamma}u\right| \leq C .$$

We therefore deduce, in the same way as in Theorem 2.1, and using the notation introduced there, that at x_0 ,

$$(2.51)\qquad \begin{aligned} D_{\tau\tau\beta}u &\le \left[a^2 - \frac{b^2 f}{(D_{\xi\xi}u)^2} - \frac{2abc}{d}\right] D_{\xi\xi\beta}u \\ &\quad + \left[\frac{2abc^2}{d} + \frac{2abdf}{(D_{\xi\xi}u)^2}\right] D_{\xi\xi\xi}u + C \\ &\le \left[a^2 - \frac{b^2 f}{(D_{\xi\xi}u)^2} - \frac{2abc}{d}\right] D_{\xi\xi\beta}u \\ &\quad + C_1\left[1 + c^2 D_{\xi\xi}u + \frac{1}{D_{\xi\xi}u}\right] \end{aligned}$$

by virtue of (2.49), since the coefficient of $D_{\xi\xi\xi}u$ is nonnegative and bounded, assuming that (2.25) is satisfied. Assuming now that

$$(2.52)\qquad |c| \le \mu_0^3/64$$

and

$$(2.53)\qquad (D_{\xi\xi}u(x_0))^2 \ge M_0 = \frac{16e^{\mu}}{\mu_0^2} ,$$

we obtain

$$(2.54)\qquad \frac{\mu_0^2}{8} \le a^2 - \frac{b^2 f}{(D_{\xi\xi}u)^2} - \frac{2abc}{d} \le 1 + \frac{\mu_0^2}{16} ,$$

and hence, assuming further that

$$(2.55) \qquad |c| \le c_0 = \min\left\{\frac{\mu_0^3}{64}\,,\ \left[\frac{\mu_0^2}{8C_1}\,\alpha\sigma\rho\right]^{1/2}\right\},$$

we finally obtain, upon using (2.50) in (2.51),

$$(2.56) \qquad D_{\tau\tau\beta}u(x_0) \le C\ .$$

A bound for $D_{\tau\tau}u(x_0)$, and then also for $D_{\xi\xi}u(x_0)$, follows exactly as before, as does the case $|c| \ge c_0$. We have therefore bounded $\sup\limits_{\bar\Omega\times S^1} v$, from which the second derivative bound (2.39) follows, so the proof of Theorem 2.3 is complete.

Once we have bounded the second derivatives, the estimate (1.15) follows from the results of [3], so using the method of continuity and a standard approximation argument to overcome the requirement $u \in C^4(\Omega) \cap C^3(\bar\Omega)$ in Theorem 2.3, we deduce Theorem 1.1.

We note that the proofs of Theorems 2.1 and 2.3 carry over to solutions $u \in W^{4,n}_{loc}(\Omega) \cap C^3(\bar\Omega)$ by virtue of the Aleksandrov maximum principles [2], Theorems 9.1, 9.6. Standard linear elliptic theory [2] ensures that under our hypotheses, convex solutions $u \in C^{2,\alpha}(\bar\Omega)$, $\alpha \in (0,1)$, are in fact in $W^{4,p}_{loc}(\Omega) \cap C^{3,\delta}(\Omega) \cap C^{2,\delta}(\bar\Omega)$ for all $p < \infty$, $\delta < 1$.

An examination of the proof of Theorem 2.1 shows that it is sufficient to assume that f is nonnegative, provided we also have $f^{1/2} \in C^{1,1}(\bar\Omega)$. The proof needs to be modified only slightly. We now write the equation (2.1) in the form

$$(2.57)\qquad F(D^2u) = (\det D^2u)^{1/2} = g(x) ,$$

and observing that F is still concave, and

$$\mathcal{T} = \operatorname{trace}[F^{ij}] \geq 1 ,$$

we deduce that $v = D_{\gamma\gamma}u + K|x|^2$ attains its maximum on $\partial\Omega$ for K sufficiently large. The arguments leading to (2.14) proceed as before with the equation written in the form (2.57), while for the final part of the argument we require only $f \in C^1(\bar{\Omega})$. By a straightforward approximation argument we can then deduce the following existence result.

THEOREM 2.4 *Let* Ω *be a* $C^{2,1}$ *uniformly convex domain in* $\mathbb{R}^2$, f *a nonnegative function with* $f^{1/2} \in C^{1,1}(\bar{\Omega})$, $\varphi \in C^{1,1}(\partial\Omega\times\mathbb{R})$ *satisfy* (1.4) *and* $\beta \in C^{1,1}(\partial\Omega,\mathbb{R}^2)$ *satisfy* (1.5) *and* (1.6). *Then the boundary value problem* (2.1), (2.2) *has a unique convex solution* $u \in C^{1,1}(\bar{\Omega})$.

The Dirichlet and Neumann problems for degenerate Monge-Ampère equations, and other fully nonlinear degenerate elliptic equations, have been treated recently by Trudinger [6] in the case that the domain Ω is a ball.

REMARKS (i) The two dimensionality is used crucially in deriving the estimate (2.21). We have also used the two dimensionality in (2.48), but this could have been avoided by using in place of v the function

$$w = e^{\alpha_1|Du|^2 + \alpha_2(\psi-u)} D_{\gamma\gamma}u$$

for suitable constants α_1 , α_2 , and modifying the proof only slightly.

(ii) Minor modifications of our arguments yield second derivative bounds for oblique derivative problems of the form

$$\det[D^2u - \sigma(x,u)] = f(x,u,Du) \quad \text{in} \quad \Omega ,$$
$$D_\beta u = \varphi(x,u) \quad \text{on} \quad \partial\Omega ,$$

where $\sigma \in C^{1,1}(\bar{\Omega}\times\mathbb{R})$ is a symmetric matrix valued function with $\sigma_z \geq 0$, Ω , f , φ , β satisfy the hypotheses of Theorem 2.3, and the solution u satisfies $D^2u \geq \sigma(x,u)$. Conditions on f , φ , β and σ ensuring a maximum modulus estimate for solutions of this problem are given in [5], Section 4.

(iii) As in [5], Theorem 4.4, we can prove the existence of a limit solution as $\epsilon \to 0$ for asymptotic problems of the form

$$\det[D^2u_\epsilon - \epsilon\sigma(x)u_\epsilon I] = f(x,Du_\epsilon) \quad \text{in} \quad \Omega ,$$
$$D_\beta u_\epsilon = \epsilon\gamma(x)u_\epsilon + \varphi(x) \quad \text{on} \quad \partial\Omega ,$$

where $\sigma \in C^{1,1}(\bar{\Omega})$, $f \in C^{1,1}(\bar{\Omega}\times\mathbb{R}^2)$ is positive, γ , $\varphi \in C^{1,1}(\partial\Omega)$, $\beta \in C^{1,1}(\partial\Omega,\mathbb{R}^2)$ satisfies (1.5), (1.6), and either $\sigma \geq \sigma_0$ in Ω or $\gamma \geq \gamma_0$ on $\partial\Omega$ for positive constants σ_0 , γ_0 , and ϵ is a positive parameter which we allow to go to zero.

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