SOME REGULARITY THEORY FOR CURVATURE VARIFOLDS

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Suppose M is a smooth n-dimensional manifold in $\mathbb{R}^N$ and for each $x \in M$ let P(x) be the matrix of the orthogonal projection of $\mathbb{R}^N$ onto T_xM. Then the second fundamental form is morally given by the following N^3-tuple

$$A = A_{ijk} = (\nabla^M P_{jk})_i \, , \tag{1}$$

where $1 \leq i,j,k \leq N$ (the usual version of the second fundamental form is easily computable from A, and conversely, see [2]).

More generally, suppose V is an n-dimensional varifold in $\mathbb{R}^N$. In other words, V is a Radon measure on $\mathbb{R}^N \times G(n,N)$, where G(n,N) is the set of all orthogonal projections of $\mathbb{R}^N$ onto some n-dimensional subspace and is naturally imbedded in $\mathbb{R}^{N^2}$. Then we say $A = [A_{ijk}]_{1 \leq i,j,k \leq N}$ is the weak second fundamental form of V if

$$\begin{aligned} &\text{(a) } A_{ijk} \in L^1_{loc}(V) \text{ for } i,j,k \leq N \\ &\text{(b) } \int \left\{ P_{ij} \frac{\partial}{\partial x_j} \phi(x,P) + A_{ijk}(x,P) \frac{\partial}{\partial P_{jk}} \phi(x,P) \right. \\ &\qquad\qquad \left. + A_{jij}\, \phi(x,P) \right\} dV(x,P) = 0 \end{aligned} \tag{2}$$

for all $\phi \in C_1(x_1,\ldots,x_N,P_{11},\ldots,P_{NN})$ which are compactly supported in $x_1,\ldots,x_n$.

A calculation using the divergence theorem (see [2]) shows that if M and A are as in (1), then A is the weak second fundamental form of the varifold v(M,1) in the sense of (2).

We have the following results.

(3) **Theorem.** A is V a.e. unique (if it exists). □

The proof is an easy test function argument ([2]).

(4) **Theorem.** Suppose $\{V_k\}_{k=1}^{\infty}$ is a sequence of integer multiplicity varifolds in a bounded open $U \subset \mathbb{R}^N$, and suppose

$$\mathbf{M}(V_k) \leq M, \qquad \int |A(V_k)|^p \, dV_k \leq K,$$

for some $p > 1$ and constants M,K.

Then there exists an integer multiplicity varifold V in U with weak second fundamental form A(V) such that

(a) $V_k \to V$ (in the sense of measures).

(b) $\int \langle A(V_k),\psi\rangle \, dV_k \to \int \langle A(V),\psi\rangle \, dV$ (for all smooth vector-valued test functions $\psi : U \times G(n,N) \to \mathbb{R}^{N^3}$).

(c) $\int |A(V)|^p \, dV \leq \liminf_{k\to\infty} \int |A(V_k)|^p \, dV_k$. □

The proof uses elementary techniques involving vector-valued measures. There are more general results but the proofs are then more involved (see [2]).

(5) **Theorem.** There exists an integer multiplicity (oriented) n-dimensional varifold V with prescribed boundary which minimises:

(a) $\int |A(V)|^p \, dV$ if $1 \leq p < n$;

(b) $\int |A(V)|^n \, dV$, provided there exists some V with the same boundary satisfying $\int |A(V^*)|^n \, dV^* < \gamma = \gamma(n)$ (here $\gamma = \gamma(n)$ is an absolute positive constant computable from the isoperimetric constant). □

The proof uses a compactness theorem for oriented integer multiplicity varifolds ([2]) together with the previous theorem.

We also have the following regularity theorem.

(6) **Theorem.** If V is an integer-multiplicity varifold in an open set $U \subset \mathbb{R}^N$, V has weak second fundamental form A, and for some $p > n$ $\int |A|^p \, dV < \infty$, then V is locally a $C^{1,1-n/p}$ (in the sense of multiple valued functions) "branched" manifold. In particular, tangent cones exist everywhere, and each such cone is a finite set of n-planes with integer multiplicities. □

For a precise statement of the above theorem and the proof, see [3].

We next consider regularity properties of local minimisers of $\int |A(V)|^n \, dV$. Details of the following results will be published elsewhere.

First observe that n is a "limit" exponent for Theorems (5) and (6). Moreover, the expression $\int |A(V)|^n \, dV$ is easily seen to be invariant under dilations.

Possible regularity of such local minimisers is limited by the following example. Let M be a complex analytic variety in $\mathbb{C}^2$. By identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ we can regard M as a 2-dimensional varifold in $\mathbb{R}^4$. Using the well-known fact that M has zero mean curvature together with an appropriate form of the Gauss-Bonnet formula, one can show that M is a local minimiser of $\int |A|^2$ However, due to the possible presence of branch points, M is no better than a $C^{1,\epsilon}$ (in the sense of multiple valued functions) "branched" manifold, for some

$\epsilon > 0$.

We next need a result for varifolds V with $\int |A(V)|^n \, dV$ small.

In the following a *flat varifold* F is a finite sum of varifolds corresponding to n-dimensional affine spaces A_i with integer multiplicities n_i. Thus

$$F = \sum_{i=1}^{Q} v(A_i, n_i) \,. \tag{7}$$

We also define the *varifold distance* $d = d_{0,R}$ between varifolds V_1, V_2 in $B_R(0)$.

$$d_{0,R}(V_1,V_2) = \sup \left\{ \left| \int \phi dV_1 - \int \phi dV_2 \right| : \phi = \phi(x,P) \text{ is } C^\infty, |\phi| \le 1, R \left| \frac{\partial \phi}{\partial x} \right|_0 \le 1, \left| \frac{\partial \phi}{\partial P} \right|_0 \le 1 \right\}.$$

It is straightforward to check that the topology induced by d is the usual topology of varifold convergence (in the sense of Radon measures).

(8) **Lemma**. Suppose V is an integer multiplicity varifold in $B_1 = B_1(0)$ with weak second fundamental form A, $\int |A|^n \, d(V \lfloor B_1) \le \epsilon$, and $\mathbf{M}(V \lfloor B_1) \le M$.

Then for each $\delta > 0$ there exists $\epsilon_0 = \epsilon_0(M,\delta) > 0$ such that $\epsilon < \epsilon_0$ implies $d_{0,9/10}(V,F) < \delta$ for some flat varifold F. □

The proof uses the compactness theorem (4) together with an easy version of the regularity theorem (6).

(9) **Remarks**. If $\int |A|^n \, d(V \lfloor B_1) < \infty$ and $\|V\|$ has a finite upper n-dimensional density bound $\theta^*(\|V\|,0) < \Lambda$ at 0, then a simple scaling argument shows that for sufficiently small ρ the varifold $\tau_{\rho\#}V$ (dilate V

about 0 by the factor ρ^{-1}) satisfies the hypotheses of the lemma with $M = \omega_n \Lambda$ and $\epsilon \le \epsilon_0$.

It is not difficult to show (using the monotonicity formula) that such a density bound holds in case n = 2. Moreover, the density bound holds at 0 for arbitrary n if $\int |H|^n \, d(V \lfloor B_\rho) \le c\rho^\epsilon$ for some $\epsilon > 0$ and all $0 < \rho < 1$, where H is the mean curvature (note that $|H| \le |A|$). This follows from the monotonicity formula. A simple covering argument then shows that if $\int |A|^n \, dV$ is finite then $||V||$ has a finite upper n-dimensional density bound except for a zero dimensional set. Although I believe that $||V||$ then has a finite upper density bound *everywhere*, I do not yet have a complete proof of this fact.

Finally, we remark that $\theta^*(||V||,0) < \infty$ and $\int |A|^n \, dV \lfloor B_1 < \infty$ implies V has varifold tangent cones at 0 and these tangent cones are flat varifolds. However, they need not be unique (see [4] for a counter-example which also applies here).

The following lemma is used to construct the comparison surface needed in the proof of Theorem (11).

(10) **Lemma.** Suppose V is an integer multiplicity varifold in $B_1 = B_1(0)$ with weak second fundamental form A, $\int |A|^n \, d(V \lfloor B_1) \le \epsilon$, and $\mathbf{M}(V \lfloor B_1) \le M$.

Then for each $\delta > 0$ there exists $\epsilon_0 = \epsilon_0(M,\delta) > 0$ such that if $\epsilon \le \epsilon_0$ then for some $r \in [1/4, 3/4]$ the following are true:

(i) $T_\xi V$ exists for $||V||$ a.e. ξ satisfying $|\xi| = r$, $\xi \in \operatorname{spt} ||V||$,

(ii) for all such ξ, the affine space $T_\xi V$ satisfies $d(T_\xi V, A) < \delta$, where A is one of the affine spaces associated with F as in (8) and (7),

(iii) if $V_r = V \cap \partial B_r$ is the varifold obtained by slicing V at radius r, then $\int |A(V_r)|^n \, dV_r \leq \epsilon$ and V_r is a $C^{1,1/n}$ "branched" manifold.

□

The proof of (i) is standard. To establish (ii) we first observe that a covering argument shows that for some $1/4 < r < 3/4$ if $|\xi| = r$ then $\int_{B_\rho(\xi)} |A|^n \leq c(N)\epsilon\rho$ for all $0 < \rho < 1/4$. A monotonicity formula for tangent plane oscillation as in [3] then gives the result.

The result (iii) follows from (ii) and an argument as in [3].

Finally we have the following theorem.

(11) **Theorem.** Suppose V is an integer multiplicity varifold which locally minimises $\int |A|^n$. Suppose $\mathbf{M}(V \lfloor B_1) \leq M$.

Then there exists $\epsilon_0 = \epsilon_0(M) > 0$ and $\alpha > 0$ such that if $\int |A|^n \, d(V \lfloor B_1) \leq \epsilon_0$ then $V \lfloor B_{1/4}$ is a $C^{1,\alpha}$ (in the sense of multiple-valued functions, see [2]) "branched manifold". □

The theorem is proved by first sewing in a comparison surface in B_r, where r is as in (10). The surface construction uses a Whitney partition of unity and is quite involved.

The Widman "hole-filling" technique [1; pp.163, 164] now shows that

$$\int |A|^n \, d(V \lfloor B_{1/4}) \leq \theta \int |A|^n \, d(V \lfloor B_1)$$

for some $\theta < 1$, $\theta = \theta(\epsilon_0, M)$.

This argument can be iterated to establish that

$$\int |A|^n \, d(V \lfloor B_\rho) \leq c\rho^\alpha \int |A|^n \, d(V \lfloor B_1)$$

for all $0 < \rho < 1/4$. (In particular, the normalised mass in $B_{4\rho}$ is controlled by an appropriate version of the monotonicity formula.)

Finally, one uses the arguments of [3] to establish the theorem.

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