REMARKS ON 2ND ORDER ELLIPTIC SYSTEMS IN LIPSCHITZ DOMAINS

G.C. *Verchota*

In this talk I will discuss a method for solving the Dirichlet and Neumann boundary value problems for 2nd order strongly elliptic systems with **real** coefficients in Lipschitz domains. As it is work in progress only the most elementary case, that of two equations in two unknowns in planar domains, will be presented. However, as I hope to show, there is at least no **apparent** impediment to the generalizing of our ideas to general 2nd order systems in higher dimensions.

Consider the 2nd order differential system equation in two variables $X = (X_1, X_2) \in \mathbb{R}^2$ for m unknowns $\vec{u} = (u^1, \ldots, u^m)$ given by

$$(1) \qquad D_i a_{ij}^{rs} D_j u^s(X) = 0 \quad , \quad 1 \leq r \leq m \, .$$

Here we use summation convention, $1 \leq i,j \leq 2$, $1 \leq s \leq m$ and D_i denotes $\frac{\partial}{\partial X_i}$. The coefficients a_{ij}^{rs} are **constant** and satisfy the symmetry **condition**:

$$(2) \qquad a_{ij}^{rs} = a_{ji}^{sr} \, .$$

It is convenient to think of the a_{ij}^{rs} as forming an $m \times m$ matrix with entries, A^{rs}; for each fixed r and s, A^{rs} is a 2×2 matrix in i and j . Then r and s denote the **row** and **column** respectively of the $m \times m$ matrix and i and j the row and column respectively of the 2×2 matrices.

We will look for solutions to (1) in a domain Ω given as the area above the graph of a compactly supported Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$. This expedient simplifies some of the algebra but creates difficulties at infinity. These difficulties are not of essence, however, and will be ignored throughout this presentation.

We will be interested in two types of boundary conditions on $\vec{u}$, Dirichlet and Neumann type : letting $\vec{g} \in L^2(\partial\Omega)$

(D) $$\vec{u}|_{\partial\Omega} = \vec{g}$$

(N) $$\frac{\partial \vec{u}}{\partial \nu} \equiv (N^i a_{ij}^{1s} D_j u^s, \dots, N^i a_{ij}^{ms} D_j u^s) = \vec{g} \, .$$

In condition (N) we have the **conormal derivative** of $\vec{u}$ at the boundary. Here $N = (N^1, N^2)$ denotes the **outer** unit normal at points of $\partial\Omega$. For condition (N), $\vec{g}$ must satisfy certain compatability conditions. In both (D) and (N) the boundary values are taken in the sense of point-wise nontangential convergence at almost every (a.e.) point of $\partial\Omega$ with respect to **surface measure**, dQ. Note that if we were interested in only solving the Dirichlet problem we could **with no loss of generality** impose the additional symmetry condition $a_{ij}^{rs} = a_{ji}^{rs}$. For replacing a_{ij}^{rs} with $(a_{ij}^{rs} + a_{ji}^{rs})/2$ effects neither (1) nor (D), but it **does** effect (N). Thus in general $a_{ij}^{rs} \neq a_{ji}^{rs}$. The meaning of (2) is that the matrices on the diagonal (of the $m \times m$ matrix) are self adjoint while for the off diagonal ones the transpose of A^{rs} is A^{sr}.

In order to solve (D) and (N) in the sense of nontangential convergence at the boundary we need to establish estimates on the **nontangential maximal functions of solutions**, $\vec{u}$,

(3D) $$\|\vec{u}^*\|_{L^2(\partial\Omega)} \leq C\ \|\vec{g}\|_{L^2(\partial\Omega)}$$

(3N) $$\|\nabla\vec{u}^*\|_{L^2(\partial\Omega)} \leq C\ \|\vec{g}\|_{L^2(\partial\Omega)}$$

for (D) and (N) respectively. Here $\nabla\vec{u}$ has $2m$ components being the gradient of each component of $\vec{u}$.

Inequalities (3) follow from properties of layer potentials together with a theorem inverting the boundary potentials as operators on $L^2(\partial\Omega)$. This will not be done here, but the main idea can be found in [V]. Our technique allows us to reduce the estimates (3) to apriori estimates of the same type with $\vec{u}|_{\partial\Omega}$ and $\nabla\vec{u}|_{\partial\Omega}$ replacing $\vec{u}^*$ and $\nabla\vec{u}^*$ respectively. In order to obtain these modified estimates we need the key estimates

(4) $$\|\nabla\vec{u}\|_{L^2(\partial\Omega)} \leq C\|\frac{\partial\vec{u}}{\partial\nu}\|_{L^2(\partial\Omega)}$$

(5) $$\|\frac{\partial\vec{u}}{\partial\nu}\|_{L^2(\partial\Omega)} \leq C\|\vec{u}\|_{L^2_1(\partial\Omega)} .$$

Here $L^2_1(\partial\Omega)$ is the Sobolov space of $L^2(\partial\Omega)$ functions with first derivatives in $L^2(\partial\Omega)$. Both (4) and (5) follow from what are known as **Rellich identities** if the a^{rs}_{ij} meet certain conditions which we will now give. Actually (5) is in general easier than (4) and does not require the full force of our conditions. Therefore the remainder of this presentation is devoted to establishing (4).

The first condition we will impose on a_{ij}^{rs} is that of **strong ellipticity**, i.e. there exists a constant $C > 0$ such that for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$

(6) $$a_{ij}^{rs}\xi_i\xi_j\eta_r\eta_s \geq C|\xi|^2|\eta|^2 \ .$$

Note that (6) implies that each of the matrices A^{rr}, $1 \leq r \leq m$, is positive definite, i.e. each $\nabla\cdot A^{rr}\nabla$ is an elliptic divergence form operator.

The next condition we impose is that of **semi-positive definiteness**, i.e. for any m vectors $\xi^1, \ldots, \xi^m \in \mathbb{R}^2$

(7) $$a_{ij}^{rs}\xi_i^r\xi_j^s \geq 0 \ .$$

It is possible for a_{ij}^{rs} to satisfy (6) but **not** satisfy (7).

EXAMPLE #1

$$a_{ij}^{rs} : \begin{matrix} \begin{bmatrix} 1/\epsilon & 0 \\ 0 & \epsilon \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{bmatrix} \end{matrix} \ , \ \epsilon > 0 \ .$$

For (6) to hold it is necessary and sufficient for the matrix $(\xi \cdot A^{rs}\xi)$ to have determinant > 0 for $\xi \neq 0$, i.e. $(\xi_1)^4 + (\xi_2)^4 + (\epsilon^2 + \frac{1}{\epsilon^2}) \cdot (\xi_1\xi_2)^2 > 4(\xi_1\xi_2)^2$. For this to be true we only need $\epsilon^2 + \frac{1}{\epsilon^2} > 2$, i.e. $\epsilon^2 \neq 1$. However for $\xi^1 = (0,-1)$ and $\xi^2 = (1,0)$ $a^{rs}_{ij}\xi^r_i\xi^s_j = 2\epsilon - 2$ and (7) fails for $\epsilon < 1$ (and fails for $\epsilon > 1$ also).

A simple integration by parts using the symmetry condition (2) establishes the Rellich identity for solutions $\vec{u}$:

$$\int_{\partial\Omega} (-N^2) D_i u^r a^{rs}_{ij} D_j u^s dQ = 2\int_{\partial\Omega} (-D_2\vec{u}) \cdot \frac{\partial\vec{u}}{\partial\nu}\, dQ \,. \tag{8}$$

Recall $\frac{\partial\vec{u}}{\partial\nu}$ is the conormal derivative from (N) above. The point is that (4) follows from the Schwarz inequality applied to the right side of (8) together with the estimate

$$\int_{\partial\Omega} |\nabla\vec{u}|^2 dQ \equiv \int_{\partial\Omega} D_i u^r D_i u^r dQ \leq C\int_{\partial\Omega} D_i u^r a^{rs}_{ij} D_j u^s dQ \,. \tag{9}$$

(Recall that $-N^2 > 0$ is bounded uniformly from below away from zero since $\partial\Omega$ is Lipschitz.) Our assumption of semi-positive definiteness means that the **integrand** on the right side of (9) (i.e. on the left side of (8)) is **nonnegative.** Observe that if we had positive **definiteness** i.e.

$$a^{rs}_{ij}\xi^r_i\xi^s_j > 0 \,, \tag{10}$$

in place of (7) then the existence of a constant C independent of $\vec{u}$ would follow immediately for (9). The C would depend only on the Lipschitz norm of φ and the minimum of the expression in (10) when $\sum_{r=1}^{m} |\xi^r| = 1$. This gives us our first result:

THEOREM 1 *The problems* (1), (D), (3D) *and* (1), (N), (3N) *are solvable under conditions* (2) *symmetry,* (6) *strong ellipticity and* (10) *positive definiteness.*

Before proceeding observe that the conditions (2), (6), (7) and (10) are **invariant under conjugation** of a_{ij}^{rs} by **invertible** matrices

$$B = (b_{jk})_{1\leq j,k\leq 2}$$

or

$$B = (b_{st})_{1\leq s,t\leq m}$$

though the constants may change depending on the norm of B^{-1}. That is putting either

$$\tilde{a}_{hk}^{rs} = a_{ij}^{rs} b_{ih} b_{jk}$$

(11) or

$$\tilde{a}_{ij}^{qt} = a_{ij}^{rs} b_{rq} b_{st}$$

$\tilde{a}$ may replace a in (2), (6), (7) and (10). The first conjugation corresponds to a linear charge of variables in $\mathbb{R}^2$. The second conjugation corresponds to rewriting our system in terms of new unknowns $\vec{v} = B^{-1}\vec{u}$ and solving the appropriate problems for data $B\vec{g}$. The estimates (3) may be obtained from those for $\vec{v}$.

In order to solve the problems, i.e. obtain estimate (9), when (10) positive definiteness fails we need further arguments. To illustrate these we specialize to the case $m = 2$.

We first claim that in order to solve the Dirichlet problem it suffices to assume merely (6) strong ellipticity. As noted above symmetrizing the matrices A^{12} and A^{21} does not effect the Dirichlet problem. More generally letting

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

be the standard rotation by $-\pi/2$ in the plane and letting γ be any real number, A^{12} and A^{21} may be replaced by $A^{12} + \gamma R$ and $A^{21} - \gamma R$ respectively without effecting (1) the equations, (2) symmetry, or (6) strong ellipticity. The key estimates (4) and (5) will yield the same Dirichlet problem no matter how we have changed the Neumann boundary values (N) by this procedure. In addition to (N) the other thing that might be altered by the matrices γR is positive definiteness (10). Thus the claim will follow from Theorem 1 and the following lemma.

LEMMA 1 *Given a system with coefficients* a_{ij}^{rs} , $1 \leq i,j,r,s \leq 2$, *satisfying* (2) *and* (6) *there exists a real number* γ *such that*

$$\begin{bmatrix} A^{11} & A^{12}+\gamma R \\ A^{21}-\gamma R & A^{22} \end{bmatrix}$$

are the coefficients of a system satisfying (10) *positive definiteness in addition to* (2) *and* (6).

PROOF We will first apply conjugations (11) in order to put our system into a standard form. Subscripts ij or rs will denote the indices on which the conjugations are applied. First apply

$$\begin{bmatrix} (\det A^{11})^{-1/4} & 0 \\ 0 & (\det A^{22})^{-1/4} \end{bmatrix}_{rs}$$

so that we may take $\det A^{11} = \det A^{22} = 1$. Next apply a unitary matrix U_{ij} so that A^{11} is diagonalized to be

$$\begin{bmatrix} 1/\epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \equiv E$$

for some $\epsilon > 0$. Next apply $E_{ij}^{-1/2}$ so that A^{11} is the identity I . Next apply another unitary matrix U_{ij} so that $A^{22} = E$ for some $\epsilon > 0$ and $A^{11} = I$. Next apply $E_{ij}^{-1/4}$ so that for some other $\epsilon > 0$ we have $A^{11} = E$ and $A^{22} = E^{-1}$. Finally make an initial choice of γ so that we may take $A^{12} = A^{21} \equiv A$ where A is self adjoint. Our original system is thus equivalent under conjugations and adding

$$\gamma \begin{bmatrix} 0 & R \\ -R & 0 \end{bmatrix}$$

to the standard system

$$\begin{bmatrix} E & A \\ A & E^{-1} \end{bmatrix}$$

for some $\epsilon > 0$ and $A = A^T$.

Strong ellipticity (6) which continues to hold for our system may be written

$$(12) \qquad (\xi\cdot E\xi)(\eta_1)^2 + 2\xi\cdot A\xi\eta_1\eta_2 + (\xi\cdot E^{-1}\xi)(\eta_2)^2 > 0$$

for all $\xi,\eta \in \mathbb{R}^2$ where $\cdot$ denotes inner product. This is true iff for all $\xi \in \mathbb{R}^2$

$$(13) \qquad (\xi\cdot E\xi)(\xi\cdot E^{-1}\xi) > (\xi\cdot A\xi)^2$$

i.e. iff the discriminant of the quadratic expression in η_1/η_2 or η_2/η_1 from (12) is negative. Of course A may always be replaced in (13) by $A + \gamma R$ and (13) still holds.

Positive definiteness (10) for some γ will read

$$(14) \qquad \xi\cdot E\xi + 2\xi\cdot(A+\gamma R)\eta + \eta\cdot E^{-1}\eta > 0$$

for all $\xi,\eta \in \mathbb{R}^2$. This is equivalent to showing for all $\xi,\eta \in \mathbb{R}^2$

$$(15) \qquad (\xi\cdot E\xi)(\eta\cdot E^{-1}\eta) > (\xi\cdot(A+\gamma R)\eta)^2 .$$

Thus (13) is a special case of (15). We need to show that (13) implies the existence of a γ such that (15) holds.

Now (15) is merely the statement that the matrix

$$E^{-1/2}(A+\gamma R)E^{1/2} \equiv M_\gamma$$

has operator norm on $\mathbb{R}^2$ under the standard Euclidean metric <u>less</u> than 1. In this context strong ellipticity (13) may be written

$$(16) \qquad |f(\xi)\cdot E^{-1/2}AE^{1/2}\xi| < 1 \quad \text{for all} \quad |\xi| = 1$$

where

$$f(\xi) \equiv E\xi/|E\xi| \ .$$

The function f is best thought of as a **nonlinear** rotation of the circle. We will show that (16) implies

$$(17) \qquad \|M_\gamma\| < 1 \quad \text{for some} \quad \gamma \ .$$

Denote the set of 2×2 symmetric matrices, A, satisfying (16) (i.e. (12) or (13)) by $\mathcal{E}$. Let $\mathcal{D}$ denote that subset of $\mathcal{E}$ for which (17) (i.e. (14) or (15)) may be realized. Note that if A_0 and A_1 are in $\mathcal{E}$ then so is $A_t = (1-t)A_0 + tA_1$, $0 \leq t \leq 1$, as (12) shows. In particular $\mathcal{E}$ is a bounded, open, convex subset of the 2×2 symmetric matrices under say the operator norm topology. $\mathcal{D}$ is, clearly nonempty and open (by (17) say) in the induced topology of $\mathcal{E}$. In order to show $\mathcal{D} = \mathcal{E}$ (i.e. (16) implies (17)) it suffices to show that $\mathcal{D}$ is a closed subset of $\mathcal{E}$ under the induced topology of $\mathcal{E}$.

Suppose there is a sequence $\{A_j\} \subset \mathcal{D}$ with $\{\gamma_j\} \subset \mathbb{R}$ such that for $M_j = E^{-1/2}(A_j+\gamma_j R)E^{1/2}$ we have $\|M_j\| < 1$ and $\|A_j - A\| \to 0$ for some A in $\mathcal{E}$. The γ_j must be contained in a bounded set whence there is a subsequence that converges to say γ_0 . Put $M = E^{-1/2}(A+\gamma_0 R)E^{1/2}$. We need only examine the case $\|M\| = 1$ and show that there is a γ such that $\|M+\gamma E^{-1/2}RE^{1/2}\| < 1$ whence A is in $\mathcal{D}$. This will establish the lemma. **Fix** a vector $|\eta| = 1$ so that $|M\eta| = 1$. There are now two possibilities.

(i) $|MR\eta| \equiv a < 1$ or

(ii) $|MR\eta| = 1$.

Note that $1 = \eta \cdot M^T M\eta$ and $\|M\| = 1$ imply that η is actually an eigenvector of $M^T M$ whence

(18) $$M\eta \cdot MR\eta = 0 .$$

Consider now the first case. Ellipticity (16) shows that $f(\xi)$ and $M(\xi)$ cannot be colinear for all $|\xi| = 1$ in a small enough neighbourhood about η and $-\eta$. Since for all ξ $f(\xi) \cdot E^{-1/2}RE^{1/2}\xi = 0$ we conclude that $E^{-1/2}RE^{1/2}\xi \cdot M(\xi)$ is **uniformly** either strictly positive or strictly negative for all ξ in a small enough neighbourhood of η . For $|\xi| = 1$ **outside** this neighbourhood (i) and (18) show that $|M\xi|$ is uniformly bounded by a constant less than 1 . This will continue to be true for $|(M+\gamma E^{-1/2}RE^{1/2})\xi|$ for $|\xi| = 1$ **outside** this neighbourhood by choosing $|\gamma| > 0$ small. But by the uniform positivity or negativity of $E^{-1/2}RE^{1/2}\xi\cdot M(\xi)$ for ξ **inside** this neighbourhood it will also be true for $|\xi| = 1$ inside by choosing the sign of γ correctly.

Now consider the second case (ii). By (18) M is unitary. Assume first that $\det M = -1$. Then M is self adjoint. Thus letting e_1 and e_2 be the standard basis vectors for $\mathbb{R}^2$ and recalling that $Re_2 = e_1$ we have

$$\text{(19)} \qquad Re_2\cdot Me_2 = Me_1\cdot e_2 = -Me_1\cdot Re_1 \ .$$

Put $F(\xi) = Rf(\xi)\cdot M\xi$. Since $f(e_1) = e_1$ and $f(e_2) = e_2$ we see from (19) that $F(e_1) = -F(e_2)$, i.e. F changes sign. By continuity then $F(\xi) = 0$ for some $|\xi| = 1$. But this implies that $|f(\xi)\cdot M\xi| = 1$ contradicting ellipticity (16). Thus if M is unitary then $\det M = 1$, i.e. it is a rotation. Put

$$M = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} .$$

Now consider any vector $|\xi| = 1$ such that $(\xi_1)^2 = \frac{\epsilon^2-\epsilon^4}{1-\epsilon^4}$ and $(\xi_2)^2 = \frac{1-\epsilon^2}{1-\epsilon^4}$. A calculation then shows that $\xi\cdot f(\xi) = \frac{2}{\epsilon+\frac{1}{\epsilon}}$. (Note that this last quantity is always less than 1 unless $\epsilon = 1$, i.e. $E = I$ in which case the lemma follows immediately.) Although we do not now need this fact it is significant that this quantity is the **minimum** value of $\xi \cdot f(\xi)$, $|\xi| = 1$. The point is that for every angle θ such that $|\cos\theta| \geq \frac{2}{\epsilon + \frac{1}{\epsilon}}$ there is a vector $\xi(\theta)$ that is rotated by f by precisely θ . In other words if $|\cos\psi| \geq \frac{2}{\epsilon+\frac{1}{\epsilon}}$ then there is a $|\xi| = 1$ such that $f(\xi)\cdot M(\xi) = 1$ contradicting (16) ellipticity again. Thus $|\cos\psi| < \frac{2}{\epsilon+\frac{1}{\epsilon}}$.

We now claim that there is a γ such that $|(M+\gamma E^{-1/2}RE^{1/2})\xi| < 1$ for all $|\xi| = 1$. This will establish the lemma. A calculation shows that

$$\frac{1}{2} \frac{d}{d\gamma}\bigg|_{\gamma=0} |(M+\gamma E^{-1/2}RE^{1/2})\xi|^2 = \frac{\sin\psi}{\epsilon} (\xi_1)^2 + 2\left[\frac{\epsilon-\frac{1}{\epsilon}}{2}\right] \cos\psi \xi_1 \xi_2$$
$$+ \epsilon \sin\psi \ (\xi_2)^2 .$$

A quarter of the discriminant of this quadratic is

$$\left[\frac{\epsilon-\frac{1}{\epsilon}}{2}\right]^2 \cos^2\psi - 1 < 0 .$$

Thus depending on the sign of $\sin\psi$ the above derivative is either uniformly positive or negative and the claim follows.//

We have thus established for $m = 2$ the following theorem for the Dirichlet problem.

THEOREM 2 *Problem* (1), (D), (3D) *is solvable under conditions* (2) *symmetry and* (6) *strong ellipticity.*

Our goal is now to solve the Neumann problem under condition (7) of semi-positive definiteness

Example #2

$$a_{ij}^{rs} : \quad \begin{matrix} \begin{bmatrix} 1/\epsilon & 0 \\ 0 & \epsilon \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1/\epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \end{matrix}, \quad \epsilon > 0 .$$

This system is strongly elliptic and semi-positive definite. Any harmonic function h provides a solution to (1) by putting $u^1 = D_1 h$ and $u^2 = D_2 h$. However the conormal derivative of such a solution is always zero. Uniqueness fails for boundary conditions (N).

The difficulty with example #2 is that the 4×4 matrix a_{ij}^{rs} has a **two** dimensional null space. We therefore are left to consider those strongly elliptic, semi-positive definite systems with **one** dimensional null spaces for a_{ij}^{rs} .

A well known example of such a system is given by the system of elastostatics. The matrix which corresponds to the Neumann problem of most interest, the **traction** boundary value problem, is

$$\begin{matrix} \begin{bmatrix} 2\mu+\lambda & 0 \\ 0 & \mu \end{bmatrix} & \begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & \mu \\ \lambda & 0 \end{bmatrix} & \begin{bmatrix} \mu & 0 \\ 0 & 2\mu+\lambda \end{bmatrix} \end{matrix}$$

where λ and μ , $\mu > 0$ are called the Lamè constants.

Let now a_{ij}^{rs} be strongly elliptic, semi-positive definite with **one dimensional null space**. Without loss of generality we may take the null eigenvector to be $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$. To see this assume that the null vector is given by $\begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$, $\omega^j \in \mathbb{R}^2$. By ellipticity the ω^j are linearly independent. Now conjugate a_{ij}^{rs} in the indices r,s as in (11) with the matrix

$$\begin{bmatrix} \omega_2^1 & -\omega_1^1 \\ \omega_2^2 & -\omega_1^2 \end{bmatrix}$$

to get an equivalent system. Thus we may assume $\omega^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\omega^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is the standard null vector.

Define b_{ij}^{rs} to be

$$\begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} .$$

b_{ij}^{rs} is semi-positive definite with null space containing the null space of a_{ij}^{rs}. Thus there is a smallest $\gamma > 0$ such that $a_{ij}^{rs} - \gamma b_{ij}^{rs}$ is semi-positive definite with at least **two dimensional** null space containing the original null space of a_{ij}^{rs}.

Using the remarks about the matrix R given before Lemma 1 we are able to write our system of equations (1) in the standard form

(20) $$D_i a_{ij}^{rs} D_j u^s = \gamma \Delta u^r + D_i(a_{ij}^{rs} - \gamma b_{ij}^{rs}) D_j u^s = 0$$

where $\gamma > 0$ and $a_{ij}^{rs} - \gamma b_{ij}^{rs}$ is as described above.

LEMMA 2 *Let* $|\xi^1|^2 + |\xi^2|^2 = 1$ *and* $\xi_i^s \omega_i^s = 0$ *where* ω^s *form the standard null vector for* a_{ij}^{rs} *as described above. Then for any two-vector of functions* $\vec{v}$ *in* $\mathbb{R}^2$ *we have the* ***point-wise*** *inequality*

$$|\xi_j^s D_j v^s(Q)|^2 \le C D_i v^r(Q) a_{ij}^{rs} D_j v^s(Q)$$

where C *is independent of* $\vec{v}$ *and* ξ^s .

PROOF The vector $\begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix}$ in $\mathbb{R}^4$ belongs to the 3-dimensional subspace on which a_{ij}^{rs} is strictly positive definite.//

The next lemma states that estimate (9) holds for harmonic vectors.

LEMMA 3 *Let* $\vec{h}$ *be any* ***harmonic*** *2-vector in* $\Omega \subset \mathbb{R}^2$ *vanishing at infinity. Then*

$$\int_{\partial\Omega} |\nabla\vec{h}(Q)|^2 dQ \le C \int_{\partial\Omega} D_i h^r(Q) a_{ij}^{rs} D_j h^s(Q) dQ$$

where C *depends only on the Lipschitz constant for* $\partial\Omega$ *and* a_{ij}^{rs} .

PROOF
$$\begin{aligned}\int_{\partial\Omega} |\nabla\vec{h}|^2 &\le C \int_{\partial\Omega} |D_2\vec{h}|^2 \le C \int_{\partial\Omega} |D_2 h^1 + D_1 h^2|^2 + |\nabla h^2|^2 \\ &\le C \int_{\partial\Omega} |D_2 h^1 + D_1 h^2|^2 + |D_2 h^2|^2 \\ &\le C \int_{\partial\Omega} D_i h^r a_{ij}^{rs} D_j h^s .\end{aligned}$$

The first and third inequalities follow variously from well known theorems on the boundary values of conjugate harmonic functions or from the Rellich identities for harmonic functions. The second is the triangle inequality. The fourth is Lemma 2.//

By symmetry and semi-positive definiteness the matrix $a_{ij}^{rs} - \gamma b_{ij}^{rs}$ from (20) may be written as $U^T \Lambda \Lambda U$ where U is a unitary matrix and

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix}$$

with $\lambda_j \geq 0$. Let us rewrite (12) as

$$\gamma \Delta \mathbf{u}^r + D_i C_{ji}^{sr} C_{jk}^{st} D_k \mathbf{u}^t = 0 \tag{21}$$

where C_{jk}^{st} is ΛU and C_{ji}^{sr} is its adjoint. Note that $C_{jk}^{2t} \equiv 0$. Define now

$$\mathbf{v}_j^s = C_{jk}^{st} D_k \mathbf{u}^t \ . \tag{22}$$

We have $\mathbf{v}_j^2 \equiv 0$. The two functions $\mathbf{v}_j^1$ we call **auxillary functions**. When $\lambda_1 = \lambda_2 = 0$ our system a_{ij}^{rs} is simply a vector version of Laplace's equation. An example of $\lambda_1 > 0$ and $\lambda_2 = 0$ is the elastostatic system. Since these systems are now well understood on Lipschitz domains let us confine ourselves to $\lambda_1 > 0$, $\lambda_2 > 0$ so that our auxillary functions are in general both nontrivial.

LEMMA 4 *Let* v_j^s *be as in* (22). *Then we have the point-wise inequality*

$$|v_j^s(Q)|^2 \le CD_iu^r(Q)a_{ij}^{rs}D_ju^s(Q)$$

where C *depends only on* a_{ij}^{rs} .

PROOF The null space of C_{jk}^{st} (i.e. ΛU) contains that of a_{ij}^{rs} . Apply Lemma 2.//

Let Γ be the logarithmic fundamental solution for the Laplacian in $\mathbb{R}^2$. If F is defined in Ω define

$$\Gamma(F)(X) = \int_\Omega \Gamma(X-Y)F(Y)dY .$$

LEMMA 5 *Let* $\vec{u}$ *satisfy* (1) *in* Ω . *Let* v_j^s *be as in* (22). *Then*

$$\int_{\partial\Omega} |\nabla\vec{u}(Q)|^2 dQ \le C \int_{\partial\Omega} D_iu^r(Q)a_{ij}^{rs}D_ju^s(Q) + \sum_{j=1}^{2} |\nabla\nabla\Gamma(v_j^1)(Q)|^2 dQ$$

where C *depends only on* a_{ij}^{rs} *and the Lipschitz constant for* $\partial\Omega$.

PROOF Let $h^r = u^r + \gamma^{-1}D_iC_{ji}^{sr}\Gamma(v_j^s)$. By (21) $\vec{h}$ is harmonic. It follows that

$$\begin{aligned}\int_{\partial\Omega} |\nabla\vec{u}|^2 &\le C \int_{\partial\Omega} |\nabla\vec{h}|^2 + \sum_j |\nabla\nabla\Gamma(v_j^1)|^2 \\ &\le C \int_{\partial\Omega} D_ih^r a_{ij}^{rs} D_jh^s + \sum_j |\nabla\nabla\Gamma(v_j^1)|^2 \\ &\le C \int_{\partial\Omega} D_iu^r a_{ij}^{rs} D_ju^s + \sum_j |\nabla\nabla\Gamma(v_j^1)|^2 .\end{aligned}$$

The first inequality is the triangle inequality. The second is Lemma 3. The third follows from the definiteness of a^{rs}_{ij} and the triangle inequality.//

Lemma 5 is almost inequality (9). We need to show that integrated on the boundary, two derivatives of the Newtonian potential of our auxillary functions are bounded by the auxillary functions. This will follow ultimately from the theorem of Coifman, McIntosh, Meyer and the fact that the auxillary functions are themselves solutions to a system for which we can solve the **Dirichlet** problem. This sytem we call the **auxillary system**.

We have by (21) and (22)

$$\gamma \Delta \mathbf{u}^r + D_i C^{sr}_{ji} \mathbf{v}^s_j = 0 \ .$$

Thus taking derivatives, operating with the matrix C^{qr}_{gh} and using (22) again

$$\gamma \Delta \mathbf{v}^q_g + D_h C^{qr}_{gh} C^{sr}_{ji} D_i \mathbf{v}^s_j = 0 \ . \tag{23}$$

Recalling that the $\mathbf{v}^2_j$ are identically zero we are left to consider the second order system of two equations in two unknowns

$$\gamma \Delta \mathbf{w}^g + D_h C^{1r}_{gh} C^{1r}_{ji} D_i \mathbf{w}^j = 0 \ . \tag{24}$$

Since $C^{1r}_{gh} C^{1r}_{ji} = C^{1r}_{ji} C^{1r}_{gh}$ the system (24) meets the symmetry conditions.

Strong ellipticity follows from

$$\gamma|\xi|^2|\eta|^2 + C^{1r}_{gh}\xi_h\eta_g C^{1r}_{ji}\xi_i\eta_j > 0$$

for non zero ξ and η in $\mathbb{R}^2$. The system (24) is the auxillary system for the original system (1). By Theorem 2 we can solve the Dirichlet problem for (24). In particular since by (23) (v^1_1, v^1_2) is a solution for (24) we have estimate (3D) for our auxillary functions

$$\int_{\partial\Omega}(v_1^{1*})^2 + (v_2^{1*})^2 dQ \leq C\int_{\partial\Omega}(v_1^1)^2 + (v_2^1)^2 dQ .$$

Consider now a strongly elliptic, symmetric, positive definite system in Ω

(25) $$D_i\alpha^{rs}_{ij}D_j w^s = 0 .$$

For any $X \in \Omega$ put

$$\vec{W}(X) = -\int_{X_2}^{+\infty}\vec{w}(X_1, t)dt .$$

Both $\vec{W}$ and $D_1\vec{W}$ are also solutions to (25). By positive definiteness (8) and (9) can be obtained for $\vec{W}$ so that in particular using the Schwarz inequality on the right side of (8) we can get

(26) $$\int_{\partial\Omega}|\nabla\vec{W}|^2 \leq C\int_{\partial\Omega}|\vec{w}|^2 .$$

Also note the following (see condition (N))

$$\left[\frac{\partial}{\partial\nu}\vec{w}\right]^r \equiv N^i\alpha^{rs}_{ij}D_j w^s = N^i D_2 \alpha^{rs}_{ij} D_j W^s = (N^1D_2 - N^2D_1)\alpha^{rs}_{1j}D_jW^s \ . \tag{27}$$

The last equality follows because W is a solution. The derivatives in parenthesis are **tangential** to $\partial\Omega$.

Let $\Gamma_\alpha = (\Gamma^{st}_\alpha)_{s,t\,=\,1,2}$ be the fundamental solution matrix for system (25) Γ_α has the property that

$$\frac{\partial}{\partial X_i}\alpha^{rs}_{ij}\frac{\partial}{\partial X_j}\int_{\mathbb{R}^2}\Gamma^{st}_\alpha(X-Y)F^t(Y)dY = F^r(X)$$

for two-vectors of functions $\vec{F}$. Now the Newtonian potential of w^u , $u = 1,2$ can be written for $Z \in \Omega$

$$\begin{aligned}\Gamma(w^u)(Z) &= \int_\Omega \Gamma(Z-X)w^u(X)dX \\ &= \int_\Omega w^r(X)\frac{\partial}{\partial X_i}\ \alpha^{rs}_{ij}\frac{\partial}{\partial X_j}\int_{\mathbb{R}^n}\Gamma^{st}_\alpha(Z-X-Y)\ \delta_{tu}\ \Gamma(Y)\ dY\ dX \\ &= \int_{\partial\Omega} w^r(Q)N^i_Q\ \alpha^{rs}_{ij}\frac{\partial}{\partial Q_j}\ \Gamma^{su}_\alpha(\Gamma)(Z-Q)\ dQ \\ &= -\int_{\partial\Omega} N^j_Q\ \alpha^{rs}_{ij}\ D_i w^r(Q)\Gamma^{su}_\alpha(\Gamma)(Z-Q)\ dQ\end{aligned} \tag{28}$$

where δ_{tu} is the Kronecker delta. In the last integral the conormal derivative of $\vec{w}$ can be by (27) transfered to $\Gamma^{su}_\alpha(\Gamma)$ but with the result that $\nabla\vec{W}$ replaces $\vec{w}$. This however is of no real consequence since we have inequality (26). If we now take any two derivatives in Z of $\Gamma(w^u)$ and let Z approach $\partial\Omega$ nontangentially we find, with three derivatives on the potentials $\Gamma^{su}_\alpha(\Gamma)$, that we have produced bounded singular integrals of the type studied by Coifman, McIntosh, Meyer acting on $\nabla\vec{W}$. We have shown that on the boundary two derivatives of the Newtonian potential of any solution to (25) are bounded in L^2 by the Dirichlet boundary values of the solution. Returning to our auxillary functions v_j^1 considered as solutions to the Dirichlet problem for system (24) (which by Lemma 1 can be considered positive definite) we have proven the following lemma.

LEMMA 6 *Let* v_j^s *be as in* (22). Then

$$\int_{\partial\Omega} |\nabla\nabla\Gamma(v_j^s)(Q)|^2 dQ \le C \int_{\partial\Omega} \sum_{j=1}^{2} |v_j^1(Q)|^2 \, dQ$$

where C *depends only on* a_{ij}^{rs} *and the Lipschitz constant for* $\partial\Omega$.

For $m=2$ we have the following theorem.

THEOREM 3 *Problem* (1), (N), (3N) *is solvable under conditions* (2) *symmetry,* (6) *strong ellipticity, and* (7) *semi-positive definiteness provided that the null space of the system matrix is one dimensional.*

PROOF Inequality (9) follows by lemmas 5, 6, and 4.//

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Department of Mathematics
University of Illinois at Chicago
Chicago, Illinois
USA 60680