COMMUTATOR THEORY
AND
PARTIAL DIFFERENTIAL OPERATORS
ON
HILBERT SPACE

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1. Introduction

Commutator theory has its origins in constructive quantum field theory. It was initially developed by Glimm and Jaffe as a method of establishing self-adjointness of quantum fields and model Hamiltonians, but it has proved useful for a variety of other problems in field theory, quantum mechanics, and Lie group theory. We will describe the basic results of the theory and illustrate them with applications to first and second order partial differential operators.

The basic ideas of commutator theory and perturbation theory are very similar. One attempts to derive information about a complex system by comparison with a simpler reference system. The nature of the comparison is different, however, in the two theories. Perturbation theory applies when the difference between the systems is small. Commutator theory only requires the complex system to be relatively smooth with respect to the reference system. No small parameters enter in the latter theory.

In order to be more precise suppose H is a self-adjoint operator on a Hilbert space h with domain $D(H)$ and with C^∞ - elements $h_\infty = \bigcap_{n\geq 1} D(H^n)$, and further suppose that K is a symmetric operator from h_∞ into h . Then the simplest theorem of perturbation theory states that K is essentially self-adjoint whenever

$$\|(K-H)a\| \leq k\|Ha\|+\ell\|a\|, \; a\in h_\infty \quad ,$$

for some k , $\ell \geq 0$ with $k < 1$ i.e., the difference $K - H$ between K and H is small in comparison with H . In contrast the simplest commutator theorem states that K is essentially self-adjoint whenever $Kh_\infty \subseteq D(H)$ and the commutator $(\text{ad } H)(K) = KH - HK$ satisfies $\|(\text{ad } H)(K)a\| \leq k'\|Ha\| + \ell'\|a\|$, $a\in h_\infty$, for some k' , $\ell' \geq 0$. The bound on the commutator can be viewed as a smoothness condition as we will see in the subsequent discussion of partial differential operators. It reflects smoothness of K with respect to H , and since $(\text{ad } H)(H) = 0$, it can alternatively be viewed as smoothness of the difference $K - H$ with respect to H . Note that in contrast to perturbation theory there is no upper bound on the coefficient k' which measures the size of the commutator with respect to H .

Refinements of both of the above results occur if $K \geq 0$. Somewhat surprisingly, in this case, commutator theory gives self-adjointness and smoothness properties from a bound on the double commutator $(\mathrm{ad}\ H)^2(K)$.

2. Basic Theorems

There are two basic theorems of commutator theory on Hilbert space, both of which have more refined variants in terms of quadratic form bounds, or for the special case $H \geq 0$. For simplicity we will only describe the results in terms of operator bounds and we will use the notation $\|\cdot\|_n$ for the graph norm

$$\|a\|_n = \|H^n a\| + \|a\| \quad , \quad a \in D(H^n) \quad ,$$

on $D(H^n)$, for each $n = 1,2\ldots$

Theorem 1. *Let* K *be a symmetric operator with domain* h^∞ *and suppose*

1. $Kh^\infty \subseteq D(H)$,
2. $\|(\mathrm{ad}\ H)(K)a\| \leq k\|a\|_1 \quad , \quad a \in h_\infty$.

It follows that K *is essentially self-adjoint. Moreover the unitary group* $V_t = \exp\{it\overline{K}\}$ *generated by the closure* $\overline{K}$ *of* K *satisfies the following :*

1. $V_t D(H) = D(H) \quad , \quad t \in \mathbb{R} \quad ,$
2. $\|V_t a\|_1 \leq e^{k|t|}\|a\|_1 \quad , \quad t \in \mathbb{R} \quad , \quad a \in D(H),$
3. $V|_{D(H)}$ *is* $\|\cdot\|_1$ *- continuous.*

The first statement of this theorem is contained in Nelson's version of the original Glimm-Jaffe theorem. The latter authors also subsequently proved invariance properties of the type contained in the second statement.

The next theorem gives similar results from a weaker commutator bound but with an additional stability hypothesis $K \geq 0$.

Theorem 2. *Let* K *be a positive operator with domain* h_∞ *and suppose*

1. $Kh_\infty \subseteq D(H^2)$,
2. $\|(\mathrm{ad}\ H)^2(K)a\| \leq k\|a\|_2$, $a \in h_\infty$.

It follows that K *is essentially self-adjoint. Moreover the self-adjoint contraction semigroup* $T_t = \exp\{-t\overline{K}\}$ *generated by the closure* $\overline{K}$ *of* K *satisfies the following for* $n = 1, 2$:

1. $T_t D(H^n) \subseteq D(H^n)$, $t \geq 0$,
2. $\|T_t a\|_n \leq e^{tn^2k/2}\|a\|_n$, $t \geq 0$, $a \in D(H^n)$,
3. $T|_{D(H^n)}$ *is* $\|\cdot\|_n$ *- continuous.*

We will not attempt to prove these theorems but we will indicate the basic method for establishing essential self-adjointness. First, consider the case of Theorem 1.

In order to establish that K is essentially self-adjoint it suffices to prove that the subspaces $(I+i\alpha K)h_\infty$ are norm-dense in h for small real α . Now assume

$$(a,(I+i\alpha K)b) = 0 \quad , \quad b \in h_\infty \quad .$$

Next introduce the self-adjoint semigroup $t \geq 0 \to S_t = \exp\{-tH^2\}$. Then, by spectral theory $S_t h \subseteq h_\infty$ for all $t > 0$. Therefore

$$(a,(I+i\alpha K)S_{2t}a) = 0 \quad , \quad t > 0 \quad .$$

Consequently

$$\|S_t a\|^2 = -i\alpha(a,KS_{2t}a)$$

$$= -i\alpha \mathrm{Im}(a, KS_{2t}a)$$

$$= \frac{-i\alpha}{2}\{(KS_{2t}a,a) - (a,KS_{2t}a)\}$$

and one concludes that

$$\|a\|^2 \le (|\alpha|/2) \limsup_{t\to 0+} |(KS_{2t}a,a) - (a,KS_{2t}a)| \quad .$$

Now the key point is that the expression on the right hand side essentially involves the commutators $C_t = (\mathrm{ad}\ S_{2t})(K)$. But one deduces by standard reasoning from the hypotheses of the theorem that the C_t are bounded operators from h_∞ into h with the property that

$$c = \limsup_{t\to 0+} \|C_t\| < +\infty \quad .$$

Therefore

$$\|a\|^2 \le (|\alpha|/2)\ c\ \|a\|^2 \quad ,$$

and if $|\alpha|c < 2$ then one must have $a = 0$. This establishes the density of $(I+i\alpha K)h_\infty$ for small α .

A similar argument works if $K \ge 0$, and one adopts the hypotheses of Theorem 2. But now it suffices to prove that $(I+\alpha K)h_\infty$ is norm-dense for small positive α . Hence assume

$$(a,(I+\alpha K)b) = 0 \quad , \quad b\in h_\infty \quad .$$

Consequently

$$(a,(I+\alpha K)S_{2t}a) = 0$$

and one deduces that

$$\|S_t a\|^2 = -\alpha(a,KS_{2t}a)$$

$$= \frac{-\alpha}{2}\{(KS_{2t}a,a) + (a,KS_{2t}a)\}$$

$$\leq \frac{-\alpha}{2}\{(KS_{2t}a,a) - 2(S_t a,KS_t a) + (a,KS_{2t}a)\}$$

The last step uses the positivity of K. Now one observes that this estimate involves the double commutators $C_t = (\mathrm{ad}\ S_t)^2(K)$ and one can establish from the assumed bound on $(\mathrm{ad}\ H)^2(K)$ that the C_t are bounded operators from h_∞ into h and $\|C_t\|$ is uniformly bounded as $t \to 0+$. Thus one concludes, as before, that K is essentially self-adjoint.

3. Partial differential operators

As an illustration of the foregoing theorems we consider first and second order partial differential operators acting on $h = L^2(\mathbb{R}^n)$.

Let $H = \Delta$ where Δ denotes the usual self-adjoint Laplacian $\Delta = -\Sigma_{i=1}^n \partial^2/\partial x_i^2$. Then h_∞ consists of the functions in $L^2(\mathbb{R}^n)$ which are infinitely often differentiable with all their derivatives in $L^2(\mathbb{R}^n)$. Now if $a_1,\ldots,a_n$ denote continuously differentiable functions over $\mathbb{R}^n$ and $A_1,\ldots A_n$ denote the associated multiplication operators then

$$K = i\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_j$$

is a symmetric operator from h_∞ into h. But if the a_j are three times continuously differentiable then $Kh_\infty \subseteq D(H)$ and $(\mathrm{ad}\ H)(K)$ is a second-order partial differential operator with bounded coefficients. Hence it follows by standard estimates on the mixed derivatives of second order by the Laplacian that one has a bound of the form

$$\|(\mathrm{ad}\ H)(K)a\| \leq k\|a\|_1 \quad , \quad a\epsilon h_\infty \ .$$

Therefore Theorem 1 can be applied to K . In particular K is essentially self-adjoint and the unitary group generated by $\overline{K}$ leaves $D(H)$ invariant and is $\|\cdot\|_1$ - continuous in restriction to $D(H)$.

Next consider a second-order differential operator

$$K = i \sum_{j=1}^{n} (A_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_j) + \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} A_{jk} \frac{\partial}{\partial x_k}$$

where A_{jk} denotes an operator of multiplication by a continuously differentiable complex-valued function a_{jk} . Now if $\overline{a_{jk}(x)} = a_{kj}(x)$ then K is symmetric. Moreover, if the a_j , a_{jk} are all three-times continuously differentiable then $Kh_\infty \subseteq D(H)$ but $(\mathrm{ad}\, H)(K)$ is now a third-order partial differential operator with bounded coefficients. Hence $(\mathrm{ad}\, H)(K)$ is dominated by H in the sense of Theorem 1 if, and only if, the third-order terms vanish, i.e. the a_{jk} are constant. Nevertheless Theorem 2 can be applied if K is positive.

Assume the a_j , a_{jk} are five-times continuously differentiable then $(\mathrm{ad}\, H)(K)$ is a fourth-order partial differential operator with bounded coefficients and the double commutator bound of Theorem 2 can be verified. Thus this theorem applies whenever K is positive. But for this it suffices that K is strongly elliptic in the sense that there is an $\varepsilon > 0$ such that $(A_{jk} - \varepsilon\|A_j\|^2\delta_{jk}I)$ is positive-definite, i.e.,

$$\sum_{j,k=1}^{d} (f_j, (A_{jk} - \varepsilon\|A_j\|^2\delta_{jk}I)f_k) \geq 0 \quad , \quad f_j \text{ , } f_k \in L^2 \text{ .}$$

The foregoing discussion illustrates the different scopes of application of the two commutator theorems. The first applies to first order differential operators and the second to second order elliptic operators.

But these specific applications can be generalized in various ways, e.g. by using quadratic form estimates one can weaken the differentiability requirements on the coefficients, or by introducing a more general notion of differentiability one can consider operators with coefficients which are quite general bounded operators on $L^2(\mathbb{R}^n)$, or one can discuss differential operators on more general Lie groups than $\mathbb{R}^n$.

References

The foregoing discussion is based on the paper :

Robinson, D.W.; Commutator theory on Hilbert space, Canberra preprint No 35 - 1986.

This paper contains references to all earlier works on commutator theorems. The basic results of the theory which are largely due to Glimm, Jaffe and Nelson are described in :

Reed, M. and B. Simon; *Methods of modern mathematical physics*, Vol II, Academic Press (1975), Chapter X, Section 5; and in

Glimm, J. and A. Jaffe; *Quantum physics*, Springer (1981) Chapter 19, Section 4.

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