INTEGRATION OF CONUCLEAR SPACE VALUED FUNCTIONS

(dedicated to Professor Igor Kluvánek)

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In the classical theory of the Lebesgue integral, the space of integrable functions is introduced as the completion of the space of simple functions (or of continuous functions), with respect to the topology of convergence in mean, that is, the uniform convergence of indefinite integrals. However, if the space of simple functions taking values in a Banach space X is considered, then its completion can only be realized as the space of integrable functions with values in some locally convex space larger than X (see [6]).

In this note, it is shown that the space of integrable functions with values in a conuclear space Y is sequentially complete with respect to the topology of convergence in mean. Further, the Y-valued simple functions form a dense linear subspace. In this case, there is no necessity to integrate functions taking values in a locally convex space larger than Y . The class of conuclear spaces includes the spaces $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{S}$ and $\mathcal{S}'$ which arise in distribution theory, as well as the product space and the locally convex direct sum of countably many copies of the real line.

1. THE ARCHIMEDES INTEGRAL

The real or complex numbers will be referred to simply as scalars.

Let Ω be a non-empty set. To save subscripts and circumlocution, subsets of Ω will be identified with their characteristic functions. Let A be a σ-algebra of subsets of Ω. For a subset E of Ω, let $E\cap A = \{E\cap F : F\in A\}$. Let λ be an extended real valued non-negative measure on the σ-algebra A and let $A_\lambda = \{E\in A : \lambda(E) < \infty\}$.

Let X be a Banach space with norm $|\cdot|$. The following lemma is well-known (cf. [6,Lemma 1]).

LEMMA 1. *Suppose that* $c_j \in X$ *are vectors and* $E_j \in A$ *sets,* $j = 1,2,\ldots,$ *such that*

(1) $$\sum_{j=1}^{\infty} |c_j|\lambda(E_j) < \infty$$

and also such that the equality

$$\sum_{j=1}^{\infty} c_j E_j(\omega) = 0$$

holds for every $\omega \in \Omega$ *satisfying the relation*

(2) $$\sum_{j=1}^{\infty} |c_j| E_j(\omega) < \infty .$$

Then, the equality

$$\sum_{j=1}^{\infty} c_j \lambda(E_j) = 0$$

holds.

Recall that a strongly λ-measurable function f on Ω is said to be Bochner λ-integrable if the non-negative valued function $|f|$ is λ-integrable (cf. [2,Chapter II]).

According to [4] , a function $f:\Omega\to X$ is Bochner λ-integrable if and only if there exist vectors $c_j \in X$ and sets $E_j \in \mathcal{A}$, j = 1,2,..., such that (1) holds and such that the equality

$$f(\omega) = \sum_{j=1}^{\infty} c_j E_j(\omega)$$

holds for every $\omega \in \Omega$ satisfying (2) . For such a function f , the indefinite Bochner integral $f\lambda : \mathcal{A}\to X$ is defined by

$$(f\lambda)(E) = \sum_{j=1}^{\infty} c_j \lambda(E_j \cap E) \ , \quad E \in \mathcal{A} \ . \tag{3}$$

The set function $f\lambda$ is well-defined by Lemma 1, and it is σ-additive by the Vitali-Hahn-Saks theorem (cf. [2,Corollary I.5.10]). Furthermore, its variation $|f\lambda|$ is finite.

The proof of the following lemma is omitted because it is proved similarly to Theorem 3 of [6] .

LEMMA 2. *Suppose that* $f_n : \Omega\to X$, $n = 1,2,\ldots,$ *are Bochner* λ-*integrable functions for which*

$$\sum_{n=1}^{\infty} |f_n\lambda|(\Omega) < \infty \ .$$

Let $f:\Omega\to X$ *be a function such that*

$$f(\omega) = \sum_{n=1}^{\infty} f_n(\omega)$$

for every $\omega\in\Omega$ *for which*

$$\sum_{n=1}^{\infty} |f_n(\omega)| < \infty \ .$$

Then, the function f *is Bochner* λ-*integrable and*

$$\lim_{N\to\infty} \Big|f\lambda - \sum_{n=1}^{N} f_n\lambda\Big|(\Omega) = 0 \ .$$

Let Y be a locally convex Hausdorff space and Y' its dual space. Denote by $P(Y)$ the collection of all continuous seminorms on Y. A sequence $\{W_n\}_{n=1}^{\infty}$ of subsets of Y is called unconditionally summable if, for any choice of $w_n \in W_n$, $n = 1,2,\ldots$, the sequence $\{w_n\}_{n=1}^{\infty}$ is unconditionally summable in Y.

A function $f : \Omega \to Y$ is said to be *Archimedes* λ-integrable if there exist vectors $c_j \in Y$ and sets $E_j \in A$, $j = 1,2,\ldots$, satisfying the following two conditions:

(i) the sequence $\{c_j \lambda(E_j \cap A)\}_{j=1}^{\infty}$ of subsets of Y is unconditionally summable; and

(ii) if $y' \in Y'$, then the equality

$$\langle y', f(\omega)\rangle = \sum_{j=1}^{\infty} \langle y', c_j\rangle E_j(\omega)$$

holds for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |\langle y', c_j\rangle| E_j(\omega) < \infty .$$

The indefinite integral $f\lambda$ of such a function f with respect to the measure λ is defined as in (3). By Lemma 1, the indefinite integral $f\lambda$ is a well-defined Y-valued set function on A. It is σ-additive by the Vitali-Hahn-Saks theorem and the Orlicz-Pettis lemma.

Let $f : \Omega \to Y$ be an Archimedes λ-integrable function. For every $y' \in Y'$, let $\langle y', f\rangle$ be the function defined by

$$\langle y', f\rangle(\omega) = \langle y', f(\omega)\rangle , \quad \omega \in \Omega .$$

For a continuous seminorm p on Y, define

$$p_\lambda(f) = \sup |\langle y', f\rangle\lambda|(\Omega) ,$$

where the supremum is taken over all vectors $y' \in Y'$ satisfying

$|\langle y', y\rangle| \le p(y)$ for every $y \in Y$.

The space of all Y-valued Archimedes λ-integrable functions on Ω will be denoted by $L(\lambda, Y)$, and will be equipped with the topology given by the seminorms p_λ , $p \in P(Y)$, that is, the topology of convergence in mean. The set of all A_λ-simple functions on Ω is then dense in the space $L(\lambda, Y)$.

If Y is an infinite dimensional Banach space, then the space $L(\lambda, Y)$ is not always complete. In this case, the space $L(\lambda, Y)$ coincides with the space of all Y-valued, strongly measurable, Pettis λ-integrable functions on Ω , which is not complete if λ is a non-atomic measure (see [8,p.131]). Therefore, it is necessary to integrate functions with values in a locally convex space which is larger than the Banach space Y needed to accommodate the values of indefinite integrals (see [6]).

2. INTEGRATION OF CONUCLEAR SPACE VALUED FUNCTIONS

A balanced convex bounded subset of a locally convex space will be called disked.

Let Y be a sequentially complete locally convex Hausdorff space. The linear space Y_B generated by a closed disked subset B of Y is a Banach space, equipped with the gauge of B (cf. [1,Lemma III.3.1]).

Let C be a collection of closed disked subsets of Y . The space Y is said to be conuclear with respect to C if, for each set $B \in C$, there exists a set $C \in C$ which includes B and for which the canonical injection from Y_B into Y_C is a nuclear map. For the properties of conuclear spaces, see [7].

THEOREM 3. *Let* λ *be an extended real valued non-negative measure on a* σ*-algebra* A *of subsets of a non-empty set* Ω *. Suppose that the space* Y

is conuclear, with respect to a collection $\mathcal{C}$ *of closed disked subsets of* Y, *which has the property that every weakly compact subset of* Y *is included in a set from* $\mathcal{C}$. *Then, the space* $L(\lambda,Y)$ *is sequentially complete with respect to the topology of convergence in mean.*

Proof. Take a Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ from the space $L(\lambda,Y)$. Then, for every $E \in A$, the sequence $\{(f_n\lambda)(E)\}_{n=1}^{\infty}$ of vectors is convergent to a vector $\mu(E)$ in Y since Y is sequentially complete. It follows, from the Vitali-Hahn-Saks theorem and the Orlicz-Pettis lemma, that the so-defined set function $\mu : A \to Y$ is a vector measure. Hence, the range $\mu(A)$ of μ is a relatively weakly compact subset of Y (cf.[5]). Since every nuclear map is compact, there exists a set B belonging to $\mathcal{C}$, which contains $\mu(A)$, such that $\mu(A)$ is a relatively compact subset of Y_B . Then, the set function $\mu : A \to Y_B$ is a vector measure; that is, it is σ-additive with respect to the topology given by the gauge of B .

Choose another set C from $\mathcal{C}$ which contains B and for which the canonical injection $J : Y_B \to Y_C$ is a nuclear map. There exist , an absolutely summable sequence $\{\alpha_j\}_{j=1}^{\infty}$ of scalars, a bounded sequence $\{\xi_j\}_{j=1}^{\infty}$ of vectors in the dual of Y_B , and a bounded sequence $\{y_j\}_{j=1}^{\infty}$ of vectors in Y_C , such that the equality

$$J(y) = \sum_{j=1}^{\infty} \alpha_j \langle \xi_j, y \rangle y_j , \quad y \in Y_B ,$$

holds in Y_C (cf. [7, Chapter IV, Part II]). For every $j = 1,2,\ldots$, the scalar measure $\langle \xi_j, \mu \rangle$ defined by $\langle \xi_j, \mu \rangle(E) = \langle \xi_j, \mu(E) \rangle$, $E \in A$, vanishes outside the union of countably many sets belonging to A_λ . Thus, the Radon-Nikodým theorem ensures the existence of a scalar valued λ-integrable function g_j such that $\langle \xi_j, \mu \rangle = g_j \lambda$, $j = 1,2,\ldots$. Further, by the Nikodým boundedness theorem, there exists a positive number M for which

$$|g_j\lambda|(\Omega) \le M , \quad j = 1,2,\ldots \quad .$$

Now, the functions $h_j : \Omega \to Y_C$ defined by $h_j(\omega) = \alpha_j g_j(\omega) y_j$ for every $\omega \in \Omega$, $j = 1,2,\ldots$, are Bochner λ-integrable, with the property that

$$\sum_{j=1}^{\infty} |h_j\lambda|(\Omega) \le M \sum_{j=1}^{\infty} |\alpha_j| < \infty .$$

Define the function $f : \Omega \to Y_C$, by

$$f(\omega) = \sum_{j=1}^{\infty} h_j(\omega)$$

for each $\omega \in \Omega$ with

$$\sum_{j=1}^{\infty} |h_j(\omega)| < \infty ,$$

and by $f(\omega) = 0$ otherwise. It then follows from Lemma 2 that f is Bochner λ-integrable and that

$$\begin{aligned}(f\lambda)(E) &= \sum_{j=1}^{\infty} (h_j\lambda)(E) = \sum_{j=1}^{\infty} \alpha_j (g_j\lambda)(E) y_j = \\ &= \sum_{j=1}^{\infty} \alpha_j \langle \xi_j, \mu(E) \rangle y_j = J(\mu(E))\end{aligned}$$

for every $E \in A$.

It is now clear that the function f belongs to the space $L(\lambda,Y)$ and the sequence $\{f_n\}_{n=1}^{\infty}$ is convergent in mean to f .

If λ is σ-finite, then the arguments in the proof of the above theorem can be used to prove the following.

COROLLARY 4. *Suppose that the space* Y *satisfies the assumption in Theorem* 3 *and also that* Y *is quasi-complete (respectively complete). Then, the space* $L(\lambda,Y)$ *is quasi-complete (respectively complete) for every* σ*-finite non-negative measure* λ .

A sequentially complete locally convex Hausdorff space is called a conuclear space if it is conuclear with respect to the collection of all closed disked subsets. It follows from [7, Theorem IV.1, Part II] that the spaces $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{S}$ and $\mathcal{S}'$, used in distribution theory, are all complete conuclear spaces.

EXAMPLE 5. Let λ be the Lebesgue measure in the Euclidean space $\Omega = \mathbb{R}^n$, $n = 1,2,\ldots$. Let S be the Schwartz space of rapidly decreasing, infinitely differentiable functions on Ω, and let S' be its dual space equipped with the strong dual topology. Define the Fourier transform $\hat{\phi}$ of a function $\phi \in S$ by

$$\hat{\phi}(\theta) = \int_\Omega \phi(\omega)\exp(-i\langle\omega,\theta\rangle)\,d\lambda(\omega) ,$$

for every $\theta \in \Omega$. Let $f : \Omega \to S'$ be the function defined by

$$\langle f(\omega),\phi\rangle = \hat{\phi}(-\omega), \phi \in S ,$$

for every $\omega \in \Omega$.

If γ is a locally λ-integrable function on Ω which grows at infinity more slowly than a polynomial, then the function $\gamma f : \Omega \to S'$ is Archimedes λ-integrable. Indeed, for every $\phi \in S$ and every Borel subset E of Ω,

$$\int_E \langle\gamma(\omega)f(\omega),\phi\rangle d\lambda(\omega) = \int_E \langle f(\omega),\phi\rangle\gamma(\omega)\,d\lambda(\omega) =$$

$$= \int_E \hat{\phi}(-\omega)\gamma(\omega)\,d\lambda(\omega) .$$

Since the Fourier transform $\phi \mapsto \hat{\phi}$, $\phi \in S$, is a continuous linear map from S into itself and since γ belongs to S', there exists a vector measure μ, which is defined on the Borel σ-algebra of Ω and takes values in S', such that, for every Borel set E,

$$\langle \mu(E), \phi \rangle = \int_E \langle \gamma f, \phi \rangle d\lambda , \quad \phi \in S .$$

In view of the proof of Theorem 3, there exists an Archimedes λ-integrable function $g : \Omega \to S'$ such that $\mu = g\lambda$. Since the space S, which is the dual space of S', has a countable total subset, it follows that γf is λ-almost everywhere equal to g. Hence, γf is Archimedes λ-integrable.

EXAMPLE 6. Let λ be the Lebesgue measure, in the space $\Omega = \mathbb{R}^n$ equipped with the Euclidean norm $|\cdot|$, $n = 1,2,\ldots$. Define the function $f : \Omega \to S'$ as in Example 5. For every real number t, let

$$\gamma_t(\omega) = (2\pi)^{-n} t \sin t|\omega| , \quad (t,\omega) \in \mathbb{R} \times \Omega ;$$

then Example 5 implies that the function $\gamma_t f : \Omega \to S'$ is Archimedes λ-integrable. Moreover, for each real number t, the tempered distribution $\big((\gamma_t f)\lambda\big)(\Omega)$ equals the difference of two oscillatory integrals

$$\Phi(t,x) = (2\pi)^{-n} \int_\Omega (2i|\omega|)^{-1} e^{i(\langle x,\omega \rangle + t|\omega|)} d\lambda(\omega) -$$

$$- (2\pi)^{-n} \int_\Omega (2i|\omega|)^{-1} e^{i(\langle x,\omega \rangle - t|\omega|)} d\lambda(\omega)$$

with phase functions $(x,\omega) \mapsto \langle x,\omega \rangle + t|\omega|$ and $(x,\omega) \mapsto \langle x,\omega \rangle - t|\omega|$, $(x,\omega) \in \Omega \times \Omega$, respectively. Let Δ denote the Laplacian in Ω and δ_0 the Dirac measure at the origin of Ω. As noted in [3, Example 7.8.4], the distribution Φ is a solution of the Cauchy problem

$$\partial_t^2 \Phi - \Delta\Phi = 0 \quad \text{in } \mathbb{R} \times \Omega ; \Phi(0,\cdot) = 0, \partial_t \Phi = \delta_0 \quad \text{when } t = 0 .$$

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