Invariant Differential Operators and Representation Theory

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Dedicated to IGOR KLUVANEK

1 Introduction

In this lecture I outlined how some results in the representation theory of the noncompact semisimple Lie group $SU(n+1,1)$ were related to harmonic analysis on the Heisenberg group. The guide we use is the example of analysis on the real line viewed as the boundary of the upper half-plane. The Heisenberg group can be identified with the boundary of a Siegel domain. For each of the following ingredients of classical analysis on the upper half-plane, we seek an analogue in the setting of harmonic analysis on noncompact symmetric spaces. They are:

1. the Cauchy-Riemann operator ;
2. the fact that the real and imaginary parts of a holomorphic function are conjugate harmonic functions ;
3. the boundary values of these functions;

4. the Cauchy-Szegö integral, which takes boundary data and assigns holomorphic functions;

5. the Hilbert transform (combine items 4, 2, and 3);

Now let G be a noncompact semisimple connected Lie group with finite centre, acting transitively as a group of isometries on a noncompact Riemannian symmetric space $\mathbb{X}$. Fix an element x_0 in $\mathbb{X}$, which we will treat as the origin, and let K denote its isotropy subgroup in G. We let G act on the right, so that $\mathbb{X} = K\backslash G$. Furthermore, fix an Iwasawa decomposition $G = ANK$ and let M denote the centralizer of A in K. Take an irreducible representation (τ, V_τ) of K. Functions on $\mathbb{X}$ with values in V_τ can be identified with τ-covariant functions on G. Items (1) through (5) above suggest the following apparatus.

1. Fix a G-invariant first order differential operator ∂_τ acting on V_τ-valued functions on $\mathbb{X}$, determined by the location of τ in the dual object of K.

2. Under the action of $\tau(M)$, V_τ splits into irreducible M-components. The M-components of an element $F \in ker(\partial_\tau)$ should be eigenfunctions of the Casimir operator and play the role of conjugate functions.

3. The boundary of $\mathbb{X}$ is approached by moving towards infinity along orbits of A in $\mathbb{X}$. Weighted boundary values of M-components of elements of $ker(\partial_\tau)$ provide a means of imbedding $ker(\partial_\tau)$ into a principal series representation.

4. The Cauchy-Szegö map, with suitable parameters, exhibits the K-finite part of $ker(\partial_\tau)$ as a quotient of a certain principal series representation.

5. Intertwining operators.

For item (1) in this list, see [5,17,3]. The remark about the location of τ is explained in Section 2 of [15]. Item (2) is connected with Corollary (3.2) in [13] and [6]. Boundary behaviour, as referred to in item (3), is described in [1,8,10,3]. My work [15] is concerned with realizations of end of complementary series representations in this setting (see section 15 in [12] for a definition of end of complementary series). I have been guided in this research by the work of John Gilbert, R. A. Kunze, Bob Stanton, and Peter Tomas, who have treated the case of the Lorentz groups $SO(n,1)$.

2 Domains in Projective Space

Fix $n \geq 2$ and recall that $G = SU(n+1,1)$ is the subgroup of $SL(n+2,\mathbf{C})$ which preserves the sesquilinear form on $\mathbf{C}^{n+2}$ given by

$$z\Gamma_1 w^* = z_1\overline{w}_1 + \ldots + z_{n+1}\overline{w}_{n+1} - z_{n+2}\overline{w}_{n+2}.$$

$SL(n+2,\mathbf{C})$ acts transitively on $P^{n+1}(\mathbf{C})$, where we represent an element of projective space by means of a row of homogeneous coordinates $[z]$ and matrices multiply on the right-hand side. In particular, G acts transitively on the domain

$$\mathbf{B} = \left\{[z] \in P^{n+1}(\mathbf{C}) : z\Gamma_1 z^* < 0\right\}$$

and its boundary $\partial\mathbf{B} = \{[z] \in P^{n+1}(\mathbf{C}) : z\Gamma_1 z^* = 0\}$.

Consider another sesquilinear form on $\mathbf{C}^{n+2}$, given by

$$z\Gamma_2 w^* = -z_1\overline{w}_{n+2} - z_{n+2}\overline{w}_1 + z_2\overline{w}_2 + \ldots + z_{n+1}\overline{w}_{n+1}.$$

Here the $(n+2)\times(n+2)$ matrices Γ_1 and Γ_2 are related by the equation $\Gamma_2 = \gamma\Gamma_1\gamma^{-1}$, where

$$\gamma = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & I_n & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$

The Siegel domain $\mathbf{X}$ is $\mathbf{X} = \{[z] \in P^{n+1}(\mathbf{C}) : z\Gamma_2 z^* < 0\}$, which is the same as $\mathbf{B}\gamma^{-1}$. If $[z] \in \mathbf{B}$ then $z_{n+2} \neq 0$ and so $\mathbf{B}$ can be identified with the unit ball in $\mathbf{C}^{n+1}$ and $\partial\mathbf{B}$ with the unit sphere. Similarly, if $[z] \in \mathbf{X}$ then $z_1 \neq 0$ and $z_{n+2} \neq 0$. We can identify $\mathbf{X}$ with the domain in $\mathbf{C}^{n+1}$ described by

$$\left\{ \zeta \in \mathbf{C}^{n+1} : Im(\zeta_1) > \frac{1}{2}(|\zeta_2|^2 + \ldots + |\zeta_{n+1}|^2) \right\}$$

and the identification is achieved by the map $\varphi : \mathbf{C}^{n+1} \to P^{n+1}(\mathbf{C})$ described by $\varphi(\zeta) = [\zeta_1, \zeta_2, \ldots, \zeta_{n+1}, i]$. The boundary of this domain in $\mathbf{C}^{n+1}$ is the set

$$\left\{ \zeta \ : \ Im(\zeta_1) = \frac{1}{2}(|\zeta_2|^2 + \ldots + |\zeta_{n+1}|^2) \right\}$$

and it is known that this is a realization of the Heisenberg group (see [2]). Its image with respect to φ consists of the open dense subset of $\partial\mathbf{X}$ consisting of those elements $[z]$ with $z_{n+2} \neq 0$. The action of G on $\mathbf{X}$ and $\partial\mathbf{X}$ is defined by $[z]\cdot g = [z\gamma g\gamma^{-1}]$ and the action on the corresponding domain in $\mathbf{C}^{n+1}$ is $\zeta \mapsto \varphi^{-1}(\varphi(\zeta)\cdot g)$. This means that G acts by fractional linear transformations. Equip $\mathbf{X}$ with the hermitian hyperbolic metric. It is known that G acts isometrically.

3 Special subgroups

Fix $\mathbf{x}_0 = [1, 0, \ldots, 0, 1]$ in $\mathbf{X}$. Its isotropy subgroup in G is the compact subgroup $K = S(U(n+1) \times U(1))$. On the boundary, take $\mathbf{x}_1 = [0, 0, \ldots, 0, 1]$ as the origin. For every $t \in \mathbf{R}$ let

$$a(t) = \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_n & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix}.$$

The geodesic half-line from $\mathbf{x}_0$ to $\mathbf{x}_1$ is traced out by $\mathbf{x}_0 \cdot a(t)$ as t varies over $0 \leq t < \infty$. Let $A = \{a(t) : t \in \mathbf{R}\}$. The centralizer of A in K is

$$M = \left\{ \begin{pmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{11} \end{pmatrix} : u_{11} \in \mathbf{T}, u_{22} \in U(n), \text{and } u_{11}^2 det(u_{22}) = 1 \right\}.$$

The isotropy subgroup of $\mathbf{x}_1$ in G is ANM, where N is the subgroup

$$N = \left\{ \begin{pmatrix} 1 - \frac{1}{2}|z|^2 + ir & z & \frac{1}{2}|z|^2 - ir \\ -z^* & I_n & z^* \\ ir - \frac{1}{2}|z|^2 & z & 1 + \frac{1}{2}|z|^2 - ir \end{pmatrix} : z \in \mathbf{C}^n \text{ and } r \in \mathbf{R} \right\}.$$

and $\partial\mathbf{X} = ANM\backslash G$. The *Heisenberg group* is $V = \{(n^{-1})^* : n \in N\}$ and so a typical element of V is of the form

$$v(z, r) = \begin{pmatrix} 1 + ir - \frac{1}{2}|z|^2 & z & ir - \frac{1}{2}|z|^2 \\ -z^* & I_n & -z^* \\ \frac{1}{2}|z|^2 - ir & -z & 1 + \frac{1}{2}|z|^2 - ir \end{pmatrix}.$$

A direct calculation shows that $\mathbf{x}_1 \cdot V$ is an open dense subset of the boundary and it follows that $ANMV$ is an open dense subset of G. The Iwasawa decomposition determined by this set up is $G = ANK$. Taking the conjugate transpose of this, we can also write $G = KVA$ so that $\mathbf{X} = \mathbf{x}_0 \cdot VA = \mathbf{x}_0 \cdot AV$. It is also known that K acts transitively on $\partial\mathbf{X}$, which means that $\partial\mathbf{X} = M\backslash K$.

4 Covariant Functions

Now we must describe some irreducible representations of K. Fix p and q, non-negative integers, and let $\mathcal{V}_{p,q}$ denote the space of spherical harmonics of bidegree (p,q) on $\mathbb{C}^{n+1}$. The group K acts by $\tau_{p,q}$ on $\mathcal{V}_{p,q}$, and this action is described by

$$\tau_{p,q}\left(\begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix}\right) f(\xi,\xi^*) = k_{22}^{q-p} f(\xi k_{11}, k_{11}^{-1}\xi^*),$$

where $k_{11} \in U(n+1)$ and $k_{22} = \det(k_{11})^{-1}$. Given a function $f : \mathbb{X} \to \mathcal{V}_{p,q}$ we can extend it to be a $\tau_{p,q}$-covariant function on G by assigning

$$f^\sharp(kav) = \tau_{p,q}(k) f(\mathrm{x}_0 \cdot (av)),$$

for all $kav \in KAV$. Similarly, if F is a $\tau_{p,q}$- covariant function on G, set

$$F^\flat(\mathrm{x}_0 \cdot av) = F(av),$$

so that $F^\flat$ is a $\mathcal{V}_{p,q}$-valued function on $\mathbb{X}$ and $(F^\flat)^\sharp = F$. The action of G on $\tau_{p,q}$-covariant functions is by right- translation, and so there is an action of G on $\mathcal{V}_{p,q}$-valued functions on $\mathbb{X}$.

The Lie algebra $\underline{g}$ of G has a Cartan decomposition $\underline{g} = \underline{k} + \underline{s}$, where $\underline{s}$ is isomorphic with the tangent space of $\mathbb{X}$ at x_0. Hence, $\underline{s}$ carries the action of $Ad(K)$ as a group of rotations. This action can be extended to yield a unitary representation of K on the complexification $\underline{s}_{\mathbb{C}}$. In fact, as a K-module, $\underline{s}_{\mathbb{C}}$ is isomorphic with $\mathcal{V}_{1,0} \oplus \mathcal{V}_{0,1}$. Each element $E \in \underline{s}_{\mathbb{C}}$ produces a right-translation invariant vector field on G, given by $f \mapsto E * f$. Now fix an orthonormal basis $E_1, E_2, \ldots, E_{2n+2}$ of $\underline{s}_{\mathbb{C}}$. There is the canonical invariant differential operator (see

[12]) ∇ acting on $\mathcal{V}_{p,q}$-valued functions on $\mathbf{X}$ and given by

$$\nabla f = (\sum_{j=1}^{2n+2} E_j * (f^{\sharp}) \otimes \overline{E}_j)^{\flat}.$$

Notice that $\nabla f : \mathbf{X} \to \mathcal{V}_{p,q} \otimes \underline{s}_c$ and that $(\nabla f)^{\sharp}$ is $(\tau_{p,q} \otimes Ad|_K)$- covariant.

Each K-equivariant projection P of $\mathcal{V}_{p,q} \otimes \underline{s}_c$ onto a K-invariant subspace gives a G-invariant first order differential operator $P \circ \nabla$, acting on $\mathcal{V}_{p,q}$-valued functions on $\mathbf{X}$. It is known that the decomposition of $\mathcal{V}_{p,q} \otimes \underline{s}_c$ into irreducible K-modules is

$$\mathcal{V}_{p,q} \otimes \underline{s}_c \cong \mathcal{V}_{p+1,q} \oplus \mathcal{V}_{p,q+1} \oplus \mathcal{V}_{p-1,q} \oplus \mathcal{V}_{p,q-1} \oplus (\text{other representations}).$$

In [15] I define such a differential operator, say $\partial_{p,q}$, by taking P to be the projection onto the orthogonal complement of $\mathcal{V}_{p+1,q} \oplus \mathcal{V}_{p,q+1}$. We say that a function on G is K-finite if its right translates by elements of K generate a finite-dimensional vector space. Clearly $ker(\partial_{p,q})$ is a G-invariant subspace of the space of smooth $\mathcal{V}_{p,q}$-valued functions on $\mathbf{X}$. Although the subspace of K-finite vectors is not G-invariant, it is a $(\underline{g}, K)-$module. The following theorem is proved in [15].

Theorem 1 *If $p = q \geq 2$ then the K-finite vectors in $ker(\partial_{p,p})$ form an irreducible $(\underline{g}, K)$-module.*

Let Ω denote the Casimir operator for G and $\Box$ the canonical Laplace-Beltrami operator acting on $\mathcal{V}_{p,p}$- valued functions on $\mathbf{X}$, as described in [6]. From the results in section 6 of [15], Corollary 3.2 in [12], and the theorem in [6], we see that elements of $ker(\partial_{p,p})$ have the following eigenfunction property.

Corollary 1 *For* $p \geq 2$ *,every element* $f \in ker(\partial_{p,p})$ *satisfies,* $\Box f = (p+n)f$ *and* $\Omega f = 2(p-1)(p+n)f$.

5 Principal Series Representations

Next we must consider the form of G-invariant spaces of vector-valued functions on $\partial\mathbf{X}$, which will provide the boundary values of elements of $ker(\partial_{p,q})$. As in the previous section, we deal with covariant functions, but now the isotropy subgroup is ANM rather than K. Fix an irreducible representation $(\sigma, \mathcal{H}_\sigma)$ of M, occuring as a subrepresentation of $(\tau_{p,q}|_M, \mathcal{V}_{p,q})$. Let $C^\infty(K,\sigma)$ denote the space of all smooth functions $f : K \rightarrow \mathcal{H}_\sigma$ with the covariance property,

$$f(mk) = \sigma(m)f(k) \text{ , } \forall m \in M \text{ and } k \in K.$$

For each complex number λ let $\mathbf{I}_{\sigma,\lambda}$ denote the space of all elements of $C^\infty(K,\sigma)$, extended to all of G by requiring that $f(a(t)nk) = e^{(\rho+\lambda)t}f(k)$ for all $t \in \mathbf{R}, n \in N$, and $k \in K$. Here $\rho = n+1$. This space is invariant under right translation by elements of G and this representation of G is called a (nonunitary) *principal series representation* . The normalisation $\rho+\lambda$ is arranged so that this is a unitary representation of G when λ is purely imaginery, see [4]. The fact that

$$G = ANMV \cup (\text{a set of measure } 0)$$

means that elements of $\mathbf{I}_{\sigma,\lambda}$ are completely determined by their restriction to V. This tells us how to equip spaces of $\mathcal{H}_\sigma$-valued functions on $\partial\mathbf{X}$ with actions of G, depending on which value we take for λ.

The passage from elements of $\mathbb{I}_{\sigma,\lambda}$ to $\tau_{p,q}$-covariant functions is achieved using Cauchy-Szegö maps, at the level of generality defined by Gilbert, Kunze, Stanton, and Tomas in [3,4]. This explicitly depends on the imbedding of $\mathcal{H}_\sigma$ as an M-invariant subspace of $\mathcal{V}_{p,q}$. Fix $\sigma, \mathcal{H}_\sigma$, and λ as above, and for every $f \in \mathbb{I}_{\sigma,\lambda}$ let

$$\mathcal{S}_{\sigma,\lambda} f(g) = \int_K \tau_{p,q}(k^{-1}) f(kg) dk, \quad \forall g \in G.$$

This operator, called a *Cauchy-Szegö map* , intertwines the principal series representation of G on $\mathbb{I}_{\sigma,\lambda}$ and right translation on $\tau_{p,q}$-covariant functions. Starting with a smooth function, say $F : \partial\mathbb{X} \to \mathcal{H}_\sigma$,which can be extended to be an element $F \in \mathbb{I}_{\sigma,\lambda}$, applying $\mathcal{S}_{\sigma,\lambda}$, and then forming $(\mathcal{S}_{\sigma,\lambda}(F))^\flat$, is a G-equivariant linear operator into the space of smooth $\mathcal{V}_{p,q}$-valued functions on $\mathbb{X}$. In particular cases we can show that it actually maps into $ker(\partial_{p,q})$.

As we said above, it is important to know the decomposition of $\mathcal{V}_{p,q}$ into M-invariant subspaces. This is described in [14]. Among the cases considered in [15] are the following two representations of M. First, let $(1, \mathcal{H}_1)$ denote the trivial representation acting on the one-dimensional subspace in $\mathcal{V}_{p,q}$ generated by the spherical harmonic

$$\varphi_{p,q}(\xi, \xi^*) = \xi_1^p \overline{\xi}_1^q \sum_{k=0}^{\infty} \frac{(-p)_k(-q)_k}{k!(n)_k} \left(\frac{|\xi_1|^2 - |\xi|^2}{|\xi_1|^2} \right)^k .$$

The second representation we consider is $(\sigma_2, \mathcal{H}_2)$, where $\mathcal{H}_2$ is the M-invariant subspace of $\mathcal{V}_{p,q}$ generated by $\xi_2^p \overline{\xi}_{n+1}^q$ and $\sigma_2(m)f = \tau_{p,q}(m)f$ for all $m \in M$ and $f \in \mathcal{H}_2$. If $\mathcal{E}$ is a space of functions on G, let $\mathcal{E}_K$ denote the subspace of K-finite vectors. The following result is a combination of Theorems 6.3.1 and 6.6.1 in [15].

Theorem 2 *Fix $p = q \geq 2$ and let $(1, \mathcal{H}_1)$ and $(\sigma_2, \mathcal{H}_2)$ be the subrepresentations of $(\tau_{p,p}|_M, \mathcal{V}_{p,p})$, as above. Then the Cauchy-Szegö maps $\mathcal{S}_{1,1-n-2p}$ and $\mathcal{S}_{\sigma_2,n-1}$ both have their images contained in $ker(\partial_{p,p})$. Furthermore, when restricted to acting on K-finite vectors, they satisfy*

$$\mathcal{S}_{1,1-n-2p}(\mathbb{I}_{1,1-n-2p})_K = ker(\partial_{p,p})_K = \mathcal{S}_{\sigma_2,n-1}(\mathbb{I}_{\sigma_2,n-1})_K.$$

This shows how the space of K-finite vectors in $ker(\partial_{p,p})$ occurs as quotients of principal series. This is analogous to the description of discrete series representations by Knapp and Wallach in [13].

6 Boundary Values

In section 2 we saw that $\partial\mathbf{X} = \partial\mathbf{B}\gamma^{-1}$ This means that if $[\zeta] \in \partial\mathbf{X}$ then

$$\frac{|\zeta_1 - \zeta_{n+2}|^2}{2} + |\zeta_2|^2 + \ldots + |\zeta_{n+1}|^2 = \frac{|\zeta_1 + \zeta_{n+2}|^2}{2}.$$

For an element $v(z,r) \in V$ the corresponding element in the boundary of $\mathbf{X}$ is $\mathrm{x}_1 \cdot v(z,r) = [|z|^2 - 2ir, -\sqrt{2}z, 1]$. Suppose we start with a scalar-valued function on V and extend it to be an element of $\mathbb{I}_{1,\lambda}$, then we would like to know its restriction to K. For this reason, we must determine the Iwasawa components of an element of V. Every $v(z,r)$ can be written as a product, $v(z,r) = \mathrm{a}(z,r) n\, \mathrm{k}(z,r)$, for some $n \in N, \mathrm{a}(z,r) \in A$, $\mathrm{k}(z,r) \in K$. Here the coset $M\mathrm{k}(z,r)$ is uniquely determined by requiring that $\mathrm{x}_1 \cdot v(z,r) = \mathrm{x}_1 \cdot \mathrm{k}(z,r)$. For an element $a(t) \in A$ and a complex number μ set $a(t)^\mu = e^{\mu t}$. With this notation we see that if $f \in \mathbb{I}_{1,\lambda}$ then

$$f(m\,\mathrm{k}(z,r)) = \mathrm{a}(z,r)^{-(\rho+\lambda)} f(v(z,r)),$$

for all $m \in M$ and $v(z,r) \in V$.

When $\mathbf{x} \in \mathbf{X}$ is of the form $\mathbf{x} = \varphi(\zeta)$ let the *height* of $\mathbf{x}$ be defined by

$$h(\mathbf{x}) = Im(\zeta_1) - \frac{1}{2}\left(|\zeta_2|^2 + \ldots + |\zeta_{n+1}|^2\right).$$

In particular, $h(\mathbf{x}_0) = 1$ and the height of points on the boundary is zero.

Lemma 1 *If* $\mathbf{x} \in \mathbf{X}$ *and* $v(z,r) \in V$*, then* $h(\mathbf{x} \cdot v(z,r)) = h(\mathbf{x})$*. If* $t \in \mathbf{R}$ *then* $h(\mathbf{x} \cdot a(t)) = e^{2t} h(\mathbf{x})$.

This tells us how to find the term $\mathrm{a}(z,r)$ in the Iwasawa decomposition of $v(z,r)$. That is, measure the height of $\mathbf{x}_0 \cdot v(z,r)^*$, which is $((1+|z|^2)^2 + 4r^2)^{-2}$. We saw earlier that $\partial\mathbf{X}$ could be identified with the unit sphere S^{2n+1}, in which case the point $\mathbf{x}_1 \cdot v(z,r) = \mathbf{x}_1 \cdot \mathrm{k}(z,r)$ corresponds to the unit vector

$$\left(\frac{|z|^2 - 1 - 2ir}{1 + |z|^2 - 2ir}, \frac{-2z}{1 + |z|^2 - 2ir}\right)$$

in $\mathbf{C}^{n+1}$, and this correspondence is K-equivariant.

Proposition 1 *If* $f \in \mathbf{I}_{1,\lambda}$*, then for all* $v(z,r) \in V$ *and* $m \in M$*,*

$$f(m\,\mathrm{k}(z,r)) = \left((1+|z|^2)^2 + 4r^2\right)^{\rho+\lambda} f(v(z,r)).$$

Furthermore, f *will be a* K*-finite vector if and only if there is a finite sequence of spherical harmonics* $Y_{j,k} \in \mathcal{V}_{j,k}$ *such that*

$$f(v(z,r)) = \left((1+|z|^2)^2 + 4r^2\right)^{-(\rho+\lambda)} \sum_{j,k} Y_{j,k}\left(\frac{|z|^2 - 1 - 2ir}{1 + |z|^2 - 2ir}, \frac{-2z}{1 + |z|^2 - 2ir}\right).$$

In Theorem 1 we saw that if f is a K-finite vector in $\mathbf{I}_{1,1-n-2p}$ then the $\mathcal{S}_{1,1-n-2p}(f)$ is in the image of the Cauchy-Szegö map $\mathcal{S}_{\sigma_2,n-1}$. The K-finite part of the kernel of $\mathcal{S}_{1,1-n-2p}$ is the extension to $\mathbf{I}_{1,1-n-2p}$ of the direct sum of the spaces $\mathcal{V}_{j,k}$, taken over all pairs (j,k) with either $j<p$ or $k<p$.

Let w_0 denote the matrix

$$\begin{pmatrix} i & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & -i \end{pmatrix},$$

so that for every $a(t) \in A$, $w_0 a(t) w_0^{-1} = a(-t)$. Then w_0 generates the Weyl group for $(\underline{g}, \underline{a})$, and there is a G-invariant pairing between $\mathbf{I}_{\sigma_2,\lambda}$ and $\mathbf{I}_{w_0\sigma_2,-\lambda}$ described in [12]. This latter space is equal to $\mathbf{I}_{\sigma_2,-\lambda}$. Furthermore, let Q denote the M-equivariant projection of $\mathcal{V}_{p,p}$ onto $\mathcal{H}_2$. In [1,3] the following result is demonstrated.

Proposition 2 *If $F \in \mathbf{I}_{\sigma_2,n-1}$ then the following limit exists,*

$$\lim_{t\to\infty} e^{2t} Q\left(\mathcal{S}_{\sigma_2,n-1} F(a(t))\right)$$

and is equal to $A(w_0, \sigma_2, n-1)F(1)$.

Here $A(w_0,\sigma_2,n-1)$ is the intertwining operator from $\mathbf{I}_{\sigma_2,n-1}$ to $\mathbf{I}_{\sigma_2,1-n}$. This means that for every $\Phi \in \mathcal{S}_{\sigma_2,n-1}(\mathbf{I}_{\sigma_2,n-1})$ (which is a subspace of $ker(\partial_{p,p})$) the boundary value operator

$$\mathcal{B}\Phi(v(z,r)) = \lim_{t\to\infty} e^{2t} Q(\Phi(a(t)w_0 v(z,r)))$$

converges to an element of $A(w_0,\sigma_2,n-1)\mathbf{I}_{\sigma_2,n-1}$ resticted to V. This is then true for the image of the K-finite vectors in $\mathbf{I}_{1,1-n-2p}$. It is also known [12] that the intertwining operator provides a means of equipping its image with a

G-invariant quadratic form. That is, if $F \in \mathbf{I}_{\sigma_2,n-1}$ then the value of this form on $A(w_0, \sigma_2, n-1)F$ is

$$\| A(w_0, \sigma_2, n-1)F \|^2 = \langle A(w_0, \sigma_2, n-1)F , F \rangle.$$

In fact, the results of [9] show that this is positive definite for the case with which we are dealing. Starting with f, a K-finite element of $\mathbf{I}_{1,1-n-2p}$, there will be a coset $F + ker(A(w_0, \sigma_2, n-1))$ such that $A(w_0, \sigma_2, n-1)F = \mathcal{BS}_{1,1-n-2p}(f)$ and we can assign a seminorm $\| f \| = \| \mathcal{BS}_{1,1-n-2p}(f) \|$. The completion of the quotient of $(\mathbf{I}_{1,1-n-2p})_K$ modulo the null space of this seminorm is a candidate for a Hardy space. For more on this see [4,3,8,10,11]. The problem remains to explicate the operator $f \mapsto \mathcal{BS}_{1,1-n-2p}(f)$ as a vector-valued convolution operator on V and to understand $\|f\|$ in terms of the Heisenberg group Fourier transform.

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