FOURIER TRANSFORM OF SURFACE CARRIED MEASURES.

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This note, which reports on results from [3], is concerned with estimates of the decay of the Fourier transform of measures supported on hypersurfaces of vanishining curvature. Let S be a hypersurface in $\mathbb{R}^{n+1}$ with Gaussian curvature K and area measure dS. Let $w \in C_c^\infty(S)$. We are seeking estimates of the Fourier transform $\widehat{d\mu}$ of the finite Borel measure $d\mu = w dS$ or, more generally, of the measures $d\mu_\alpha = |K|^\alpha w\, dS$, for $\alpha \geq 0$. Such estimates are important in a number of problems, such as counting lattice points inside dilates of S, proving a priori inequalities for maximal averages of functions over dilates and translates of S [2] [10], and in the study of certain operators related to hyperbolic differential operators [8].

The problem of estimating $\widehat{d\mu}$ has a long history. The one dimensional case, i.e. when S is a curve in $\mathbb{R}^2$, has been investigated by van der Corput for number theoretical reasons. More recently estimates for higher dimensional hypersurfaces have been given by Hlwaka [5], Herz [4], Littman [7], Randol [9] and Svensson [12] . If the Gaussian curvature of S does not vanish on the support of w, the method of stationary phase [6] applied to the oscillatory integral

$$\widehat{d\mu}(\varrho\vartheta)=\int_S \exp(i\varrho\langle x,\vartheta\rangle)\, w(x)\, dS$$

$\varrho>0$, $|\vartheta|=1$, yields the estimate

$$(1) \qquad \widehat{d\mu}(\varrho\vartheta) \sim C\,\varrho^{-n/2} \sum_{\vartheta\perp T(x_j)} \exp(i\varrho\langle x_j,\vartheta\rangle)\; w(x_j)\, K(x_j)^{-1/2}$$

as $\varrho\rightarrow\infty$, where the sum is extended over the finite set of points x_j in the support of w, such that ϑ is normal to the tangent hyperplane $T(x_j)$ to S at x_j. Thus the estimate

$$(2) \qquad |\widehat{d\mu}(\varrho\vartheta)|\leq C\,\varrho^{-n/2}$$

as $\varrho\rightarrow\infty$ holds uniformly in ϑ, $|\vartheta|=1$. However if the curvature of S vanishes at some point in the support of w then, in general, $\widehat{d\mu}(\varrho\vartheta)$ decays slower than $\varrho^{-n/2}$ as $\varrho\rightarrow\infty$ in some direction ϑ, $|\vartheta|=1$. The asymptotic expansion (1) suggests that in these cases the optimal decay in (2) could be recovered by multiplying the measure $d\mu$ by a suitable power of the curvature. Thus one seeks minimal conditions on the surface S and α which guarantee that the measure $d\mu_\alpha=|K|^\alpha\, d\mu$ satisfies the estimate

$$|\widehat{d\mu}_\alpha(\varrho\vartheta)| \leq C\, \varrho^{-n/2}$$

as $\varrho \to \infty$, uniformly for ϑ in the unit sphere of $\mathbb{R}^{n+1}$. A first result in this direction has been obtained by the authors [1] [2], who proved that $\widehat{d\mu}_{1/2}$ has optimal decay when S, is one of the surfaces in $\mathbb{R}^3$ obtained by revolving around the z axis the curve of equation $x^{2a}+z^{2b}=1$, $a,b \geq 1$. Later Sogge and Stein [10] proved that for any smooth hypersurface S in $\mathbb{R}^{n+1}$ $\widehat{d\mu}_{2n}$ has optimal decay. This result can be improved for certain convex hypersurfaces. Let S be a smooth convex hypersurface in $\mathbb{R}^{n+1}$.

We shall say that S is of **finite type** if at every point x of S every tangent line to S at x makes a contact of finite order with S.

THEOREM *Let* S *be a smooth convex hypersurface of finite type in* $\mathbb{R}^{n+1}$. *Let* α *be the integer part of* $(n+3)/2$. *Then* $\widehat{d\mu}_\alpha$ *satisfies the estimate*

$$|\widehat{d\mu}_\alpha(\varrho\vartheta)| \leq C\, \varrho^{-n/2}$$

uniformly with respect to ϑ *in* $\mathbb{R}^{n+1}$, $|\vartheta| = 1$.

Sketch of the proof. Fix ϑ in $\mathbb{R}^{n+1}$, $|\vartheta|=1$. Then it is well known (see for instance [7]) that one needs only to examine the contribution to $\widehat{d\mu}_\alpha(\varrho\vartheta)$ coming from a small neighborhood of the points where ϑ is normal to S. Let x_0 be one such point. After a rotation and a translation we can assume that x_0 coincides with the origin and that in a neighborhood of x_0 S is the graph of a smooth convex function $f_\vartheta: \mathbb{R}^n \to \mathbb{R}$ such that $f_\vartheta(0)=0$, $\nabla f_\vartheta(0)=0$. Moreover since the function f_ϑ depends continuously on ϑ all the estimates will be uniform in ϑ and we shall forget the dependence on ϑ altogether. Thus matters reduce to estimating an oscillatory integral of the form

$$(3) \qquad I(\varrho)= \int_{\mathbb{R}^n} \exp(i\varrho f(x)) \det(f''(x))^\alpha \, w_1(x)\, dx$$

where $w \in C_c^\infty(\mathbb{R}^n)$. Introducing polar coordinates in $\mathbb{R}^n$, we can write $I(\varrho)$ as an average of one-dimensional oscillatory integrals

$$I(\varrho)= \int_{S_{n-1}} d\omega \int_0^{+\infty} \exp(i\varrho\varphi(t,\omega)) \; \psi^\alpha(t,\omega) \; t^{n-1} \, u(t,\omega)\, dt$$

where $\varphi(t,\omega)=f(t,\omega)$, $\psi(t,\omega)=\det(f''(t\omega)$, for (t,ω) in $\mathbb{R}_+ \times S_{n-1}$. Moreover the functions $t \to \varphi(t,\omega)$, $t \to \psi(t,\omega)$, $t \to u(t,\omega)$ satisfy the following assumption, uniformly in ω.

Assumption A. There exist $q \geq 2$, $\varepsilon > 0$ and constants C_0, M such that for every $p \geq 1$

i) $\varphi \in C^{p+1}$, $\psi \in C^{p-1}$, $u \in C^p$

ii) φ is convex, $\varphi(0)=\varphi'(0)=0$ and $\max\{\, |\varphi^{(i)}(0)| : 2 \leq i \leq q\} \geq \varepsilon$

iii) $0 \leq \psi(t) \leq C_0 \varphi''(t)$ for $0 \leq t \leq 1$

iv) $\|\varphi\|_{(p+1)} + \|\psi\|_{(p-1)} \leq M$

v) $u(t)=1$ if $0\leq t\leq 1/3$, $u(t)\leq 0$ if $2/3\leq t\leq 1$.

Thus the estimate of the oscillatory integral (3) follows from the following van Der Corput type lemma.

LEMMA *Let* k *be an integer* ≥ 1. *There exists* $p_0=p_0(q,k)$ *such that if* φ, ψ, u *satisfy Assumption* A *for* $p\geq p_0$ *then the one dimensional oscillatory integral*

$$I_k(\varrho)=\int_0^1 \exp(i\varrho\varphi(t))\ \psi^{k+1}(t)\ u(t)\ t^{2k-1}\,dt$$

satisfies the estimate

$$|I_k(\varrho)|\leq C\,\varrho^{-k}.$$

Full details shall appear in [3].

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