SOLVABILITY OF DIFFERENTIAL OPERATORS I:

DIRECT AND SEMIDIRECT PRODUCTS OF LIE GROUPS

F.D. Battesti

1. INTRODUCTION

Let G be a Lie group. The group G acts on itself by left (or right) translations. A linear differential operator P on G is said to be left (or right) invariant if it commutes with the left (or right) action of G, i.e. if it satisfies

$$P(f \circ L_g) = (Pf) \circ L_g \quad (\text{or } P(f \circ R_g) = (Pf) \circ R_g)$$

for all $g \in G$, $f \in C^\infty(G)$, where for $x \in G$

$$L_g(x) = gx \quad \text{and} \quad R_g(x) = xg.$$

The operator P is said to be bi-invariant if it is left and right invariant.

All the Lie groups considered in this paper are real.

We identify the algebra of left invariant linear differential operators on G with the complexified universal enveloping algebra

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$U(\underline{\underline{g}})$, where $\underline{\underline{g}}$ is the Lie algebra of G. Bi-invariant differential operators correspond then to elements of the centre $Z(\underline{\underline{g}})$ of $U(\underline{\underline{g}})$.

A distribution $E \in \mathcal{D}'(G)$ on G is a fundamental solution of the operator P if it satisfies the equation $PE = \delta$, where δ is the Dirac distribution at the unit of G.

Left invariant differential operators on a Lie group in general do not possess a global fundamental solution, but under additional conditions either on the operator or on the group, we can prove the existence of such solutions.

2. RESULTS

Let us first recall the main results concerning this problem. In 1955 Ehrenpreis [6] and Malgrange [7] proved that every nonzero differential operator with constant coefficients on R^n has a fundamental solution on R^n. Raïs [8] established the existence of a fundamental solution for every bi-invariant operator on a simply connected nilpotent Lie group and Duflo and Raïs [5] extended this result to simply connected exponential Lie groups. For bi-invariant differential operators on simply connected solvable Lie groups, Duflo and Raïs [5] showed the existence of a local fundamental solution and Rouvière [9] proved the existence of semiglobal fundamental solutions (i.e. on every relatively compact open subset).

In all the above cases, the group G is assumed to be simply connected. This property is essential, as can be seen from the result of Cerezo and Rouvière [4]. They gave a necessary and sufficient condition for left invariant operators on the direct product $G = K \times R^n$, where K is a compact connected Lie group.

For results on semisimple Lie groups we refer the reader to [3].

In the next section we shall recall the results of [1] and [2] concerning the existence of global fundamental solutions for bi-invariant operators on the direct product $H \times K$ where H and K are Lie groups and K is compact connected. We give a necessary condition which coincides with the necessary and sufficient condition of Cerezo and Rouvière when $H = R^n$, and, when the group H is solvable and simply connected, we obtain a sufficient condition very similar to the necessary one. In particular, we prove that every nonzero bi-invariant operator on a simply connected solvable Lie group admits a global fundamental solution.

In the following section we generalize some of these results to certain semidirect products of Lie groups. We define the partial Fourier coefficients of a differential operator on a Cartan motion group $V \rtimes K$ and study the action of K on the elements of the universal enveloping algebra $U(V \oplus \underline{\underline{k}})$. Then, use of the partial Fourier transform on $V \rtimes K$ allows us to translate the problem on $V \rtimes K$ to an equivalent problem on the group V and to prove a necessary condition of existence of a fundamental solution for K-bi-invariant, $V \rtimes K$-left invariant differential operators on $V \rtimes K$.

In the last section, we apply these results to the Euclidean motion group $M(2)$. In this case we explicitly determine K-bi-invariant and $M(2)$-bi-invariant differential operators respectively. We also show that every nonzero bi-invariant operator on $M(2)$ has a global fundamental solution.

3. DIRECT PRODUCT $H \times K$

Let H and K be two Lie groups, K compact connected, and $G = H \times K$ be the direct product. Since K is compact we have a partial Fourier transform on G. Let $\hat{K}$ denote the dual of K. In each equivalence class $\Lambda \in \hat{K}$, we choose an element also denoted Λ: Λ is an unitary irreducible representation of K. The partial Fourier coefficients of an operator P are defined by

$$(P_\Lambda f)(x) = P(f \otimes \Lambda)(x, e_K)$$

for every $f \in \mathcal{D}(H)$ and $\Lambda \in \hat{K}$ ($\mathcal{D}(H)$ denotes the space of compactly supported C^∞-functions on H with its usual topology).

The P_Λ's are differential operators on H, with coefficients in $\mathrm{End}(H_\Lambda)$, where H_Λ is the representation space of Λ and $\mathrm{End}(H_\Lambda)$ is the space of endomorphisms of H_Λ. It is easily seen (using Schur's lemma) that if the operator P is G-bi-invariant then the P_Λ's are H-bi-invariant scalar operators.

THEOREM 1: *Let* H,K *be Lie groups,* K *compact connected,* H *arbitrary and* U *an open subset in* H. *Let* P *be a linear differential operator, bi-invariant on the product* $H \times K$.

Then P *has a fundamental solution on* $U \times K$ *if and only if there exists for each* $\Lambda \in \hat{K}$ *a distribution* E_Λ *on* U *such that*

$$P_\Lambda E_\Lambda = \delta_H$$

and

$$\begin{cases} \text{for every compact subset } C \text{ in } U, \text{ there exist a constant} \\ A > 0 \text{ and positive integers } a \text{ and } b \text{ such that} \\ \forall f \in \mathcal{D}(C),\ \forall \Lambda \in \hat{K}, \quad |\langle E_\Lambda, f\rangle| \leq AN(\Lambda)^a \|f\|_b \end{cases}$$

Here the seminorms $\| \ \|_b$, $b \in \mathbb{N}$, define the topology of $\mathcal{D}(H)$ and N is a positive function on $\hat{K}$.

So it is equivalent to study the existence of a family of fundamental solutions for the partial Fourier coefficients P_Λ, satisfying a growth condition.

From this theorem we deduce a necessary condition for the existence of a fundamental solution for P.

Let $X_1,\ldots,X_n$ be a basis of the Lie algebra $\underline{\underline{h}}$ of H. Then, according to the Poincaré-Birkhoff-Witt theorem, the $X^\alpha = X_1^{\alpha_1}\ldots X_n^{\alpha_n}$,

$\alpha \in N^n$, form a basis of the complexified universal enveloping algebra $\underline{\underline{U}}(\underline{\underline{h}})$ of $\underline{\underline{h}}$.

We define a norm on $\underline{\underline{U}}(\underline{\underline{h}})$, for the operator $Q = \sum_{\alpha \in N^n} a_\alpha X^\alpha$, as

$$\|Q\| = \sum_{\alpha \in N^n} |a_\alpha|,$$

where $a_\alpha \in C$.

THEOREM 2: *Let* H *and* K *be real Lie groups,* K *compact connected, and* U *an open neighbourhood of* e_H *in* H. *Let* P *be a bi-invariant linear differential operator on* H × K *and* P_Λ *its partial Fourier coefficients. The condition*

$$\exists A > 0, \exists a \in N, \forall \Lambda \in \hat{K}, \|P_\Lambda\| > AN(\Lambda)^{-a}$$

is a necessary condition for existence of a fundamental solution for P *on* U × K.

If $H = R^n$, this condition is exactly the necessary and sufficient condition of Cerezo and Rouvière. To prove that this condition is sufficient, Cerezo and Rouvière [4] used the Fourier inversion formula (generalizing Hormander's construction) and they chose a new contour of integration which avoids the singularities. In the general case, this condition is probably sufficient but we don't know how to prove it. I proved a sufficient condition, very close to this one, using different methods.

The notion of P-convexity enables us to obtain global solutions from semiglobal solutions (i.e. solutions on every relatively compact open subset).

DEFINITIONS: Let G be a Lie group, Ω an open subset of G and P a linear differential operator on G. Ω is P-convex if for every compact subset $L \subset \Omega$, there exists a compact subset $L' \subset \Omega$ such that

$$\operatorname{supp} {}^tP\psi \subset L \implies \operatorname{supp} \psi \subset L'$$

for all $\psi \in \mathcal{D}(\Omega)$.

A compact subset $L \subset G$ is said to be P-full if

$$\operatorname{supp} P\psi \subset L \implies \operatorname{supp} \psi \subset L$$

for all $\psi \in \mathcal{D}(G)$.

EXAMPLE In R^n, the convex subsets are P-full. In the definition of P-convex, for L' one can take the convex hull of L.

THEOREM 3: *Let* G *be a second countable Lie group and* P *a linear differential operator on* G *such that*

(i) Every compact subset of G *is contained in a* P*-full compact subset of* G.

(ii) P *has a fundamental solution on every relatively compact open subset of* G.

Then P *has a global fundamental solution on* G *and*

$PC^\infty(G) = C^\infty(G)$.

The proof of this theorem uses general methods in functional analysis.

COROLLARY 4: *Let* G *be a simply connected solvable Lie group. Every nonzero bi-invariant linear differential operator* P *on* G *has a global fundamental solution on* G *and* $PC^{\infty}(G) = C^{\infty}(G)$.

Therefore, to study the existence of global solutions it is enough to study the existence of semiglobal solutions and P-convexity.

The P-convexity of $H \times K$ corresponds by Fourier transform to a property of uniform convexity of H for the P_{Λ}'s. I proved this property when H is solvable simply connected.

To show the semiglobal solvability of P, I used Rouvière's method which generalizes to solvable Lie groups Hormander's method based on L^2-inequalities on R^n. In the case of one linear differential operator on a solvable Lie group, Rouvière [9] proved some inequalities and deduced from them the existence of semiglobal fundamental solutions for this operator. I calculated explicitly the constants in the inequalities and studied their dependance on the operator.

The "winning coefficients" of a nonzero operator P are the coefficients which may occur in the final inequality. These coefficients turn out to be the maximal elements (in the sense of a particular order on N^n) $\alpha \in N^n$ such that $\partial^{\alpha} P \neq 0$.

Applying this method to the partial Fourier coefficients P_Λ of the operator P, together with theorems 1 and 3, allows us to prove a sufficient condition for the existence of a global fundamental solution for P on $H \times K$.

THEOREM 5: *Let* H *and* K *be real Lie groups,* H *solvable simply connected,* K *compact connected and* G *be the direct product* $H \times K$. *Let* P *be a bi-invariant linear differential operator on* G *and* P_Λ, $\Lambda \in \hat{K}$, *its partial Fourier coefficients.*

If for each $\Lambda \in \hat{K}$ *the operator* ${}^t(P_\Lambda)$ *is nonzero and if the following condition is satisfied*

$$\exists A > 0, \exists a \in \mathbb{N}, \forall \Lambda \in \hat{K}, \ \|{}^t(P_\Lambda)\|' \geq AN(\Lambda)^{-a}$$

then the operator P *has a fundamental solution on* G *and* $PC^\infty(G) = C^\infty(G)$.

If $Q = \sum_{\alpha \in \mathbb{N}^n} a_\alpha X^\alpha \in \underline{\underline{U}}(h)$ is nonzero, then $\|Q\|' = \sum |a_\beta|$ for a_β winning coefficient.

4. CARTAN MOTION GROUPS

Some of the above results can be extended to semidirect products of Lie groups. In this section we study the existence of fundamental solutions for left invariant linear differential operators on Cartan motion groups.

Let G be a Lie group. A subgroup K of G is said to be reductive in G if there exists a vector space V such that

$$\underline{\underline{g}} = \underline{\underline{k}} \oplus V \quad \text{and} \quad Ad(K)V \subset V$$

where $\underline{\underline{g}}$ and $\underline{\underline{k}}$ denote the Lie algebras of G and K respectively and Ad is the adjoint representation of G.

When the group G is semisimple, connected, with finite centre and K is a maximal compact subgroup of G, the semidirect product $V \rtimes K$ of V by K relative to this action is called the Cartan motion group associated to the pair (G,K).

The multiplication law in $V \rtimes K$ is given by $(v,k)(v',k') = (v+k.v',kk')$ for all $v,v' \in V$ and $k,k' \in K$, where

$$k.v' = Ad(k)(v').$$

In this situation, the relationship between existence of fundamental solutions for differential operators on G and $V \rtimes K$ respectively is studied in [3], using contraction maps.

Since the group K is compact, we have a partial Fourier transform on $V \rtimes K$.

Let P be a left invariant linear differential operator on $V \rtimes K$. As in the case of a direct product, the partial Fourier coefficients P_Λ, $\Lambda \in \hat{K}$, of P are defined by

$$(P_\Lambda f)(v) = P(f \otimes \Lambda)(v,e_K)$$

for every $f \in \mathcal{D}(V)$ and $\Lambda \in \hat{K}$. They are left invariant differential operators on V.

Let $(X_1,\dots,X_n)$ and $(T_1,\dots,T_p)$ be bases of V and $\underline{\underline{k}}$ respectively. Then $(X_1,\dots,X_n,T_1,\dots,T_p)$ is a basis of the Lie algebra $V \oplus \underline{\underline{k}}$ of $V \rtimes K$ (here $V \oplus \underline{\underline{k}}$ is a direct sum of vector spaces).

Let $X_{V_1},\dots,X_{V_n}$, $T_{K_1},\dots,T_{K_p}$, $X_{V\rtimes K_1},\dots,X_{V\rtimes K_n}$, $T_{V\rtimes K_1},\dots,T_{V\rtimes K_p}$ denote the corresponding left invariant vector fields on V, K and $V \rtimes K$ respectively. For $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ and $\beta = (\beta_1,\dots,\beta_p) \in \mathbb{N}^p$, we set

$$X_V^\alpha = X_{V_1}^{\alpha_1}\dots X_{V_n}^{\alpha_n}, \quad X_{V\rtimes K}^\alpha = X_{V\rtimes K_1}^{\alpha_1}\dots X_{V\rtimes K_n}^{\alpha_n}$$

and

$$T_K^\beta = T_{K_1}^{\beta_1}\dots T_{K_p}^{\beta_p}; \quad T_{V\rtimes K}^\beta = T_{V\rtimes K_1}^{\beta_1}\dots T_{V\rtimes K_p}^{\beta_p}.$$

LEMMA 6: *(i) For all* $X' \in V$, $T' \in \underline{\underline{k}}$, $f \in \mathcal{D}(V)$ *and* $g \in C^\infty(K)$ *we have*

$$X'_{V\rtimes K}(f \otimes g)(v,k) = (k.X'_V f)(v)g(k)$$

and

$$T'_{V\rtimes K}(f \otimes g)(v,k) = f(v)(T'_K g)(k)$$

(ii) For all $X' \in V$, $f \in \mathcal{D}(V)$, $k \in K$, $\alpha \in \mathbb{N}$ *and* $v \in V$, *we have*

$$(k.X'_V)^\alpha f(v) = [X'^{\alpha}_V(f \circ Ad(k)) \circ Ad(k^{-1})](v).$$

PROOF (i) We have, for $X' \varepsilon V$,

$$X'_{V \rtimes K}(f \otimes g)(v,k) = \frac{d}{dt}\Big|_{t=0} (f \otimes g)((v,k)(tX',e_K))$$

$$= \frac{d}{dt}\Big|_{t=0} (f \otimes g)(v+tk.X',k)$$

$$= \frac{d}{dt}\Big|_{t=0} [f(v+tk.X')g(k)] = \frac{d}{dt}\Big|_{t=0} (f(v+tk.X'))g(k)$$

$$= (k.X'_V f)(v)g(k)$$

and, for $T' \varepsilon \underline{\underline{k}}$,

$$T'_{V \rtimes K}(f \otimes g)(v,k) = \frac{d}{dt}\Big|_{t=0} (f \otimes g)((v,k)(0_V,\exp(tT')))$$

$$= \frac{d}{dt}\Big|_{t=0} (f \otimes g)(v,k\exp(tT'))$$

$$= \frac{d}{dt}\Big|_{t=0} [f(v)g(k\exp(tT'))]$$

$$= f(v) \frac{d}{dt}\Big|_{t=0} g(k\exp(tT'))$$

$$= f(v)(T'_K g)(k).$$

(ii) Let $f \varepsilon \mathcal{D}(V)$ and put $g(v) = f(k.v)$ for all $v \varepsilon V$, i.e. $g = f \circ Ad(k)$ $(k \varepsilon K)$.

Then, for $X' \varepsilon V$,

$$(k.X'_V f)(v) = \frac{d}{dt}\Big|_{t=0} f(v+tk.X')$$

$$= \frac{d}{dt}\Big|_{t=0} f(k.(k^{-1}.v+tX'))$$

$$= \frac{d}{dt}\Big|_{t=0} g(k^{-1}.v+tX')$$

$$= (X'_V g)(k^{-1}.v).$$

So

$$k.X'_V f = X'_V(f \circ Ad(k)) \circ Ad(k^{-1})$$

and, by induction,

$$(k.X_V')^{\alpha} f = X_V'^{\alpha}(f \circ Ad(k)) \circ Ad(k^{-1}).$$

q.e.d.

Let $P \in U(V \oplus k)$. According to the Poincaré-Birkhoff-Witt theorem, P can be written

$$P = \sum_{\alpha \in N^n, \beta \in N^p} a_{\alpha\beta} X^{\alpha}_{V \rtimes K} T^{\beta}_{V \rtimes K}$$

where $a_{\alpha\beta} \in C$.

Then, for every $f \in \mathcal{D}(V)$ and $\Lambda \in \hat{K}$, we have

$$P(f \otimes \Lambda)(v,k) = \sum_{\alpha,\beta} a_{\alpha\beta}[(k.X_V)^{\alpha} f](v)\Lambda(k)T^{\beta}_{\Lambda}$$

and

$$P_{\Lambda} = \sum_{\alpha,\beta} a_{\alpha\beta} X^{\alpha}_V T^{\beta}_{\Lambda}$$

where $T_{\Lambda} = \frac{d}{dt}\Big|_{t=0} \Lambda(\exp(tT_K))$.

PROOF: For $f \in \mathcal{D}(V)$ and $\Lambda \in \hat{K}$ we have

$$P(f \otimes \Lambda)(v,k) = [(\sum_{\alpha,\beta} a_{\alpha\beta} X^{\alpha}_{V \rtimes K} T^{\beta}_{V \rtimes K})(f \otimes \Lambda)](v,k)$$

$$= \sum_{\alpha,\beta} a_{\alpha\beta} X^{\alpha}_{V \rtimes K}(T^{\beta}_{V \rtimes K}(f \otimes \Lambda))(v,k)$$

and applying lemma 6

$$P(f \otimes \Lambda)(v,k) = \sum_{\alpha,\beta} a_{\alpha\beta} X^{\alpha}_{V \rtimes K}(f \otimes (T^{\beta}_K \Lambda))(v,k).$$

But $T^{\beta}_K \Lambda(k) = \Lambda(k) T^{\beta}_{\Lambda}$ where $T_{\Lambda} = \frac{d}{dt}\Big|_{t=0} \Lambda(\exp(tT_K))$

So
$$P(f \otimes \Lambda)(v,k) = \sum_{\alpha,\beta} a_{\alpha\beta} X^{\alpha}_{V \rtimes K}(f \otimes (\Lambda T^{\beta}_{\Lambda}))(v,k)$$

$$= \sum_{\alpha,\beta} a_{\alpha\beta} X^{\alpha}_{V \rtimes K}(f \otimes \Lambda)(v,k) T^{\beta}_{\Lambda}$$

and using lemma 6 again

$$P(f \otimes \Lambda)(v,k) = \sum_{\alpha,\beta} a_{\alpha\beta}[(k.X_V)^{\alpha}f](v)\Lambda(k)T_{\Lambda}^{\beta}.$$

Now $(P_{\Lambda}f)(v) = P(f \otimes \Lambda)(v,e_K)$

$$= [\sum_{\alpha,\beta} a_{\alpha\beta}[(k.X_V)^{\alpha}f](v)\Lambda(k)T_{\Lambda}^{\beta}]_{|k=e_K}$$

$$= \sum_{\alpha,\beta} a_{\alpha\beta}(X_V^{\alpha}f)(v)T_{\Lambda}^{\beta}$$

hence

$$P_{\Lambda} = \sum_{\alpha,\beta} a_{\alpha\beta}\, X_V^{\alpha}\, T_{\Lambda}^{\beta}.$$

q.e.d.

Furthermore, if we choose a system of coordinates $(x_1,\dots,x_n)$ on V, the partial Fourier coefficients P_{Λ}, $\Lambda \in \hat{K}$, of P can be written

(1) $$P_{\Lambda} = Q_{\Lambda}(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$$

where Q_{Λ} is a polynomial of n variables, with coefficients in $\mathrm{End}(H_{\Lambda})$.

Let U be an open subset in V and E be a distribution on $U \rtimes K$.

The partial Fourier coefficients $\hat{E}(v,\Lambda)$ of E are defined as follows:

$$\langle\hat{E}(v,\Lambda),f(v)\rangle = \langle E(v,k),f(v)\Lambda(k)\rangle$$

for all $\Lambda \in \hat{K}$ and $f \in \mathcal{D}(U)$.

They are distributions on U with values in $End(H_\Lambda)$. Let $\mathcal{D}'(U,End(H_\Lambda))$ denote the space of distributions on U with values in $End(H_\Lambda)$.

If u is an endomorphism of a vector space, $\|u\|_{HS}$ denotes the Hilbert-Schmidt norm of u, i.e.

$$\|u\|_{HS} = \sqrt{tr(uu^*)}$$

where u^* is the adjoint of u and $tr(uu^*)$ is the trace of uu^*.

We have the following characterization of distributions on $U \rtimes K$:

PROPOSITION 7: *Let* $(\hat{E}_\Lambda)_{\Lambda \in \hat{K}}$ *be a family of distributions in* $\mathcal{D}'(U,End(H_\Lambda))$. *The distributions* E_Λ *are the partial Fourier coefficients* $\hat{E}(v,\Lambda)$ *of a distribution* E *on* $U \rtimes K$ *if and only if for every compact subset* C *in* U, *there exist a constant* $A > 0$ *and positive integers* a *and* b *such that*

$$\|\langle E_\Lambda, f\rangle\|_{HS} \leqslant AN(\Lambda)^a \|f\|_b$$

for all $\Lambda \in \hat{K}$ *and* $f \in \mathcal{D}(U)$ *such that* $\operatorname{supp} f \subset C$.

This result was proved in [4] in the case of the direct product $R^n \times K$. The proof given in [4] can be easily adapted to the case of the semidirect product $V \rtimes K$.

PROPOSITION 8: *Let* P *be a left invariant linear differential operator on the Cartan motion group* $V \rtimes K$.

If the operator P *is* K*-bi-invariant, then we have*

$$\widehat{PE}(v,\Lambda) = {}^t[({}^tP)_\Lambda]\hat{E}(v,\Lambda) \tag{2}$$

for every $E \in \mathcal{D}'(V \rtimes K)$ *and* $\Lambda \in \hat{K}$.

PROOF: Let $E \in \mathcal{D}'(V\times K)$, $\Lambda \in \hat{K}$ and $f \in \mathcal{D}(V)$.

By definition of the partial Fourier coefficients of a distribution, we have

$$\langle \widehat{PE}(v,\Lambda), f(v)\rangle = \langle PE(v,k), f(v)\Lambda(k)\rangle$$

$$= \langle E(v,k), {}^{t}P(f \otimes \Lambda)(v,k)\rangle.$$

But ${}^{t}P(f \otimes \Lambda)(v,k) = [{}^{t}P(f \otimes \Lambda)] \circ R_{(0_V,k)}(v,e_K)$

$$= {}^{t}P[(f \otimes \Lambda) \circ R_{(0_V,k)}](v,e_K)$$

since the operator P is K-bi-invariant.

Now,

$$[(f \otimes \Lambda) \circ R_{(0_V,k)}](v',k') = (f \otimes \Lambda)(v',k'k)$$

$$= f(v')\Lambda(k'k)$$

$$= f(v')\Lambda(k')\Lambda(k)$$

$$= (f \otimes \Lambda)(v',k')\Lambda(k).$$

Therefore

$${}^{t}P[(f \otimes \Lambda) \circ R_{(0_V,k)}](v,e_K) = [{}^{t}P(f \otimes \Lambda)](v,e_K)\Lambda(k)$$

$$= [({}^{t}P)_{\Lambda}f](v)\Lambda(k)$$

and

$$\langle \widehat{PE}(v,\Lambda), f(v)\rangle = \langle E(v,k), [({}^{t}P)_{\Lambda}f](v)\Lambda(k)\rangle$$

$$= \langle \hat{E}(v,\Lambda), [({}^{t}P)_{\Lambda}f](v)\rangle$$

$$= \langle {}^{t}[({}^{t}P)_{\Lambda}]\hat{E}(v,\Lambda), f(v)\rangle.$$

q.e.d.

From these two propositions we deduce the following theorem:

THEOREM 9: *Let* $P \in U(V \oplus \underline{\underline{k}})$ *be* K-*bi-invariant and let* U *be an open subset in* V.

Then P *has a fundamental solution on* $U \rtimes K$ *if and only if, for each* $\Lambda \in \hat{K}$, *there exists a distribution* $E_\Lambda \in \mathcal{D}'(U, \mathrm{End}(H_\Lambda))$ *on* U *such that*

(3) $$^t[(^tP)_\Lambda]E_\Lambda = \delta_V$$

and

(4) $$\begin{cases} \text{for every compact subset } C \subset U, \text{ there exist a constant} \\ A > 0 \text{ and positive integers } a \text{ and } b \text{ such that} \\ \forall f \in \mathcal{D}(C),\ \forall \Lambda \in \hat{K},\quad \| \langle E_\Lambda, f \rangle \|_{HS} \leq AN(\Lambda)^a \|f\|_b . \end{cases}$$

PROOF By partial Fourier transform on K, the equation $PE = \delta_{V \rtimes K}$ is equivalent to

$$\forall \Lambda \in \hat{K} \qquad \widehat{PE}(v,\Lambda) = \hat{\delta}_{V \rtimes K}(v,\Lambda).$$

But $\hat{\delta}_{V \rtimes K}(v,\Lambda) = \delta_V(v)$ and since the operator P is K-bi-invariant, according to proposition 8, we have

$$\widehat{PE}(v,\Lambda) = {}^t[(^tP)_\Lambda]\hat{E}(v,\Lambda).$$

So, P has a fundamental solution on $U \rtimes K$ if and only if, for each $\Lambda \in \hat{K}$, there exists a distribution $E_\Lambda \in \mathcal{D}'(U, \mathrm{End}(H_\Lambda))$ on U such that

$$^t[(^tP)_\Lambda]E_\Lambda = \delta_V$$

and the distributions E_Λ, $\Lambda \in \hat{K}$, are the partial Fourier coefficients of a distribution on $U \rtimes K$.

By proposition 7, this condition is equivalent to the inequalities (4).

q.e.d.

This theorem allows us to give a necessary condition for the existence of a fundamental solution for K-bi-invariant differential operators on the Cartan motion group $V \rtimes K$.

If M is a matrix in $End(H_\Lambda)$, let us introduce the comatrix ${}^{co}M$ of M, that is the transpose of the matrix of the cofactors of M. We have

$$ {}^{co}M.M = M.{}^{co}M = \det M.Id_{H_\Lambda} $$

where $\det M$ is the determinant of the matrix M.

Let $M(\xi)$ be a matrix in $End(H_\Lambda)$ such that its coefficients with respect to an orthonormal basis of H_Λ are polynomials $m_{ij}(\xi)$. Then $\tilde{M}(\xi)$ denotes the matrix whose coefficients with respect to this basis are $\tilde{m}_{ij}(\xi)$, where

$$ \tilde{m}_{ij}(\xi) = \Big(\sum_{\alpha \in N} |m_{ij}^{(\alpha)}(\xi)|^2 \Big)^{1/2}. $$

This definition does not depend on the choice of the orthonormal basis in H_Λ and one has

$$ \|\tilde{M}(\xi)\|_{HS}^2 = \sum_{\alpha \in N} \|M(\xi)^{(\alpha)}\|_{HS}^2. $$

THEOREM 10: *Let* $V \rtimes K$ *be a Cartan motion group and* U *an open neighbourhood of the origin in* V. *Let* P *be a* K-*bi-invariant,* $V \rtimes K$-*left invariant linear differential operator on* $V \rtimes K$ *and,* *for* $\Lambda \in \hat{K}$, *let* Q_Λ *be the polynomial defined by* (1) *for the operator* ${}^t[({}^tP)_\Lambda]$.

The following condition

$$\exists A > 0, \exists a \in \mathbb{N}, \forall \Lambda \in \hat{K}$$

(5) $$\det Q_\Lambda(\xi) \not\equiv 0 \quad \text{and} \quad \left\| \frac{({}^{co}Q_\Lambda(0))^\sim}{(\det Q_\Lambda(0))^\sim} \right\|_{HS} \leq AN(\Lambda)^a$$

is a necessary condition for the existence of a fundamental solution for P *on* $U \rtimes K$.

PROOF Let $P \in U(V \oplus \underline{\underline{k}})$ be K-bi-invariant.

If P has a fundamental solution on $U \rtimes K$, then, according to theorem 9, for each $\Lambda \in \hat{K}$, there exists a distribution $E_\Lambda \in \mathcal{D}'(U,\mathrm{End}(H_\Lambda))$ on U satisfying (3) and (4).

Since the operators ${}^t[({}^tP)_\Lambda]$ are differential operators on the vector space V, we can use the same arguments as in [4] to prove the necessity of the condition (5). The idea of the proof is to apply the inequalities (4) to particular functions. For the details we refer the reader to [4], p.576-577.

q.e.d.

5. THE EUCLIDEAN MOTION GROUP

In this section we apply the above results to differential operators on the Euclidean motion group $M(2)$.

The group $M(2)$ is the Cartan motion group $V \rtimes K$ when $V = R^2$ and $K = SO(2)$. In this case the group K acts by rotations on the Euclidean plane.

If we choose a system of coordinates (x,y,θ) on $M(2)$, with $x,y, \in R$ and $\theta \in T = R/2\pi Z$, then the multiplication is given by

$$(x,y,\theta)(x',y',\theta') = (x+x'\cos\theta-y'\sin\theta, y+y'\cos\theta+x'\sin\theta, \theta+\theta')$$

Let $\mathfrak{m}(2)$ be the Lie algebra of $M(2)$, and let (X,Y,T) be a basis of $\mathfrak{m}(2)$ with the brackets

$$\begin{cases} [X,Y] = 0 \\ [X,T] = -Y \\ [Y,T] = X \end{cases}$$

$(X,Y \in V$ and $T \in \mathfrak{so}(2))$.

The corresponding left invariant vector fields are given by,

for $f \in \mathcal{D}(M(2))$,

$$\begin{cases} (X_{M(2)}f)(x,y,\theta) = \cos\theta \dfrac{\partial f}{\partial x} + \sin\theta \dfrac{\partial f}{\partial y} \\ (Y_{M(2)}f)(x,y,\theta) = -\sin\theta \dfrac{\partial f}{\partial x} + \cos\theta \dfrac{\partial f}{\partial y} \\ (T_{M(2)}f)(x,y,\theta) = \dfrac{\partial f}{\partial \theta} \end{cases}$$

and for $g \in \mathcal{D}(V)$, $h \in \mathcal{D}(K)$,

$$\begin{cases} (X_V g)(x,y) = \dfrac{\partial g}{\partial x} \\ (Y_V g)(x,y) = \dfrac{\partial g}{\partial y} \\ (T_K h)(\theta) = \dfrac{\partial h}{\partial \theta} \end{cases}$$

The dual $\hat{K}$ of K is here isomorphic to Z and we have, for $n \in Z$,

$$\Lambda_n(\theta) = e^{in\theta}.$$

Thus, if

$$P = \sum_{\alpha,\beta,\gamma \in \mathbb{N}} a_{\alpha\beta\gamma} X^{\alpha}_{M(2)} Y^{\beta}_{M(2)} T^{\gamma}_{M(2)} \in \underset{=}{U}(m(2))$$

the partial Fourier coefficients of P are given by

$$P_n = \sum_{\alpha,\beta,\gamma \in \mathbb{N}} a_{\alpha\beta\gamma}(in)^{\gamma} X^{\alpha}_V Y^{\beta}_V$$

for all $n \in \mathbb{Z}$.

LEMMA 11: *Let* P *be a left invariant linear differential operator on* $M(2)$.

(i) *If* P *is* K*-bi-invariant then*

$$P = \sum_{\alpha,\beta \in \mathbb{N}} a_{\alpha\beta}(X^2_{M(2)} + Y^2_{M(2)})^{\alpha} T^{\beta}_{M(2)} .$$

(ii) *If* P *is* $M(2)$*-bi-invariant then*

$$P = \sum_{\alpha \in \mathbb{N}} a_{\alpha}(X^2_{M(2)} + Y^2_{M(2)})^{\alpha}.$$

PROOF: Since there is no ambiguity here (we consider only vector fields on $M(2)$), we shall omit the subscript $M(2)$ in this proof.

(i) P is K-bi-invariant if and only if it satisfies $[P,T] = 0$. In the symmetric algebra, this condition can be written

$$[P,T] = \frac{\partial P}{\partial X}[X,T] + \frac{\partial P}{\partial Y}[Y,T]$$

$$= -\frac{\partial P}{\partial X}Y + \frac{\partial P}{\partial Y}X = 0 .$$

This implies that P is of the form

$$P = Q(X^2+Y^2,T)$$

where Q is a polynomial of two variables, i.e.

$$P = \sum_{\alpha,\beta\varepsilon N} a_{\alpha\beta}(X^2_{M(2)} + Y^2_{M(2)})^{\alpha}\, T^{\beta}_{M(2)}.$$

(ii) P is M(2)-bi-invariant if and only if it satisfies $[P,X] = [P,Y] = [P,T] = 0$.

In the symmetric algebra, we have

$$[P,X] = \frac{\partial P}{\partial Y}[Y,X] + \frac{\partial P}{\partial T}[T,X]$$
$$= \frac{\partial P}{\partial T}Y = 0.$$

Therefore $\frac{\partial P}{\partial T} = 0$.

Also $$[P,Y] = 0 \Rightarrow \frac{\partial P}{\partial T} = 0$$

and since P is K-bi-invariant, we have

$$P = \sum_{\alpha\varepsilon N} a_{\alpha}(X^2_{M(2)} + Y^2_{M(2)})^{\alpha}.$$

q.e.d.

Let $P \varepsilon \underline{\underline{U}}(m(2))$ be K-bi-invariant.

Then

$$P = \sum_{\alpha,\beta\varepsilon N} a_{\alpha\beta}(X^2_{M(2)} + Y^2_{M(2)})^{\alpha}\, T^{\beta}_{M(2)}$$

and

$${}^tP = \sum_{\alpha,\beta\varepsilon N} a_{\alpha\beta}(X^2_{M(2)} + Y^2_{M(2)})^{\alpha}({}^tT_{M(2)})^{\beta}$$

(since $(X^2_{M(2)} + Y^2_{M(2)})^{\alpha}$ is M(2)-bi-invariant and ${}^t(X^2_{M(2)} + Y^2_{M(2)}) = X^2_{M(2)} + Y^2_{M(2)}$).

So, for all $n \varepsilon Z$, we have

$$({}^tP)_n = \sum_{\alpha,\beta\varepsilon N} a_{\alpha\beta}(-in)^{\beta}(X_V^2+Y_V^2)^{\alpha}$$
$$= P_{-n}$$

and

$$^{t}[(^{t}P)_n] = {}^{t}(P_{-n}) = P_{-n}.$$

Therefore, (2) can be written

$$\widehat{PE}(v,n) = P_{-n}\hat{E}(v,n).$$

THEOREM 12: *Every nonzero bi-invariant linear differential operator on the Euclidean motion group has a global fundamental solution on* M(2).

PROOF: Let P be a nonzero bi-invariant linear differential operator on $M(2)$. By lemma 11(ii), we have

$$P = \sum_{\alpha \varepsilon \mathbb{N}} a_\alpha (X^2_{M(2)} + Y^2_{M(2)})^\alpha .$$

So, for every $n \varepsilon \mathbb{Z}$

$$P_n = \sum_\alpha a_\alpha (X_V^2 + Y_V^2)^\alpha = \sum_\alpha a_\alpha \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^\alpha = P.$$

Now, P (considered as an operator on V) has a global fundamental solution F on V.

Putting $E_n = F$ for every $n \varepsilon \mathbb{Z}$, the family $(E_n)_{n\varepsilon\mathbb{Z}}$ satisfies (4) since F is a distribution: for every compact subset C in V, there exist a constant $A > 0$ and a positive integer b such that

$$\forall f \varepsilon \mathcal{D}(C), \quad |\langle F,f\rangle| \leq A\|f\|_b$$

(take $U = V$, and $a = 0$ in (4)).

Hence we have the existence of a global fundamental solution for P on $M(2)$ by theorem 9.

q.e.d.

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Françoise Battesti
School of Mathematics
University of New South Wales
Kensington 2033 N.S.W.
Australia.