INTEGRATION FOR THE SPECTRAL THEORY

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Let E be a complex Banach space. Let $B(E)$ be the algebra of all bounded linear operators on E. Then $B(E)$ is a Banach algebra with respect to the operator (uniform) norm defined by $\|T\| = \sup\{|Tx| : |x| \le 1, x \in E\}$, for every $T \in B(E)$. By I is denoted the identity operator.

A spectral measure is a multiplicative and σ-additive (in the strong operator topology) map $P : \mathcal{Q} \to B(E)$, whose domain, $\mathcal{Q}$, is a σ-algebra of sets in a space Ω, such that $P(\Omega) = I$. An operator $T \in B(E)$ is said to be of scalar type if there exists a spectral measure P and a P-integrable function f such that

(1) $$T = \int_\Omega f dP.$$

This notion, due to N. Dunford, extends to arbitrary Banach space the idea of an operator with diagonalizable matrix on a finite-dimensional space. It proved to be very fruitful as shows the exposition in the monograph [3]. Many powerful techniques in which scalar operators play a role are based on the requirements that $\mathcal{Q}$ be a σ-algebra and that P be σ-additive. But precisely these requirements are reponsible for excluding many operators of prime interest from the class of scalar-type operators. Suggestions for extending this class lead to new interesting theories.

So, C. Foias introduced the notion of a generalized scalar operator, replacing the algebra of all bounded measurable functions by some other, possibly poorer algebras of functions and the integration map by certain homomorphisms of such algebras into $B(E)$. The resulting theory is systematically presented in [1].

The theory of well-bounded operators, having its origin in the work of D.R. Smart and J.R. Ringrose, is discussed in the monograph [2]; see also the relevant part of Section XV.16 in [3]. It uses the fact that, even if the set function P is not σ-additive and is not defined on a σ-algebra, it may still be possible to introduce the integral with respect to P, based on strong operator convergence, for sufficiently many functions.

The theory of extended spectral operators, due to W. Ricker [7], is not yet available in a monograph form. Its point of departure is the observation that the failure of an operator T to be of scalar type may be, so to say, not the fault of the operator T itself but, rather, of the space E. Indeed, there often exist a space F, continuously and densely containing E, and an extension, S, of the operator T, by continuity, onto the whole of F such that S is a scalar-type operator.

The purpose of this note is to propose still another generalization of the notion of a scalar-type operator. It is suggested by the well-known fact that, if the integral (1) exists, then there exist $\mathcal{Q}$-simple functions f_j, $j = 1,2,\ldots,$ such that

(2) $$\sum_{j=1}^{\infty} \left\| \int_{\Omega} f_j \, dP \right\| < \infty$$

and the equality

(3) $$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

holds for every $\omega \in \Omega$ for which

(4) $$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty.$$

In that case,

(5) $$\int_{\Omega} f \mathrm{d}P = \sum_{j=1}^{\infty} \int_{\Omega} f_j \, \mathrm{d}P.$$

So, the integral (1) can be defined purely in terms of the operator norm convergence. Consequently, it is not necessary to assume that the set function P be bounded, let alone σ-additive, nor that $\mathcal{Q}$ be a σ-algebra. These assumptions can be replaced by less stringent ones which nevertheless guarantee that the integral (1) is defined unambiguously, that the operator T can be approximated by linear combinations of disjoint projection operators - values of P - that the spectrum of T is equal to the essential range of the function f and that the family of all operators so expressed, with fixed P but varying f, is a semisimple commutative Banach algebra.

Let Ω be a non-emtpy set to be called the space. To save subscripts and circumlocution, subsets of Ω will be identified with their characteristic functions. Let $\mathcal{Q}$ be an algebra of sets in the space Ω. The vector space of all $\mathcal{Q}$-simple functions is denoted by sim($\mathcal{Q}$).

An additive and multiplicative map $P : \mathcal{Q} \to B(E)$ such that $P(\Omega) = I$ will be called a $B(E)$-valued spectral set function on $\mathcal{Q}$. A spectral set function is not distinguished in the notation from its unique linear $B(E)$-valued extension onto the whole of sim($\mathcal{Q}$).

Given a spectral set function P, let us call P-null any set $Y \subset \Omega$ for which there exist sets $X_j \in \mathcal{Q}$ such that $P(X_j) = 0$, for every $j = 1,2,\ldots$, and

$$Y \subset \bigcup_{j=1}^{\infty} X_j.$$

For a function f on Ω, let

$$\|f\|_{\infty} = \inf\{\sup\{|f(\omega)| : \omega \in \Omega \backslash Y\} : Y \in N\},$$

where N is the family of all P-null sets. Then $0 \le \|f\|_\infty \le \infty$. Following the custom, we shall call P-null any function f on Ω such that $\|f\|_\infty = 0$. The P-equivalence class of a function f will be denoted by $[f]$. To be sure, $[f]$ is the set of all functions g on Ω such that $\|f - g\|_\infty = 0$. Of course, when there is no danger of confusion, we use the usual licence which allows us not to distinguish between a function and its equivalence class.

Let $L^\infty(P)$ be the family of all functions f on Ω such that, for every $\varepsilon > 0$, there exists a function $g \in \mathrm{sim}(Q)$ for which $\|f - g\|_\infty < \varepsilon$. Then $L^\infty(P)$ is an algebra under the point-wise operations.

Let $L^\infty(P) = \{[f] : f \in L^\infty(P)\}$. Then $L^\infty(P)$ is a Banach algebra with respect to the operations induced by the operations in the algebra $L^\infty(P)$ and the norm, $\|\cdot\|_\infty$, induced by the seminorm $f \mapsto \|f\|_\infty$, $f \in L^\infty(P)$.

The spectral set function $P : Q \to B(E)$ will be called closable if

$$\lim_{n\to\infty} \Big\| \sum_{j=1}^{n} P(f_j) \Big\| = 0$$

for any functions $f_j \in \mathrm{sim}(Q)$, $j = 1,2,\ldots$, satisfying condition (2), such that

$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which the inequality (4) holds.

PROPOSITION 1. *Let* $P : Q \to B(E)$ *be a spectral set function. Let* $A(P)$ *be the closure of the algebra of operators* $\{P(f) : f \in \mathrm{sim}(Q)\}$ *in* $B(E)$.

The spectral set function P *is closable if and only if there exists an injective map* $\Phi : A(P) \to L^\infty(P)$ *such that* $\|\Phi(T)\|_\infty \le \|T\|$, *for every* $T \in A(P)$, *and* $\Phi(P(f)) = [f]$, *for every* $f \in \mathrm{sim}(Q)$.

Let $P : Q \to B(E)$ be a closable spectral set function. The range of the map Φ from Proposition 1 will be denoted by $L(P)$. Furthermore, we shall write $L(P) = \{f : [f] \in L(P)\}$ and

$$P(f) = \int_{\Omega} f dP = \Phi^{-1}([f]),$$

for every $f \in L(P)$. Functions belonging to $L(P)$ will be called P-integrable.

PROPOSITION 2. *Let* $P : \mathcal{Q} \to B(E)$ *be a closable spectral set function.*

A function f *on* Ω *is P-integrable if and only if there exist functions* $f_j \in \mathrm{sim}(\mathcal{Q})$, $j = 1,2,\ldots,$ *satisfying condition* (2), *such that the equality* (3) *holds for every* $\omega \in \Omega$ *for which the inequality* (4) *does. In that case, the equality* (5) *holds.*

Furthermore, $\mathrm{L}(P) \subset \mathrm{L}^{\infty}(P)$ *and* $\|f\|_{\infty} \le \|P(f)\|$, *for every* $f \in \mathrm{L}(P)$. *If* $f \in \mathrm{L}(P)$ *and* $g \in \mathrm{L}(P)$, *then* $fg \in \mathrm{L}(P)$ *and* $P(fg) = P(f)P(g)$. *So,* $L(P)$ *is an algebra of functions.*

If $f \in \mathrm{L}(P)$, *then the spectrum of the operator* (1) *is equal to the P-essential range of the function* f, *that is, the set*

$$\bigcap_{Y \in N} \{f(\omega) : \omega \in \Omega \setminus Y\}^{-},$$

where N *is the family of all P-null sets and the bar indicates the closure in the complex plane.*

$A(P)$ *is a semisimple Banach algebra. The integration map* $P = \Phi^{-1} : L(P) \to A(P)$ *is an isomorphism of the algebra* $L(P)$ *onto the algebra* $A(P)$.

So, operators $T \in B(E)$, for which there exist a space Ω, an algebra $\mathcal{Q}$ of its subsets, a closable spectral set function $P : \mathcal{Q} \to B(E)$ and a function $f \in L(P)$ such that $T = P(f)$, can be considered natural generalizations of scalar operators in the sense of Dunford, in particular operators with diagonalizable matrix on a finite-dimensional vector space.

Let us call such operators scalar in a wider sense.

It turns out that an operator $T \in B(E)$ is scalar in wider sense if and only if there exists a Boolean algebra of projections belonging to $B(E)$ such that the Banach algebra of operators it generates is semisimple and contains T.

To demonstrate the viability of the introduced concepts, we use them to obtain new information about some multiplier operators in L^p spaces. We show, in particular, that, for any $p \in (1,\infty)$, translations are scalar operators in the indicated wider sense. This is particularly significant if $p \in (2,\infty)$ because, as proved in [4], in this case, translations are not extended spectral operators in the sense of W. Ricker, [7].

Let G be a locally compact Abelian group and Γ its dual group. The value of a character $\xi \in \Gamma$ on an element $x \in G$ is denoted by $\langle x,\xi \rangle$.

Let $1 < p < \infty$ and let $E = L^p(G)$, with respect to a fixed Haar measure on G.

Let $M^p(\Gamma)$ be the family of all individual functions on Γ which determine multiplier operators on E. That is, $f \in M^p(\Gamma)$ if and only if there exists an operator $T_f \in B(E)$ such that $(T_f\varphi)^\wedge = f\hat{\varphi}$, for every $\varphi \in L^2 \cap L^p(G)$. Here, of course, $\hat{\varphi}$ denotes the Fourier-Plancherel transform of an element φ of $L^2(G)$.

Functions belonging to $M^p(\Gamma)$ are essentially bounded. In fact, $\|f\|_\infty \leq \|T_f\|$, for every $f \in M^p(\Gamma)$, where $\|f\|_\infty$ is the essential supremum norm of f with respect to the Haar measure. The operator T_f depends only on the equivalence class of a function f. That is, if $f \in M^p(\Gamma)$ and if g is a function on Γ such that $g(\xi) = f(\xi)$ for almost every $\xi \in \Gamma$, relative to the Haar measure, then $g \in M^p(\Gamma)$ and $T_g = T_f$.

It is well-known that an operator $T \in B(E)$ commutes with all translations of G if and only if there exists a function $f \in M^p(\Gamma)$ such that

$T = T_f$. So, $\{T_f : f \in M^p(\Gamma)\}$ is a commutative algebra of operators, containing the identity operator, which is closed in $B(E)$. Clearly, $M^p(\Gamma)$ is an algebra of functions and the map $f \mapsto T_f$, $f \in M^p(\Gamma)$, is multiplicative and linear.

Let $R^p(\Gamma)$ be the family of all sets $X \subset \Gamma$ such that $X \in M^p(\Gamma)$. Let $P^p_\Gamma(X) = T_X$, for every $X \in R^p(\Gamma)$.

PROPOSITION 3. *The family* $R^p(\Gamma)$ *is an algebra of sets in* Γ *and* $P^p_\Gamma : R^p(\Gamma) \to B(L^p(G))$ *is a closable spectral set function.*

The usefulness of this proposition depends of course on how rich is the algebra of sets $R^p(\Gamma)$. A result of T.A. Gillespie implies that it is rich enough to permit complete spectral analysis of translation operators. Let us introduce the necessary relevant notation.

Let $\mathbb{T}$ be the circle group, $\{z \in \mathbb{C} : |z| = 1\}$, with its usual topology of a subset of the complex plane. Connected subsets of $\mathbb{T}$ will be called arcs. For an element x of the group G and an arc $Z \subset \mathbb{T}$, let

$$X_{Z,x} = \{\xi \in \Gamma : \langle x,\xi \rangle \in Z\}.$$

Let $K_1(\Gamma)$ be the family of all sets $X_{Z,x}$ corresponding to arcs $Z \subset \mathbb{T}$ and elements x of G. The classes of sets $K_n(\Gamma)$, $n = 2,3,\ldots$, are then defined recursively by requiring that $K_n(\Gamma)$ consist of all sets $X \cap Y$ such that $X \in K_{n-1}(\Gamma)$ and $Y \in K_1(\Gamma)$.

For $n = 1$, the following lemma is a simple re-formulation of Lemma 6 of [5]. (See also Lemma 20.15 in [2].) By induction, the result follows for every $n = 2,3,\ldots$.

LEMMA 4. *The inclusion* $K_n(\Gamma) \subset R^p(\Gamma)$ *is valid for every* $p \in (1,\infty)$ *and every* $n = 1,2,\ldots$. *Moreover, for every* $p \in (1,\infty)$, *there exists a constant* $C_p \geq 1$ *such that* $\|P^p_\Gamma(X)\| \leq C_p^n$, *for every* $X \in K_n(\Gamma)$, *every*

$n = 1,2,\ldots$ *and every locally compact Abelian group* Γ.

Now, each element x of the group G is interpreted as a function on Γ - the character it generates - that is, the function $\xi \mapsto \langle x,\xi \rangle$, $\xi \in \Gamma$. Then $x \in M^p(\Gamma)$ and T_x is the operator of translation by x.

It can be shown that, for every $x \in G$, there exist numbers c_j and sets $X_j \in K_2(\Gamma)$, $j = 1,2,\ldots$, depending on x but not on p, such that

$$\sum_{j=1}^{\infty} |c_j| \, \| P_\Gamma^p(X_j) \| < \infty,$$

the equality

$$\langle x,\xi \rangle = \sum_{j=1}^{\infty} c_j X_j(\xi)$$

holds for every $\xi \in \Gamma$, and

$$T_x = \sum_{j=1}^{\infty} c_j P_\Gamma^p(X_j),$$

for every $p \in (1,\infty)$. Consequently, $x \in L(P_\Gamma^p)$ and

$$T_x = \int_\Omega \langle x,\xi \rangle \, P_\Gamma^p(d\xi).$$

For $p = 2$, this is of course an instance of Stone's theorem ([6], 36E). It might be of interest to note that, for each $p \in (1,\infty)$, the translation operator, T_x, can be expressed as the sum of the same multiples of the projections $P_\Gamma^p(X_j)$, $j = 1,2,\ldots$; only the underlying space, $E = L^p(G)$, varies with p.

More generally, if u is a function of bounded variation on the circle, $\mathbb{T}$, such that the continuous singular component of u is zero, and $f(\xi) = u(\langle x,\xi \rangle)$, for some $x \in G$ and every $\xi \in \Gamma$, then $f \in L(P_\Gamma^p)$, for every $p \in (1,\infty)$.

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