ON ISOMORPHISMS OF ALGEBRAS OF OPERATORS

Michael Cowling

The starting point of the investigations described here is Pontryagin duality. If G is a locally compact abelian group, and $\hat{G}$ is its character group, i.e. the group $\mathrm{Hom}(G,\mathbb{T})$, then $\hat{\hat{G}} = G$, and $G \to \hat{G}$ is a contravariant functor on the category LCAG of locally compact abelian groups, with morphisms being continuous homomorphisms. This theorem, together with its analytic versions, concerning the Fourier transformation, inspired substantial research on general locally compact abelian groups, and at the same time begged the question of what analogues hold for other groups. It is generally accepted that the right answer to this question involves the continuous unitary representations of G, as the natural analogue of $\mathrm{Hom}(G,\mathbb{T})$, but the structures involved are more complicated.

To describe some further developments, a number of group algebras and spaces should be described. For a general locally compact group, $\hat{G}$ denotes the space of continuous irreducible unitary representations π of G on a Hilbert space, H_π, modulo unitary equivalence. If G is abelian, this coincides with the space $\hat{G}$ described before, but the fact that $\hat{G}$ is a group is lost unless one considers tensor products of representations (corresponding to multiplication of characters), which is unpleasant in the non-abelian situation, since in general the tensor product of two irreducible representations is not irreducible. Some kind of (generalised) function space on G is needed. The standard spaces include:

(a) $(L^1(G),+,*)$, where the convolution product * is given by

$$f*g(x) = \int_G dy\, f(y)\, g(y^{-1}x)$$

[here dy is a left-invariant Haar measure on G]; $(L^1(G),+,*)$ is a Banach algebra which is commutative if and only if G is;

(b) $(M(G),+,*)$, the space of bounded measures on G, with * appropriately defined;

(c) $(VN(G),+,*)$, the von Neumann algebra of G, obtained by taking the weak closure of $L^1(G)$ or $M(G)$ in $\mathcal{L}(L^2(G))$, where f in $L^1(G)$ acts on $L^2(G)$ by the left regular representation, λ:

$$(\lambda(f)h)(x) = f*h(x).$$

These algebras incorporate the group multiplication in the convolution product. Other algebras, which are always commutative, are defined using representations of G:

(d) $(A(G),+,.)$ is the function algebra (with pointwise operations) consisting of all <u>coefficient</u> <u>functions</u> of the regular representation λ:

$$u \in A(G) \Leftrightarrow u(x) = \langle\lambda(x)h,k\rangle \qquad \forall x \in G$$

for some appropriate h,k in $L^2(G)$.

(e) $(B(G),+,.)$ is the function algebra of all coefficient functions of all unitary representations:

$$u \in B(G) \Leftrightarrow u(x) = \langle\pi(x)\xi,\eta\rangle \qquad \forall x \in G$$

for some unitary representation π (not necessarily irreducible) and vectors ξ, η in H_π.

All these spaces can be naturally normed, e.g.

$$\|u\|_B = \inf\{\|\xi\|\|\eta\|: u = \langle \pi\xi, \eta\rangle\}.$$

It is obvious that $B(G)$ is a Banach algebra (+ and . correspond to sums and tensor products of unitary representations), but the proof that $A(G)$ is an algebra involves some non-trivial operator theory. For a locally compact abelian group, G, we have some correspondences under the Fourier transformation:

$$L^1(G) \longleftrightarrow A(\hat{G}) \qquad A(G) \longleftrightarrow L^1(\hat{G})$$

$$M(G) \longleftrightarrow B(\hat{G}) \qquad B(G) \longleftrightarrow M(\hat{G})$$

$$VN(G) \longleftrightarrow L^\infty(\hat{G}).$$

One can prove that $VN(G)$ is always the dual space of $A(G)$, which in the abelian case boils down to the familiar duality:

$$L^\infty(\hat{G}) = (L^1(\hat{G}))^*.$$

This duality preserves the Banach space structure, but multiplication is lost.

In the first half of this century, duality for compact groups was developed (Peter-Weyl theorem; Tannaka-Krein duality). In this half century, we have:

WENDEL'S THEOREM (1952): $(L^1(G), +, *)$ *determines* G.

Note that $(VN(G), +, *)$ does not determine G; for example,

$$VN(Z_2 \times Z_2) \cong \ell^\infty(\{1,2,3,4\}) \cong VN(Z_4).$$

In the 1960's, it was observed that $(VN(G)+,*,c)$, where c stands for co-multiplication, does determine G. Knowing the co-multiplication c is equivalent to knowing the pointwise multiplication in the predual $A(G)$; according to P. Eymard (1964), G "is" the Gelfand spectrum of $A(G)$, i.e. the set of (continuous) multiplicative linear functionals on $A(G)$, so G is a subspace of $VN(G)$, which gets its multiplication from convolution in $VN(G)$.

M. Walter (1974) showed that $(A(G),+,\cdot)$ determines G, as does $(B(G),+,\cdot)$. The idea is that A is a special ideal in B, and that $\mathrm{Aut}(A(G),+,\cdot) \cong \mathrm{Aut}(G) \times G$. Walter picks out the elements of G as the translations in $\mathrm{Aut}(A(G),+,\cdot)$). This result seems to have satisfied many mathematicians, though some gluttons for punishment (French, Luukainen and Price (1982), McMullen (1984), ...) have continued working on duality.

We now come to the main part of this discussion. One of the most pervasive puns in mathematics was perpetrated by M.M. Day (1957) when he called a group <u>amenable</u> if there existed an invariant <u>mean</u> on $L^\infty(G)$. In the 1960's much work was done on amenability, and the following characterisation emerged:

G is amenable if and only if $A(G)$ has an approximate identity, i.e. there exists a net (u_α) in $A(G)$ with

$$\|u_\alpha\|_A \text{ bounded}$$

$$u_\alpha \to 1 \text{ uniformly on compacta}$$

(equivalently, $u_\alpha v \to v$ in $A(G)$ for all v in $A(G)$);

for example, compact and solvable groups are amenable. These ideas filtered into Banach algebras and von Neumann algebras in the 1970's.

The next idea was discovered by D.A. Kazhdan (1967), and called "Property T". He showed that some non-amenable groups have the trivial representation isolated in $\hat{G}$. The constants are a direct summand in B(G), and this property is equivalent to saying that there is no approximate identity in B(G) - C; it is a strong form of non-amenability.

Kazhdan's applications of this idea were to the structure of lattices in simple Lie groups, i.e. large (cocompact or of cofinite volume) discrete subgroups Γ of groups like SL(n,R). He shows that if the rank of G is at least 2 (i.e. $n \geq 3$ for SL(n,R)), then $\Gamma/[\Gamma : \Gamma]$ is finite. His ideas led to Margulis' Field's medal winning work on rigidity of lattices in simple Lie groups. Recently, A. Connes has defined property T for arbitrary von Neumann algebras.

I now want to describe some joint work with U. Haagerup. We use the Banach algebra $M_0(A(G))$ of completely bounded multipliers of the Banach algebra A(G). These are, basically, the functions v on G with the property that for any $u \in A(G)$, $u.v \in A(G)$, and some extra stability properties. It is known that, if M(A(G)) is the space of multipliers of A(G), then

$$B(G) \subseteq M_0(A(G)) \subseteq M(A(G)),$$

with equality for amenable G only (V. Losert, (1984)) and it is likely that all inequalities are strict if G is not amenable. We define

$$\Lambda_G = \inf\{\sup_\alpha \|u_\alpha\|_{M_0A} : u_\alpha \in M_0A \cap C_c(G),\ u_\alpha \to 1 \text{ unif. on compacta}\}.$$

We can compute Λ_G for some groups: for G amenable, $\Lambda_G = 1$, and for non-amenable G,

$G = SO(n,1)$ $(n\geq 2)$	$\Lambda_G = 1$	not T	De Cannière and Haagerup (1985)
$G = SU(n,1)$ $(n\geq 2)$	$\Lambda_G = 1$	not T	Cowling (1983)
$G = Sp(n,1)$ $(n\geq 2)$	$\Lambda_G = 2n-1$	T	Cowling and Haagerup (1986)
$G = SL(n,\mathbb{R})$ $(n\geq 3)$	$\Lambda_G = +\infty$	T	Haagerup (1986)

Haagerup (1986) defines $\Lambda_{\mathcal{A}}$ similarly for an arbitrary von Neumann algebra $\mathcal{A}$, and by using ideas from Kazhdan's paper, he shows that, if Γ is a lattice in a simple Lie group G, then $\Lambda_\Gamma = \Lambda_G$, and further he shows that if $\mathcal{A} = VN(\Gamma)$, $\Lambda_{\mathcal{A}} = \Lambda_\Gamma$. Then Λ is a possibly-Property-T-related index which distinguishes certain von Neumann algebras $(VN(\Gamma)$'s) of type II_1 which, up to now, were not known to be different. In particular we have the following result.

THEOREM (Cowling and Haagerup (1986)): *The von Neumann algebras of lattices in* $SL(2,\mathbb{R})$ *and* $Sp(n,1)$ $(n\geq 2)$ *are all distinct.*

The last development I want to mention is current research. If G is a connected simple Lie group, non-compact, then I showed (1979) that

$$B(G) = \mathbb{C} \oplus B_0(G), \quad \text{where} \quad B_0(G) = B(G) \cap C_0(G),$$

and that, if G is not locally isomorphic to $SO(n,1)$ or $SU(n,1)$, there exists an index N_G so that

$$B_0(G)^{N_G} \subseteq A(G).$$

R. Howe (1980) showed that, for $G = \widetilde{Sp}(n,\mathbb{R})$, $N_G = 2n$, and what we know about general simple groups indicates that, probably

$$N_G \simeq \text{rank}(G)$$

($N/r \in [a,b]$, $a,b \in \mathbb{R}^+$). I am presently trying, on one hand, to push these results to lattices and from there to von Neumann algebras; on the other hand, it seems possible that one can show

$$M_0A(G) = \mathbb{C} \oplus (M_0A(G) \cap C_0(G))$$

and that $$(M_0A(G) \cap C_0(G))^{N_G} \subseteq A(G);$$

one should then identify $A(G)$ in $M_0A(G)$, and pass to von Neumann algebras. (Actually, some parts of this programme already work).

Last, but not least, let us ask: what is an invariant? Is the "cohomology functor" or the "n^{th} Betti number" the "invariant"? Is one entitled to call Λ_G or N_G an invariant? Or is there a new theory for which Λ_G and N_G are the tips of the iceberg?

A few words about the proofs of these results will be in order. For a simple group G, there is always a maximal compact subgroup K, and harmonic analysis of K-bi-invariant functions is easier. For example, we set

$$C_c(K\backslash G/K) = \{f \in C_c(G) : f(kxk') = f(x) \ \forall x \in G \ \forall k, k' \in K\};$$

then there exists an approximate identity in $M_0(A(G)) \cap C_c(G)$ if and only if there exists one in $M_0(A(G)) \cap C_c(K\backslash G/K)$; also, if $(B_0(G) \cap C(K\backslash G/K))^n \subseteq A(G)$ then we know that $B_0(G)^{2n} \subseteq A(G)$. Harmonic analysis of K-bi-invariant functions is easier. For instance, $L^1(G)$ is not commutative, but $L^1(K\backslash G/K)$ is, for $*$. Finally, $*$ gets easier for $L^1(K\backslash G/K)$, as follows.

The "Iwasawa decomposition" expresses G as $ANK = SK$, say, where $S = AN$ is solvable. If $f \in C_c(K\backslash G/K)$, then $f|_S$ determines f; the left-K-invariance means that $f|_S$ is constant on certain algebraic sets in S. Further, we may write Haar measure on G as

$$dx = dsdk,$$

where ds is left-invariant Haar measure on S, and dk is the Haar measure of K. For s in S, k in K, and f, f' in $C_c(K\backslash G/K)$,

$$\begin{aligned} f * f'(sk) &= f * f'(x) \\ &= \int_G f(x)\, f'(x^{-1}s)\, dx \\ &= \int_S \int_K f(s'k)\, f'(k^{-1}s'^{-1}s)\, dkds \\ &= \int_S f(s')\, f'(s'^{-1}s)\, ds \\ &= f|_S * f'|_S(s); \end{aligned}$$

convolution on the smaller group S holds all the secrets.

For calculating $M_0(G)$ norms, we use the following result:

PROPOSITION: *If* $f \in C(K\backslash G/K)$, *then* f *is in* $M_0(A(G))$ *if and only if* $f \in M(A(G))$ *if and only if* $f|_S$ *is in* $B(S)$; *the norms also coincide.*

Finally, the problems related to passing to Γ are closely related to the problem of harmonic analysis on trees and graphs which have been described recently, by A. Figà-Talamanca and M.A. Picardello (1983), et al..

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School of Mathematics,

University of New South Wales,

Kensington, N.S.W. 2033.