ON CONTINUATION OF QUASI-ANALYTIC SOLUTIONS OF PDE'S TO COMPACT CONVEX SETS

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The aim of this talk is to present and explain to non-specialists in this subject an extension theorem for regular solutions of systems of PDE's with constant coefficients. The theorem was obtained by Kaneko [2,3,4] in the real analytic case and was then generalised by the author [1] to quasi-analytic solutions.

Let us first describe the problem considered. Let K be a compact subset of an open set $\Omega \subset \mathbb{R}^n$, $n > 1$, and let $P(D)$ be a matrix of complex polynomials $P_{jk}(D)$, $j = 1,\dots,J$, $k = 1,\dots,K$, in the variable $D = (D_1,\dots,D_n)$, $D_\nu = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_\nu}$, $\nu = 1,\dots,n$. Let u be a distribution solution of the homogeneous system

$$(1) \qquad P(D)u = 0$$

defined on $\Omega\setminus K$. One may now ask the following questions:

Q1. Can u be continued to Ω as a distribution solution $[u]$ of (1)?

Q2. Does it help to assume u "regular"? Does there then exist a "regular" extension $[u]$?

The answers to these questions can be formulated in terms of a family of irreducible algebraic varieties $\{N_\lambda(P)\}_{\lambda = 0,1,\ldots,L}$ associated with the matrix $P(\zeta)$. Let us recall some definitions.

DEFINITION 1 (cf. [5,Ch.iv]). Consider $P(\zeta)$ as a matrix multiplication operator between the $\mathcal{P}$-modules $\mathcal{P}^K$ and $\mathcal{P}^J$ (here $\mathcal{P}$ denotes the ring of complex polynomials in the variable $\zeta = (\zeta_1,\ldots,\zeta_n))$, let $P(\zeta) \cdot \mathcal{P}^K = \mathfrak{p}_0 \cap ,\ldots, \cap \mathfrak{p}_L$ be a reduced decomposition of the range of $P(\zeta)$ into primary components, and, finally, let $N_\lambda(P)$ be the set of common zeros of all polynomials in $\mathcal{P}$ which multiply $\mathfrak{p}_\lambda$ into $P(\zeta) \cdot \mathcal{P}^K$, $\lambda = 0,1,\ldots,L$.

The family $\{N_\lambda(P)\}_{\lambda=0,1,\ldots,L}$ is then uniquely determined by $P(\zeta)$ and provides a decomposition of the set $N(P) = \{ \zeta \in \mathbb{C}^n : \text{rank } P(\zeta) < J\}$ into irreducible components.

DEFINITION 2 (cf. [5,Ch.viii]) The operator $P(D)$ is called **determined** if $N(P') \overset{\text{def}}{=} \{ \zeta \in \mathbb{C}^n : \text{rank } P(\zeta) < K\}$ is not all of $\mathbb{C}^n$, and **overdetermined** if, in addition, the only non-empty component in $\{N_\lambda(P)\}_{\lambda=0,1,\ldots,L}$ is equal to $\mathbb{C}^n$.

REMARK 1 The "degree of determinacy" of $P(D)$ is best defined in terms of the vanishing of $\text{Ext}^i = \text{Ext}^i(\frac{\mathcal{P}^K}{P(z)\cdot\mathcal{P}^J} , \mathcal{P})$ modules up to some order, or, equivalently, in terms of the dimension of $N(P')$; here P' is the transpose of P . Thus $P(D)$ is determined iff $\text{Ext}^0 = 0$ (alt. iff $\dim N(P') < n$) and $P(D)$ is overdetermined iff $\text{Ext}^0 = \text{Ext}^1 = 0$ (alt, iff $\dim N(P') < n-1$) . In order to have a definition which is well adapted to our purposes and avoids introducing Ext-modules we have used here the observation [5, ch.viii, §14, 4^0, Proof of Cor. 4] that, except for the n-dimensional component, the family of varieties associated with the module Ext^1 is the same as the family associated with the matrix $P(\zeta)$.

EXAMPLE 1 If $K = 1$, i.e. the matrix $P = (P_1, \ldots, P_J)$ is a column of polynomials, then $N(P')$ is the set of common zeros of $P_1, \ldots, P_J$ and $N(P)$ decomposes into $\mathbb{C}^n$ and the zero-set of the greatest common factor of $P_1, \ldots, P_J$ (cf [5, Ch viii, §13 Prop.3] and Remark 1). Hence, in this case, $P(D)$ is determined unless it is the zero operator and it is overdetermined if the polynomials $P_1, \ldots, P_J$ have no common non-trivial factor. A very well known and important example of an (elliptic) overdetermined operator of this form is the Cauchy-Riemann operator in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, $n > 1$.

Returning to our problem, let us give an answer to the first question (Q1).

THEOREM 1 (Ehrenpreis; cf [5, Ch viii, §14, 3^0]). *Let* Ω *be a neighbourhood of a* ***convex*** *and compact set* K . *In order that every distribution solution of* (1) *on* $\Omega \setminus K$ *have a unique extension to* Ω *as a distribution solution of* (1), *it is necessary and sufficient that the operator* $P(D)$ *be* ***overdetermined***.

REMARK 2 The condition is no longer necessary for the existence of non-unique extensions; see the quoted reference.

It turns out however that **regular** solutions of (1) on $\Omega \setminus K$ do have extensions to Ω as solutions of (1) under a **weaker** hypothesis on $P(D)$. Before stating the result which provides an answer to Q2 let us recall the notion of ellipticity of an algebraic variety.

DEFINITION 3 Let V be an irreducible algebraic variety in $\mathbb{C}^n$. We consider $\mathbb{C}^n$ imbedded in the complex projective space $\mathbb{P}^n$ by means of a mapping φ which to a point $\zeta = (\zeta_1, \ldots, \zeta_n)$ associates the point in $\mathbb{P}^n$ with homogeneous coordinates $(1, \zeta_1, \ldots, \zeta_n)$. The points in the closure of $\varphi(V)$ (in the metric of $\mathbb{P}^n$) but not in $\varphi(V)$ itself are called the **points at infinity** of V. The variety V is called **elliptic** if none of its points at infinity has a real coordinate representation. The operator $P(D)$ is called elliptic if all the varieties $N_\lambda(P')$ associated to the transpose P' of P are elliptic.

THEOREM 2 ([1-4]) *Let* Ω *be a neighbourhood of a convex and compact set* K. *In order that every solution of* (1) *on* $\Omega \setminus K$ *in a given quasi-analytic class have a (necessarily unique) extension to* Ω *as a solution of* (1) *in the same quasi-analytic class it is necessary and sufficient that* $P(D)$ *be* ***determined*** *and that* ***none*** *of the irreducible varieties* $N_\lambda(P)$ *associated with the matrix* $P(\zeta)$ *be* ***elliptic****. Moreover, if the quasi-analytic class considered is the real analytic class, the convexity assumption on* K *may be replaced by the condition that* $\mathbb{P}^n \setminus K$ *be connected.*

REMARK 3 Again, the condition is no longer necessary for the existence of a non-regular (e.g. hyperfunction) extension; see the quoted references.

EXAMPLE 2 When $P = (P_1, \ldots, P_J)$ is a non-zero matrix with a single column, the condition in Theorem 1 means that the polynomials $P_1, \ldots, P_J$ have no common factor, while the conditions of Theorem 2 require only that the polynomials $P_1, \ldots, P_J$ have no common **elliptic** factor. In particular we must have $J > 1$ in the first case but not in the second.

REMARK 4 Concerning the convexity assumption on K. The method of proof of both Theorem 1 and 2 outlined below involves solving a system $P(D)v = f$ for v with support lying in an arbitrarily given neighbourhood of the support of f (which is essentially equal to K). While for general $P(D)$ this can only be done if K is convex, for elliptic $P(D)$, K may be arbitrary with connected complement. It follows that this is the right assumption on K (easily seen to be optimal) in Theorem 1 if $P(D)$ is assumed elliptic. In Theorem 2 elliptic $P(D)$ are excluded; however, in the real analytic case, a cohomological argument [4] using the triviality of the cohomology group $H^1(U,A)$ for the real analytic sheaf A in $\mathbb{R}^n$ and any open $U \subset \mathbb{R}^n$ shows that Theorem 2 with K convex yields the stronger (optimal) version which only requires that the complement of K is connected. It is still not known to the author whether $H^1(U,C^L) = 0$ for a quasi-analytic sheaf C^L and, consequently, whether the corresponding stronger version of Theorem 2 holds in the quasi-analytic case.

REMARK 5 The quasi-analyticity of u in Theorem 2 cannot be relaxed if K contains interior points, cf [1, Remark 3]. If K is a single point, however, the hypothesis of no elliptic component implies that K is removable for (non-quasi-analytic) u in a Gevrey class determined by the operator $P(D)$ (private communication by A. Kaneko).

Outline of proof of Theorem 2 For the **necessity part** for general $P(D)$ see [1,3]; for the particular $P(D)$ of the form considered in Examples 1 and 2, if $P_1,\dots,P_J$ have an elliptic factor Q let u be a fundamental solution of $Q(D)$ translated to a point $x_0 \in K$. Clearly u is real analytic (hence quasi-analytic) on $\Omega\setminus K$ and, by the uniqueness of quasi-analytic continuation, any quasi-analytic extension $[u]$ must agree with u on $\mathbb{R}^n\setminus\{x_0\}$. Hence, if $[u]$ was smooth at x_0, the singularity of u at x_0 would be removable, which it clearly is not.

Turning now to the **sufficiency part**, proceed in three steps as follows:

STEP 1 Extend u as a distribution U in Ω, not necessarily satisfying (1) there. Of course, this will not always be possible unless we agree to modify u near K. So let ω be any open neighbourhood of K and assume that $U = u$ on $\Omega - \bar{\omega}$; observe that $P(D)U$ has compact support contained in $\bar{\omega}$, and that U can be chosen smooth if u is smooth.

STEP 2 Solve $P(D)V = P(D)U$ for V with compact support $K_1 \subset \Omega$, K_1 being arbitrarily close to $\bar{\omega}$ (for general $P(D)$ this cannot usually be done if $\bar{\omega}$ is not convex, hence the convexity assumption on K). Then $[u] = U - V$ thus constructed clearly satisfies (1) in Ω and agrees with u on $\Omega\setminus K_1$.

STEP 3 Invoke uniqueness and regularity results for solutions of homogeneous determined systems to conclude that the extensions [u] obtained in STEP 2 are actually independent of K_1 and thus define a single unique extension, which, being regular near the boundary of Ω, must necessarily be regular also in the interior.

The non-trivial part of the above procedure is, of course, the second step. In general, when f is a (vector) distribution with compact support, the solvability of the system

$$P(D)v = f \tag{2}$$

for v with compact support depends on the behaviour of the Fourier-Laplace transform $\hat{f}$ of f (and of its derivatives) on the varieties $N_\lambda(P)$ $\lambda = 0,1,\ldots,L$ associated with the matrix P (cf Definition 1). More precisely, a necessary and sufficient condition is the following: to each $N_\lambda(P)$ there is associated a (matrix) differential operator with polynomial coefficients $\partial_\lambda = \partial_\lambda(\zeta, D_\zeta)$ (a so-called "Noetherian operator"), which applied to $\hat{f}$ should produce a (vector) function vanishing on $N_\lambda(P)$. Moreover, as the Noetherian operator associated with the component, say N_0, of dimension n (i.e. $N_0 = \mathbb{C}^n$) one can take the operator of multiplication by a matrix of polynomials P_1 (a "compatibility matrix"), the rows of which (by definition) generate the module of relations between the rows of P (thus $P_1P = 0$ and if $P_2P = 0$ for some matrix of polynomials P_2, then the rows of P_2 are generated over the ring of polynomials by the rows of P_1). Consequently, if the right side in (2) is of the form $F = P(D)U$ considered in STEP 2 above, then

$$\partial_0 \hat{f}(\zeta) = P_1(\zeta)(P(D)U)^\wedge(\zeta) = (P_1(D)P(D)U)^\wedge(\zeta) = 0$$

for all $\zeta \in \mathbb{C}^n$. Hence, what remains to be shown in order to carry out STEP 2 are the equalities:

$$(3) \qquad \partial_\lambda (P(D)U)^\wedge|_{N_\lambda(P)} = 0 \qquad \lambda = 1,2,\dots,L .$$

As a side comment we may at this stage recall Definition 2 to observe that **for over-determined $P(D)$ the varieties $N_\lambda(P)$, $\lambda = 1,\dots,L$, are empty,** hence in this case the equalities (3) are trivially satisfied, ultimately yielding the sufficiency part of Theorem 1.

Returning to the proof of Theorem 2, let us now indicate how the hypothesis implies the inequalities (3). The regularity of U outside of a neighbourhood of K causes the functions $\partial_\lambda (P(D)U)^\wedge|_{N_\lambda(P)}$ to have more restricted growth at infinity than what should be expected knowing that $(P(D)U)^\wedge$ is the Fourier transform of a (smooth) distribution with compact support. Specifically, the estimates

$$(4) \qquad \log |\partial_\lambda (P(D)U)^\wedge(\zeta)| \le C(1+|\mathrm{Im}\zeta|) - \log L\left[\frac{|\zeta|}{C}\right]$$

hold for $\zeta \in N_\lambda(P)$, $\lambda = 1,\dots,L$. Here C is a positive constant and the function L determines the quasi-analytic class of U outside a neighbourhood of K ; in particular

$$(5) \qquad \int_{-\infty}^{\infty} \frac{\log L(t)}{1+t^2}\, dt = +\infty .$$

To show (4) observe that the restriction of $\partial_\lambda(P(D)U)^\wedge$ to $N_\lambda(P)$ actually does not depend on the extension U of u chosen in STEP 1 (because the difference $U_1 - U_2$ of any two such extensions U_i has compact support and therefore the difference

$$\partial_\lambda(P(D)U_1)^\wedge - \partial_\lambda(P(D)U_2)^\wedge = \partial_\lambda(P(D)(U_1-U_2))^\wedge$$

is zero on $N_\lambda(P)$ by the defining property of the Noetherian operators). This allows us to take as an estimate for the growth of $\partial_\lambda(P(D)U_0)^\wedge$ on $N_\lambda(P)$, for any fix extension U_0 , the infimum of the usual growth estimates of $\partial_\lambda(P(D)U)^\wedge$ over all extensions U which agree with U_0 outside a given neighbourhood of K . A "minimizing sequence" of such extensions U_k , $k = 1,2,\dots$ can be obtained by cutting off u near K by smooth functions ψ_k , the derivatives of which up to order k grow like those of a real analytic function, cf [4]. Having established (4), the next step is a Liouville type result ([4, Lemma 2]) stating that if an algebraic variety V has a real point at infinity, then no non-trivial analytic function on V can satisfy an estimate of the kind (4) if (5) is to hold. Hence, if none of the varieties $N_\lambda(P)$, $\lambda = 1,\dots,L$, is elliptic and u is in a quasi-analytic class on $\Omega\setminus K$ then the equalities (3) hold and the proof of Theorem 2 can be carried out as indicated above.

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