EVOLUTION OPERATORS OF PARABOLIC EQUATIONS IN CONTINUOUS FUNCTION SPACE

A. Yagi

1. INTRODUCTION

Let

$$(P)\quad \begin{cases} \partial u/\partial t + \sum_{|\alpha|\le 2m} a_\alpha(t,x)D^\alpha u = f(t,x) & \text{in } (0,T]\times\Omega \\ \sum_{|\beta|\le m_j} b_{j\beta}(t,x)D^\beta u = 0 & \text{on } (0,T]\times\partial\Omega,\ j=1,\cdots,m \\ u(0,x) = u_0(x) & \text{in } \Omega \end{cases}$$

be the initial value problem of a parabolic partial differential equation in a (bounded or unbounded) region Ω in $\mathbb{R}^n$. This Note studies the construction of an evolution operator (fundamental solution) for (P) in the continuous function space $\mathscr{C}(\overline{\Omega})$ on $\overline{\Omega}$. In the L_p $(1 < p < \infty)$ space case the construction has been studied by several authors, including Kato et al.[1], Tanabe [4] and Yagi [6]. Recently Tanabe [8] and his student Park [2] showed existence of the evolution operator for (P) even in a "worse" function space $L^1(\Omega)$ (recall that there is no a priori estimate for elliptic operators in L^1 space). We are then interested to work in another "worse" function space $\mathscr{C}(\overline{\Omega})$.

For $0 \le t \le T$ let $A(t)$ denote the operator

$\sum_{|\alpha|\leq 2m} a_\alpha(t,x)D^\alpha$ acting in $\mathscr{C}(\overline{\Omega})$ with boundary conditions $\sum_{|\beta|\leq m_j} b_{j\beta}(t,x)D^\beta u = 0$ on $\partial\Omega$ for $1 \leq j \leq m$. According to Stewart [3], $A(t)$ are shown under suitable assumptions to be the generators of analytic semigroups on $\mathscr{C}(\overline{\Omega})$, therefore (P) can be formulated as an abstract evolution equation

$$(E)\quad \begin{cases} du/dt + A(t)u = f(t), & 0 < t \leq T \\ u(0) = u_0 \end{cases}$$

in the space $\mathscr{C}(\overline{\Omega})$. In the present case, however, we have to notice that the domains $\mathscr{D}(A(t))$ of $A(t)$ may be no longer dense in $\mathscr{C}(\overline{\Omega})$ (for example, consider the Dirichelet condition $u = 0$ on $\partial\Omega$ for second order operators in Ω, clearly the space $\{u \in \mathscr{C}(\overline{\Omega});\ u = 0 \text{ on } \partial\Omega\}$ is not dense in $\mathscr{C}(\overline{\Omega})$).

2. ABSTRACT EVOLUTION EQUATION (E)

Let X be a Banach space. In this section we study the construction of an evolution operator for an abstract evolution equation

$$(E)\quad \begin{cases} du/dt + A(t)u = f(t), & 0 < t \leq T \\ u(0) = u_0 \end{cases}$$

in X. (E) is of parabolic type, this means that each $A(t)$, $0 \leq t \leq T$, is the generator of an analytic semigroup on X, but the domain $\mathscr{D}(A(t))$ of $A(t)$ is not assumed to be dense in X. $f:[0,T] \to X$ and $u_0 \in X$ are given, $u:[0,T] \to X$ is unknown.

In the case where $A(t)$ are densely defined, there is already a large literature on the present problem. Some of them, especially we are concerned with [6], can be generalized to the case of non dense domain. According to [6] let us make the following hypotheses:

(I) The resolvent sets $\rho(A(t))$ of $A(t)$ contain a sector $\Sigma = \{\lambda \in \mathbb{C};\ |\arg \lambda| \geq \pi/2 - \delta\}$ where $\delta > 0$, and there the resolvents $(\lambda - A(t))^{-1}$ satisfy

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(X)} \leq M/(|\lambda| + 1)\ , \quad \lambda \in \Sigma.$$

(II) The function $A(\cdot)^{-1}$ is strongly continuously differentiable on $[0,T]$: $A(\cdot)^{-1} \in \mathscr{C}^1([0,T];\mathcal{L}_s(X))$.

(III) The derivatives $dA(t)^{-1}/dt$, $0 \leq t \leq T$, satisfy

$$\|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt\|_{\mathcal{L}(X)} \leq N/(|\lambda| + 1)^{\nu}\ , \quad \lambda \in \Sigma$$

with some constants $0 < \nu \leq 1$ and $N \geq 0$.

Then we can prove:

THEOREM 2.1 *There exists a family* $U(t,s)$, $0 \leq s \leq t \leq T$, *of bounded linear operators on* X *which have the properties: a)* $U(t,s)U(s,r) = U(t,r)$ *for* $0 \leq r \leq s \leq t \leq T$, $U(s,s) = 1$ *for* $0 \leq s \leq T$; *b)* $U(t,s)$ *is strongly continuous for* $0 \leq s < t \leq T$ *with an estimate* $\|U(t,s)\|_{\mathcal{L}(X)} \leq C_1$; *c) the ranges* $\mathcal{R}(U(t,s))$ *are contained in* $\mathcal{D}(A(t))$ *for all* $0 \leq s < t \leq T$, *and* $A(t)U(t,s)$ *is strongly continuous for* $0 \leq s < t \leq T$ *with an estimate* $\|A(t)U(t,s)\|_{\mathcal{L}(X)} \leq C_2(t - s)^{-1}$; *and d)* $U(t,s)$ *is strongly continuously differentiable in* t *for* $0 \leq s < t \leq T$, *and* $\partial U(t,s)/\partial t = -A(t)U(t,s)$.

$U(t,s)$ is called the evolution operator for (E). In fact, an existence and uniqueness result of strict solution u (i.e. $u \in \mathscr{C}^1((0,T];X)$, $A(\cdot)u(\cdot) \in \mathscr{C}((0,T];X)$, and $\lim_{t\to 0} A(0)^{-1}(u(t) - u_0) = 0$ in X) for the problem (E) is obtained by using the operator $U(t,s)$.

THEOREM 2.2 *For any* $f \in \mathscr{C}^{\sigma}([0,T];X)$, $\sigma > 0$, *and any* $u_0 \in X$, *the function* u *defined by*

$$(2.1) \qquad u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)f(\tau)d\tau , \qquad 0 \le t \le T,$$

gives a strict solution of (E). *Conversely, let* u *be any strict solution of* (E) *where* $f \in \mathscr{C}([0,T];X)$ *and* $u_0 \in X$ *are arbitrary, and assume that* u *satisfies a growth condition:* $\|u(t)\|_X \le Ct^{-\gamma}$ *near* $t = 0$ *with some* $\gamma < \nu$*; then, necessarily* u *must be equal to the function given by* (2.1) *for all* $0 \le t \le T$.

The spirit of proof of these two theorems is quite similar to that in [6] where the theorems have been proved in the case where $\mathscr{D}(A(t))$ are dense. We have to recover, however, a technical difficulty that the Yosida regularization $n(n + A(t))^{-1}$ of $A(t)$ converges to the identity mapping no longer on the whole space X but only on the closure of $\mathscr{D}(A(t))$, which results from lack of the density of the domains. Full proof will be seen in the forthcoming paper [7].

3. INITIAL VALUE PROBLEM (P)

Let us observe in this Section how to apply the abstract result in the previous Section to the problem (P).

Let Ω be a (possibly unbounded) region in $\mathbb{R}^n$ with the boundary $\partial\Omega$, $x = (x_1,\cdots,x_n) \in \overline{\Omega}$. For each integer $k \ge 0$, $\mathscr{C}^k(\overline{\Omega})$ (resp. $\mathscr{C}^k(\partial\Omega)$) is the Banach space of all continous bounded functions on $\overline{\Omega}$ (resp. $\partial\Omega$) which have smooth and bounded derivatives on $\overline{\Omega}$ (resp. $\partial\Omega$) up to the order k; $\mathscr{C}^0(\overline{\Omega})$

(resp. $\mathscr{C}^0(\partial\Omega)$) will be abbreviated to $\mathscr{C}(\overline{\Omega})$ (resp. $\mathscr{C}(\partial\Omega)$).

For $0 \le t \le T$, let

$$A(t,x;D) = \sum_{|\alpha|\le 2m} a_\alpha(t,x)D^\alpha$$

be differential operators in Ω of order $2m$, where $D_1 = i^{-1}\partial/\partial x_1, \cdots, D_n = i^{-1}\partial/\partial x_n$, and $D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ for multi index $\alpha = (\alpha_1,\cdots,\alpha_n)$. And let

$$B_j(t,x;D) = \sum_{|\beta|\le m_j} b_{j\beta}(t,x)D^\beta , \quad j = 1,\cdots, m$$

be boundary differential operators on $\partial\Omega$ of order $m_j \le 2m-1$.

We assume the following conditions:

(R1) The boundary $\partial\Omega$ is uniformly regular of class $\mathscr{C}^{2m}$.

(A1) $a_\alpha \in \mathscr{C}^1([0,T];\mathscr{C}(\overline{\Omega}))$ for $|\alpha| \le 2m$, moreover $a_\alpha(t,\cdot)$ are uniformly continuous on $\overline{\Omega}$ for $|\alpha| = 2m$.

(A2) $A(t,x;D)$ are uniformly strongly elliptic, i.e.

$$\sum_{|\alpha|\le 2m} a_\alpha(t,x)\xi^\alpha \ge E|\xi|^{2m} \quad (E > 0) \quad \text{for} \quad \xi \in \mathbb{R}^n,\ x \in \overline{\Omega},\ 0\le t\le T.$$

(B1) $b_{j\beta} \in \mathscr{C}^1([0,T];\mathscr{C}^{2m-m_j}(\partial\Omega))$ for $|\beta| \le m_j$, $1 \le j \le m$; and $D^\gamma b_{j\beta}(t,\cdot)$ are uniformly continuous on $\partial\Omega$ for $|\gamma| = 2m-m_j$.

(B2) For $|\theta| \ge \pi/2 - \delta$ $(\delta > 0)$, $A(t,x;D) - e^{i\theta}D_y^{2m}$ and $B_j(t,x;D)$ satisfy the complementing condition on a product region $\overline{\Omega} \times \mathbb{R}_y$ (specifically see e.g.[8,p.251]).

Set

$$\begin{cases} X = \{f \in \mathscr{C}(\overline{\Omega});\ \lim_{x\in\Omega,|x|\to\infty} f(x) = 0\} \\ (X = \mathscr{C}(\overline{\Omega}) \text{ if } \Omega \text{ is a bounded region}) \\ \|f\|_X = \|f\|_{\mathscr{C}(\overline{\Omega})}. \end{cases}$$

And define, for each $0 \le t \le T$, a linear operator $A(t)$ acting in X by

$\Big\{$ $\mathscr{D}(A(t)) = \{u \in \bigcap_{n<q<\infty} W_q^{2m}(\Omega);\ A(t,x;D)u \in X$ and

$$\begin{cases} \qquad\qquad B_j(t,x;D)u = 0 \text{ on } \partial\Omega \text{ for } 1 \le j \le m\}, \\ A(t)u = A(t,x;D)u - \lambda_0 u. \end{cases}$$

Then it is verified that:

THEOREM 3.1 $A(t)$, $0 \le t \le T$, *satisfy the Hypotheses* (I), (II) *and* (III) *in Section 2 (we shall assume if necessary that the constant* λ_0 *is sufficiently positive).*

Proof In fact, (I) has been already verified by Stewart [3]. To verify (II) and (III) we use a priori estimates in L^p_{loc} space for $1 < p < \infty$. For $x \in \overline{\Omega}$ and $r > 0$, $\Omega(x,r) = \{y \in \Omega;\ |y - x| < r\}$. For $0 \le j \le 2m$, $\|\cdot\|_{j,p,\omega}$ is the usual norm of the Sobolev space $W^j_p(\omega)$ on $\omega \subset \Omega$.

LEMMA 3.2 *For any* $1 < p < \infty$ *there are two positive constants* C_p *and* R_p *such that, if* $|\arg \lambda| \ge \pi/2 - \delta$ *and* $|\lambda| \ge C_p$ *and if* $r \ge R_p$, *then*

$$(3.1)\quad \sum_{j=0}^{2m} |\lambda|^{1-j/2m} \sup_{x\in\overline{\Omega}} \|u\|_{j,p,\Omega(x,r)} \le C_p\{\sup_{x\in\overline{\Omega}} \|(\lambda - A(t,x;D))u\|_{0,p,\Omega(x,r)} + \sum_{j=1}^{m} |\lambda|^{1-m_j/2m} \sup_{x\in\overline{\Omega}} \|g_j\|_{0,p,\Omega(x,r)} + \sum_{j=1}^{m} \sup_{x\in\overline{\Omega}} \|g_j\|_{2m-m_j,p,\Omega(x,r)}\}$$

for all $u \in W^{2m}_p(\Omega)$, *here* g_j $(1 \le j \le m)$ *are arbitrary functions in* $W^{2m-m_j}_p(\Omega)$ *provided* $g_j = B_j(t,x;D)u$ *on* $\partial\Omega$.

We take some $n/2m < p < \infty$, and assume that $\lambda_0 \ge C_p$. Let $f \in \mathscr{E}(\overline{\Omega})$ be a function with compact support; since $f \in L^p(\Omega)$, $A(t)^{-1}f$ belongs to $W^{2m}_p(\Omega)$ and satisfies

$$(3.2)\qquad (A(t,x;D) - \lambda_0)A(t)^{-1}f = f \quad \text{in } \Omega, \text{ and}$$

$$(3.3)\qquad B_j(t,x;D)A(t)^{-1}f = 0 \quad \text{on } \partial\Omega,\ 1 \le j \le m.$$

Then, by using the a priori estimates in $L^p(\Omega)$, it is shown from (A1) and (B1) that $A(\cdot)^{-1}f$ is, as a $W^{2m}_p(\Omega)$-valued function, continuously differentiable on $[0,T]$ and the derivative $dA(t)^{-1}/dtf$ is specified by

(3.4) $(A(t,x;D) - \lambda_0)dA(t)^{-1}/dtf =$

$$- \sum_{|\alpha|\le 2m} \partial a_\alpha(t,x)/\partial t D^\alpha A(t)^{-1}f \quad \text{in } \Omega$$

(3.5) $B_j(t,x;D)dA(t)^{-1}/dtf =$

$$- \sum_{|\beta|\le m_j} \partial b_{j\beta}(t,x)/\partial t D^\beta A(t)^{-1}f \quad \text{on } \partial\Omega.$$

This then shows by the Sobolev imbedding theorem ($W_p^{2m}(\Omega) \subset \mathscr{C}(\bar{\Omega})$) that $A(\cdot)^{-1}f \in \mathscr{C}^1([0,T];X)$. Take an arbitrary point $x_0 \in \bar{\Omega}$, and let $\phi_0(x) = \phi(x-x_0)$ be a function such that $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with supp $\phi \subset \{|x| < R_p\}$ and $\phi(0) = 1$. Then

$|\{dA(t)^{-1}/dtf\}(x_0)|$

$$\le \{\|\phi_0 dA(t)^{-1}/dtf\|_{2m,p,\Omega}\}^\mu \{\|\phi_0 dA(t)^{-1}/dtf\|_{0,p,\Omega}\}^{1-\mu}$$

with $\mu = n/2mp$, so that

$$\le C_p\{\|dA(t)^{-1}/dtf\|_{2m,p,\Omega(x_0,R_p)}\}^\mu\{\|dA(t)^{-1}/dtf\|_{0,p,\Omega(x_0,R_p)}\}^{1-\mu}.$$

We here use the local a priori estimate (3.1) with $\lambda = \lambda_0$, then it follows from (3.4) and (3.5) that

$$\le C_p \sup_{x\in\bar{\Omega}}\|A(t)^{-1}f\|_{2m,p,\Omega(x,R_p)}.$$

We use again (3.1), then (3.2) and (3.3) yield

(3.6) $$\sup_{x\in\bar{\Omega}}\|A(t)^{-1}f\|_{2m,p,\Omega(x,R_p)} \le C_p \sup_{x\in\bar{\Omega}}\|f\|_{0,p,\Omega(x,R_p)} \le C_p\|f\|_{\mathscr{C}(\bar{\Omega})}.$$

Hence we have proved that

$$\|dA(t)^{-1}/dtf\|_{\mathscr{C}(\bar{\Omega})} \le C_p\|f\|_{\mathscr{C}(\bar{\Omega})},$$

the constant C_p being independent of f. (II) then follows easily from the fact that functions in $\mathscr{C}(\bar{\Omega})$ with compact support are dense in X.

Verification of (III) is now an easy analogue to the

L^p case (cf. [5] or [6]). For $|\arg\lambda| \geq \pi/2 - \delta$ and $0 \leq t \leq T$, we denote the operator $A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt$ by $D(\lambda,t)$. Let $f \in \mathscr{C}(\overline{\Omega})$ be again with compact support; $D(\lambda,t)f$ is a function in $W_p^{2m}(\Omega) \subset \mathscr{C}(\overline{\Omega})$; in the same way as above it is seen that

$$(3.7)\ \|D(\lambda,t)f\|_{\mathscr{C}(\overline{\Omega})}$$
$$\leq C_p\{\sup_{x\in\overline{\Omega}}\|D(\lambda,t)f\|_{2m,p,\Omega(x,R_p)}\}^{\mu}\{\sup_{x\in\overline{\Omega}}\|D(\lambda,t)f\|_{0,p,\Omega(x,R_p)}\}^{1-\mu}$$

with $\mu = n/2mp$. But, since

$(\lambda + \lambda_0 - A(t,x;D))D(\lambda,t)f = (A(t,x;D) - \lambda_0)dA(t)^{-1}/dtf$ in Ω

and (3.4), and since

$B_j(t,x;D)D(\lambda,t)f = -B_j(t,x;D)dA(t)^{-1}/dtf$ on $\partial\Omega$, $1 \leq j \leq m$

and (3.5), it follows by using (3.1) that

$$\sum_{j=0}^{2m}|\lambda|^{1-j/2m}\sup_{x\in\overline{\Omega}}\|D(\lambda,t)f\|_{j,p,\Omega(x,R_p)}$$
$$\leq C_p\{\sup_{x\in\overline{\Omega}}\|A(t)^{-1}f\|_{2m,p,\Omega(x,R_p)} +$$
$$\sum_{1\leq j\leq m,\, m_j\neq 0}|\lambda|^{1-m_j/2m}\sup_{x\in\overline{\Omega}}\|A(t)^{-1}f\|_{m_j,p,\Omega(x,R_p)}\}$$

(note that $B_j(t,x;D) = b_{j0}(t,x) \equiv 1$ if $m_j = 0$). Therefore from (3.6)

$$\leq C_p|\lambda|^{1-\nu_B}\|f\|_{\mathscr{C}(\overline{\Omega})},$$

where $\nu_B = \mathrm{Min}\{m_j>0;\ 1 \leq j \leq m\}/2m$. We therefore conclude (from (3.7)) that

$$\|D(\lambda,t)f\|_{\mathscr{C}(\overline{\Omega})} \leq C_p|\lambda|^{\mu-\nu_B}\|f\|_{\mathscr{C}(\overline{\Omega})}.$$

The density of functions with compact support provides thus

$$\|D(\lambda,t)\|_{\mathscr{L}(X)} \leq C_p|\lambda|^{\mu-\nu_B},$$

hence (III) (remember that p was arbitrarily taken in $n/2m < p < \infty$).

4. PROOF OF LEMMA 3.2

Lemma 3.2 is a slight modification of the ordinary a priori estimates in L^p space. Under (R1), (A1-2) and (B1-2) it is known (see e.g. [8, Lemma 17.6] that:

Theorem 4.1 *For any* $1 < p < \infty$ *there is a positive constant* C_p *such that, if* $|\arg\lambda| \geq \pi/2 - \delta$ *and* $|\lambda| \geq C_p$, *then*

$$(4.1)\quad \sum_{j=1}^{2m} |\lambda|^{1-j/2m} \|u\|_{j,p,\Omega} \leq C_p\{\|(\lambda - A(t,x;D))u\|_{0,p,\Omega} + \sum_{j=1}^{m} |\lambda|^{1-m_j/2m} \|g_j\|_{0,p,\Omega} + \sum_{j=1}^{m} \|g_j\|_{2m-m_j,p,\Omega}\}$$

for all $u \in W_p^{2m}(\Omega)$, *where* $g_j \in W_p^{2m-m_j}(\Omega)$ *with the condition that* $g_j = B_j(t,x;D)u$ *on* $\partial\Omega$, $1 \leq j \leq m$.

Let ψ be a function in $\mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset \{|x| < 2\}$ and $\psi \equiv 1$ on $\{|x| \leq 1\}$. For any $x_0 \in \overline{\Omega}$ and $r \geq 1$, we set $\psi_0(x) = \psi((x-x_0)/r)$ and apply (4.1) to $\psi_0 u$. Since

$$(\lambda - A(t,x;D))(\psi_0 u) = \psi_0(\lambda - A(t,x;D))u + \sum_{|\alpha|\leq 2m} \sum_{0\neq\gamma\leq\alpha} \binom{\alpha}{\gamma} a_\alpha(t,x) D^\alpha \psi_0 D^{\alpha-\gamma} u \quad \text{in } \Omega,$$

it follows that

$$\|(\lambda - A(t,x;D))(\psi_0 u)\|_{0,p,\Omega} \leq C_p\{\|(\lambda - A(t,x;D))u\|_{0,p,\Omega(x_0,2r)} + \frac{1}{r}\|u\|_{2m-1,p,\Omega(x_0,2r)}\}.$$

On the other hand, if we put

$$h_j = \psi_0 g_j + \sum_{|\beta|\leq m_j} \sum_{0\neq\gamma\leq\beta} \binom{\beta}{\gamma} b_{j\beta}(t,x) D^\gamma \psi_0 D^{\beta-\gamma} u, \quad \text{for } 1 \leq j \leq m,$$

then $h_j \in W_p^{2m-m_j}(\Omega)$, $h_j = B_j(t,x;D)(\psi_0 u)$ on $\partial\Omega$ and h_j satisfies for $0 \leq k \leq 2m - m_j$ the estimate

$$\|h_j\|_{k,p,\Omega} \leq C_p\{\|g_j\|_{k,p,\Omega(x_0,2r)} + \frac{1}{r}\|u\|_{m_j+k-1,p,\Omega(x_0,2r)}\}.$$

Hence it turns out that

$$\sum_{j=0}^{2m} |\lambda|^{1-j/2m} \|u\|_{j,p,\Omega(x_0,r)} \le \sum_{j=0}^{2m} |\lambda|^{1-j/2m} \|\psi_0 u\|_{j,p,\Omega}$$

$$\le C_p\{\|(\lambda - A(t,x;D))u\|_{0,p,\Omega(x_0,2r)}$$

$$+ \sum_{j=1}^{m} |\lambda|^{1-m_j/2m} \|g_j\|_{0,p,\Omega(x_0,2r)} + \sum_{j=1}^{m} \|g_j\|_{2m-m_j,p,\Omega(x_0,2r)}\}$$

$$+ C_p/r\{ \sum_{j=1}^{m} |\lambda|^{1-m_j/2m} \|u\|_{m_j-1,p,\Omega(x_0,2r)} + \|u\|_{2m-1,p,\Omega(x_0,2r)}\}.$$

To complete the proof it now suffices to notice a fact that for an integer N, which is independent of $x_0 \in \overline{\Omega}$ and $r \ge 1$, $\Omega(x_0,2r)$ can be covered by N number of $\Omega(x_i,r)$, $x_i \in \overline{\Omega}$, $1 \le i \le N$, and therefore

$$\|v\|_{j,p,\Omega(x_0,2r)} \le N \sup_{x\in\overline{\Omega}} \|v\|_{j,p,\Omega(x,r)}, \quad v \in W_p^j(\Omega)$$

hold for all $0 \le j \le 2m$.

REFERENCES

[1] T.Kato and H.Tanabe, *On the abstract evolution equation*, Osaka Math. J. **14** (1962), 107-133.

[2] D.Park, *Initial-boundary value problem for parabolic equation in L^1*, Proc. Japan Acad. **62** Ser.A (1986), 178-180.

[3] H.B.Stewart, *Generation of analytic semigroups by strongly elliptic operators under general boundary conditions*, Trans. Amer. Math. Soc. **259** (1980), 229-310.

[4] H.Tanabe, *Remarks on the equations of evolution in a Banach space*, Osaka Math. J. **12** (1960), 145-166.

[5] M.Watanabe, *A remark on fractional powers of linear operators in Banach spaces*, Proc. Japan Acad. **53** (1977), 4-7.

[6] A.Yagi, *On the abstract linear evolution equations in*

Banach spaces, J. Math. Soc. Japan **28** (1976), 290-303.

[7] A.Yagi, *Parabolic evolution equations in which the coefficients are the generators of infinitely differentiable semigroups*, to appear.

[8] H.Tanabe, *Functional Analysis, II*, Jikkyo Shuppan Publishing Company, Tokyo, 1981 (in Jananese).

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan