A DUALITY THEOREM FOR CROSSED PRODUCTS BY NONABELIAN GROUPS

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Let $\alpha : G \to \mathrm{Aut}\,A$ be an action of a locally compact group G on a C*-algebra A. When G is abelian, there is a canonical dual action $\hat{\alpha}$ of the dual group $\hat{G}$ on the crossed product $A \times_{\alpha} G$, and the Takai duality theorem asserts that the second crossed product $(A \times_{\alpha} G) \times_{\hat{\alpha}} \hat{G}$ is isomorphic to the tensor product $A \otimes \mathcal{K}(L^2(G))$ of A with the algebra $\mathcal{K}(L^2(G))$ of compact operators on $L^2(G)$. The usual proof of this theorem [5,8] uses spatial techniques, but we have recently given a new proof in which we exploit the universal properties of crossed products, and which we hope is a bit more transparent [7].

Imai and Takai used essentially the same spatial techniques to obtain a duality theorem for actions of nonabelian groups [1]. They replaced the dual action of $\hat{G}$ by a "coaction" of G, and defined all their crossed products spatially, so for non-amenable G their theorem concerns the reduced crossed product $A \times_{\alpha,r} G$ rather than the full one $A \times_{\alpha} G$. Here we shall show that the approach of [7] can also be adapted to the case of nonabelian groups, where it gives a duality theorem for the full crossed product.

We start by describing our notion of coaction, which is slightly different from the normal one: usually a coaction of G on A is a homomorphism of A into $M(A \otimes_{\min} C_r^*(G))$, whereas ours will be a homomorphism of A into $M(A \otimes_{\max} C^*(G))$. We have deliberately chosen to use the full group C*-algebra and the maximal tensor product because we are stressing universal properties rather than spatial ones. As a result, we have to handle crossed products by coactions differently. We first give a definition of what a crossed product should be, and then prove that one exists (Proposition 6); this follows very closely the treatment of ordinary crossed products in [7]. Once we have this sorted out, it is a relatively simple matter to adapt the argument of [7, §2] to the nonabelian setting. We feel that the resulting duality theorem is more satisfactory, particularly for non-amenable groups, and that its proof is conceptually clearer.

NOTATION. When G is a locally compact group, we shall write λ and ρ for the left and right regular representations of G on $L^2(G)$, M for the representation of $C_0(G)$ by multiplication operators on $L^2(G)$, and δ_s for the point mass at $s \in G$, viewed as an element of the multiplier algebra $M(C^*(G))$. In general, we shall use 1 for the identity element of an algebra, and i for an identity mapping between algebras. As in [7], a homomorphism ϕ of a C*-algebra A into a multiplier algebra $M(B)$ will be called <u>nondegenerate</u> if there is an approximate identity $\{e_i\}$ for A such that $\phi(e_i) \to 1$ strictly in $M(B)$: this implies that ϕ has a (unique) strictly continuous extension to $M(A)$ [4, §1], and we shall use this fact repeatedly without comment. If $f \in B^*$, the slice map $S_f : A \otimes B \to A$ is defined by $S_f(\Sigma_i\, a_i \otimes b_i) = \Sigma_i\, a_i f(b_i)$: this is bounded for the minimal (and hence any) C*-tensor product norm, and extends to a strictly continuous map of $M(A \otimes B)$ into $M(A)$.

§1. COACTIONS AND THEIR CROSSED PRODUCTS.

We begin by recalling that if A and B are C*-algebras, the maximal tensor product $A \otimes_{max} B$ is the completion of the algebraic tensor product $A \odot B$ in the norm

$$\|\Sigma\, a_i \otimes b_i\| = \sup\ \{\|\Sigma\, \pi(a_i)\rho(b_i)\| \ :\ \pi \text{ and } \rho \text{ are commuting representations of } A \text{ and } B \text{ on the same space}\}.$$

We shall often write $A \otimes B$ for $A \otimes_{max} B$. There is always a natural map of $M(A) \otimes_{max} M(B)$ into $M(A \otimes_{max} B)$, which in particular allows us to view a tensor $m \otimes n$ of multipliers as a multiplier of $A \otimes_{max} B$. If G is a locally compact group, this construction gives us a strictly continuous homomorphism $s \to \delta_s \otimes \delta_s$ of G into the unitary group $UM(C^*(G) \otimes_{max} C^*(G))$, which we can integrate to obtain a nondegenerate homomorphism

$$\delta_G : C^*(G) \to M(C^*(G) \otimes_{max} C^*(G)),$$

called the <u>comultiplication</u> of $C^*(G)$ (see [2] for more details).

DEFINITION 1. A <u>coaction</u> of a locally compact group G on a C^*-algebra A is a nondegenerate homomorphism δ of A into $M(A \otimes_{max} C^*(G))$ satisfying

1) $\delta(a)(1 \otimes z)$ and $(1 \otimes z)\delta(a)$ belong to $A \otimes_{max} C^*(G)$ for all $a \in A$, $z \in C^*(G)$;
2) $(\delta \otimes i)\circ\delta = (i \otimes \delta_G)\circ\delta$.

Of course, this requires some explanation. First of all, the nondegeneracy of δ and δ_G imply that both $\delta \otimes i$ and $i \otimes \delta_G$ extend to the multiplier algebra $M(A \otimes C^*(G))$, so the compositions in (2) at least make sense. To understand where conditions (1) and (2) come from, we consider the case of an abelian group G. Here

$$A \otimes C^*(G) \cong A \otimes C_0(\hat{G}) \cong C_0(\hat{G}, A),$$

so $C_b(\hat{G}, A)$ acts naturally as multipliers of $A \otimes C^*(G)$: condition (1) says the range of a coaction of G must lie in this subalgebra. Indeed, given an action α of $\hat{G}$ on A, the analogous coaction δ^α

of G is given as a map from A to $C_b(\hat{G},A)$ by $\delta^{\alpha}(a)(\gamma) = \alpha_{\gamma}(a)$. In general, condition (1) is technically useful because it implies $S_f(\delta(a)) \in A$ for $a \in A$, $f \in B(G) = C^*(G)^*$. The homomomorphism property of the action α of $\hat{G}$ says the two functions $(\gamma,\chi) \to \alpha_{\gamma}(\alpha_{\chi}(a))$ and $(\gamma,\chi) \to \alpha_{\gamma\chi}(a)$ agree: this translates into condition (2) on the coaction δ^{α}. In general, we refer to (2) as the <u>coaction identity</u>, and we think of coactions as an analogue for nonabelian groups of actions of the dual group.

For us, the most important example will be the dual coaction on a crossed product. Before describing it, we recall that the crossed product by an action α of G on A is a C*-algebra $A \times_{\alpha} G$ whose representation theory is "the same as" the covariant representation theory of the system (A,G,α). Formally, there are embeddings $i_A : A \to M(A \times_{\alpha} G)$ and $i_G : G \to UM(A \times_{\alpha} G)$ such that

a) $i_A(\alpha_s(a)) = i_G(s)i_A(a)i_G(s)^*$ for $a \in A$, $s \in G$;

b) if (π,U) is a covariant representation of (A,G,α) (i.e. $\pi(\alpha_s(a)) = U_s\pi(a)U_s^*$), then there is a nondegenerate representation $\pi \times U$ of $A \times_{\alpha} G$ such that $\pi = (\pi \times U)\circ i_A$ and $U = (\pi \times U)\circ i_G$;

c) the ranges of i_A, i_G generate $A \times_{\alpha} G$.

Condition (a) implies that for any representation ρ of $A \times_{\alpha} G$, the pair $(\rho\circ i_A, \rho\circ i_G)$ is covariant, so the representations of $A \times_{\alpha} G$ are precisely those of the form $\pi \times U$. In the standard construction of $A \times_{\alpha} G$ as the C*-enveloping algebra of $L^1(G,A)$, we think of $i_G(s)$ as the point mass δ_s at $s \in G$ times the identity for A, and $i_A(a)$ as the multiple $a\delta_e$ of the point mass at e (see [5, §7.6]). For a detailed discussion of this view of crossed products, and in particular for an explanation of what (c) means, we refer to [7, §1].

Now suppose α is an action of G on A, and define maps j,k of A,G into $M((A \times_\alpha G) \otimes_{max} C^*(G))$ by

$$j(a) = i_A(a) \otimes 1, \quad k(s) = i_G(s) \otimes \delta_s.$$

Then if we represent $(A \times G) \otimes C^*(G)$ concretely on a Hilbert space $\mathcal{H}$, the pair (j,k) is a covariant representation of the system (A,G,α) on $\mathcal{H}$, and hence integrates to give a homomorphism $\hat{\alpha} = j \times k$ of $A \times_\alpha G$ into $M((A \times_\alpha G) \otimes C^*(G))$. When A is absent, of course, $\hat{\alpha}$ is just the comultiplication δ_G.

LEMMA 2. *The homomorphism* $\hat{\alpha}$ *is a coaction of* G *on* $A \times_\alpha G$, *called the* *<u>dual coaction</u>*.

PROOF. For $a \in A$ and $z,w \in C_c(G)$ we have

$$\hat{\alpha}(i_A(a)i_G(z))(1 \otimes w) = (i_A(a) \otimes 1) \iint z(s)w(s^{-1}t)(i_G(t) \otimes \delta_t)dsdt.$$

We can approximate the function $(s,t) \to z(s)w(s^{-1}t)$ uniformly by a sum $\Sigma\, z_j \otimes w_j \in C_c(G) \odot C_c(G)$, and then the right-hand side has the form

$$\Sigma\, i_{A\times G}(a \otimes z_j) \otimes w_j \in (A \times_\alpha G) \odot C^*(G).$$

This gives (1). To check the coaction identity, we just have to verify that $(\hat{\alpha} \otimes i)\circ\hat{\alpha}$ and $(i \otimes \delta_G)\circ\hat{\alpha}$ agree on the generators $i_A(a)i_G(z)$ for $A \times_\alpha G$, and this is a routine computation.

Next we have to discuss covariant representations of our dual systems (A,G,δ); the crossed product $A \times_\delta G$ will then be defined as a C*-algebra with these representations. In the reduced theory, the crossed product came first, defined spatially in terms of generators, and only later was the appropriate representation theory developed [4]. We shall have to modify this theory slightly to accommodate unreduced crossed products, but the ideas are essentially the same. We first define $W_G : G \to UM(C^*(G))$ by $W_G(s) = \delta_s$; this function is strictly continuous, and so defines a unitary multiplier W_G of $C_0(G,C^*(G)) = C_0(G) \otimes C^*(G)$.

DEFINITION 3. A <u>covariant representation</u> of (A,G,δ) is a pair (π,μ) of nondegenerate representations of A and $C_0(G)$, on the same Hilbert space $\mathcal{H}$, such that, for all $a \in A$,

$$\pi \otimes i(\delta(a)) = \mu \otimes i(W_G)(\pi(a) \otimes 1)\mu \otimes i(W_G^*) \qquad (*)$$

in $M(C^*(\pi(A),\mu(C_0(G))) \otimes_{max} C^*(G))$.

In [4], the role of the unitary W_G was played by the unitary operator W'_G on $L^2(G \times G)$ given by $W'_G\xi(s,t) = \xi(s,s^{-1}t)$, which is the image under $\mu \otimes \lambda$ of our $W_G \in M(C_0(G) \otimes C^*(G))$. As in the spatial theory, we can recover the representation μ from the unitary $\mu \otimes i(W_G)$ by slicing; in fact, for $f \in A(G) \subset C^*(G)^*$, we have $S_f(W_G) = f$, and hence

$$S_f(\mu \otimes i(W_G)) = \mu(S_f(W_G)) = \mu(f) \quad \text{for} \quad f \in A(G).$$

When G is abelian, and the coaction δ^α is given by an action α of $\hat{G}$,

$$W_G \in M(C_0(G) \otimes C^*(G)) = M(C^*(\hat{G}) \otimes C_0(\hat{G})) = M(C_0(\hat{G},C^*(\hat{G})))$$

can be viewed as the function $\gamma \to \delta_\gamma$; if μ is given by the unitary representation U of $\hat{G}$, then $\mu \otimes i(W_G)$ is the function $\gamma \to U_\gamma$, and (*) reduces to the usual covariance condition on (π,U).

EXAMPLE 4. The pair (λ,M) is a covariant representation of $(C^*(G),G,\delta_G)$. To see this, we note that (M,λ) is a covariant representation of $(C_0(G),G,\tau)$, where $\tau_s(f)(t) = f(s^{-1}t)$. Thus for $s \in G$, we have

$$M \otimes i(W_G)(\lambda_s \otimes 1)M \otimes i(W_G^*) = M \otimes i(W_G(\tau_s \otimes i(W_G)))(\lambda_s \otimes 1)$$
$$= M \otimes i(1 \otimes \delta_s)(\lambda_s \otimes 1)$$
$$= \lambda \otimes i(\delta_G(\delta_s)),$$

and integrating both sides to homomorphisms on $C^*(G)$ gives the covariance condition (*).

DEFINITION 5. Let $\delta : A \to M(A \otimes C^*(G))$ be a coaction. A crossed product for (A,G,δ) is a C*-algebra B together with nondegenerate homomorphisms $j_A : A \to M(B)$, $j_{C(G)} : C_0(G) \to M(B)$ satisfying:

a) $j_A \otimes i(\delta(a)) = j_{C(G)} \otimes i(W_G)(j_A(a) \otimes 1)j_{C(G)} \otimes i(W_G^*)$ in $M(B \otimes C^*(G))$;

b) for every covariant representation (π,μ) of (A,G,δ) there is a nondegenerate representation $\pi \times \mu$ of B such that $(\pi \times \mu) \circ j_A = \pi$, $(\pi \times \mu) \circ j_{C(G)} = \mu$;

c) the span of $\{j_A(a)j_{C(G)}(f) : a \in A,\ f \in C_0(G)\}$ is a dense subspace of B.

Of course, condition (a) implies that every nondegenerate representation ρ of B gives rise to a covariant representation $(\rho \circ j_A, \rho \circ j_{C(G)})$ of (A,G,δ), so all the representations of B have the form $\pi \times \mu$. It is easy to check that up to isomorphism there is at most one such C*-algebra, and we shall soon prove there is always one, so we shall call it the crossed product and denote it by $A \times_\delta G$.

PROPOSITION 6. *Let* $\delta : A \to M(A \otimes_{max} C^*(G))$ *be a coaction of a locally compact group* G *on a C*-algebra* A. *Then there is a crossed product C*-algebra* $A \times_\delta G$, *and a dual action* $\hat{\delta}$ *of* G *on* $A \times_\delta G$ *such that*
$\hat{\delta}_s(j_A(a)j_{C(G)}(f)) = j_A(a)j_{C(G)}(\sigma_s(f))$ for $a \in A$, $f \in C_0(G)$, $s \in G$,

where $\sigma_s(f)(t) = f(ts)$.

PROOF. Take a set $\mathcal{S}$ of cyclic covariant representations containing a representative of each unitary equivalence class, let

$$(\rho, v) = \bigoplus \{(\pi, \mu) \ : \ (\pi, \mu) \in \mathcal{S}\}$$

acting in $\mathcal{H} = \bigoplus \mathcal{H}(\pi, \mu)$, and let B be the closure in $B(\mathcal{H})$ of the set $\{\rho(a)v(f) \ : \ a \in A, \ f \in C_0(G)\}$. Because ρ and v are $*$-representations, to show that B is a C*-algebra it is enough to show that each product $v(f)\rho(a)$ belongs to B; further, since $A(G)$ is dense in $C_0(G)$, we may suppose $f \in A(G)$. Consider the action of $C^*(G)$ on $A(G)$ defined by

$$\langle g \circ x, y \rangle = \langle g, xy \rangle \quad \text{for} \quad g \in A(G) \quad \text{and} \quad x, y \in C^*(G).$$

We have $(A(G) \circ C^*(G))^{-} = A(G)$, so by the Cohen factorisation theorem we can factor $f = g \circ x$ for some $x \in C^*(G)$, $g \in A(G)$. Then

$$\begin{aligned} v(f)\rho(a) &= v(S_f(W_G))\rho(a) \\ &= S_f(v \otimes i(W_G)(\rho(a) \otimes 1)) \\ &= S_{g \circ x}(\rho \otimes i(\delta(a))v \otimes i(W_G)) \qquad \text{(by covariance of } (\rho, v)) \\ &= S_g(\rho \otimes i((1 \otimes x)\delta(a))v \otimes i(W_G)). \end{aligned}$$

We know that $(1 \otimes x)\delta(a) \in A \otimes_{max} C^*(G)$, and we can therefore approximate it by a finite tensor $\Sigma\, a_i \otimes x_i \in A \quad C^*(G)$. Then

$$\begin{aligned} v(f)\rho(a) &\sim S_g(\Sigma\, (\rho(a_i) \otimes x_i)v \otimes i(W_G)) \\ &= \Sigma\, \rho(a_i) S_{g \circ x_i}(v \otimes i(W_G)) \\ &= \Sigma\, \rho(a_i) v(g \circ x_i). \end{aligned}$$

Thus $v(f)\rho(a)$ does belong to B, and B is a C*-algebra. We define $j_A = \rho$, $j_{C(G)} = v$; then (c) is satisfied by definition, (a) because (ρ, v) is a covariant representation, and it remains to check (b). But this is easy too: the usual arguments show that any covariant representation is equivalent to a direct sum of cyclic representations, and hence to a direct sum of subrepresentations of (ρ,ν). Thus $(B, j_A, j_{C(G)})$ is the required crossed product for (A,G,δ).

We next claim that if $(B, j_A, j_{C(G)})$ is a crossed product for (A,G,δ), then so is $(B, j_A, j_{C(G)} \circ \sigma_s)$. To see this, note that $\sigma_s \otimes i(W_G)$ is the function $t \to \delta_{ts}$, or, in other words, $\sigma_s \otimes i(W_G) = W_G(1 \otimes \delta_s)$. Thus

$$\begin{aligned} &((j_{C(G)} \circ \sigma_s) \otimes i)W_G)(j_A(a) \otimes 1)((j_{C(G)} \circ \sigma_s) \otimes i)(W_G^*) \\ &\quad = j_{C(G)} \otimes i(W_G)(1 \otimes \delta_s)(j_A(a) \otimes 1)(1 \otimes \delta_s^*) j_{C(G)} \otimes i(W_G^*) \\ &\quad = j_A \otimes i(\delta(a)), \end{aligned}$$

and (a) is satisfied. Similarly, if (π,μ) is a covariant representation of (A,G,δ), so is $(\pi, \mu \circ \sigma_s)$. But then

$$(\pi \times (\mu \circ \sigma_s)) \circ j_{C(G)} = \mu \circ \sigma_s = ((\pi \times \mu) \circ j_{C(G)}) \circ \sigma_s,$$

and (b) holds too. Since condition (c) follows trivially from the corresponding property for $(B, j_A, j_{C(G)})$, this justifies our claim. From the uniqueness of the crossed product, we obtain an isomorphism $\hat{\delta}_s$ of B onto B satisfying $\hat{\delta}_s \circ j_A = j_A$, $\hat{\delta}_s \circ j_{C(G)} = j_{C(G)} \circ \sigma_s$, and it is now routine to verify that $s \to \hat{\delta}_s$ is a strongly continuous action of G on B with the required property. This completes the proof of the proposition.

REMARK. We finish our discussion of coactions and crossed products by pointing out one other minor difference between our treatment and the usual spatial one: we have not assumed our coactions are injective. This is definitely not a significant issue; for one thing, all our main examples are injective! In general, if δ is a coaction of G on A, then A decomposes as a direct sum $\ker \delta \oplus B$, and δ induces an injective coaction ε of G on B such that $B \times_\varepsilon G$ is naturally isomorphic to $A \times_\delta G$. To see this, one first computes directly that

$S_1 \circ \delta_G$ is the identity on $C^*(G)$, where 1 is the identity of $B(G) = C^*(G)^*$. (This also proves that δ_G is injective, and the same argument shows any dual coaction is injective.) Then the coaction identity implies

$$\delta(S_1(\delta(a))) = S_1(\delta \otimes i(\delta(a))) = S_1(i \otimes \delta_G(\delta(a))) = \delta(a),$$

so that S_1 is a splitting for $\delta: A \to \delta(A)$. It is a homomorphism because 1 is, and A therefore decomposes as a C^*-algebraic direct sum $\ker \delta \oplus B$, as claimed. Given this, the rest of our assertion can be routinely verified.

§2. THE DUALITY THEOREM

THEOREM 7 *Let* $\alpha: G \to \operatorname{Aut} A$ *be an action of a locally compact group* G *on a* C^**-algebra* A. *Then there is an isomorphism of* $(A \times_\alpha G) \times_{\hat{\alpha}} G$ *onto* $A \otimes K(L^2(G))$ *which carries the second dual action of* G *into* $\alpha \otimes \operatorname{Ad} \rho$.

PROOF We define embeddings α, α^{-1} of A in $C_b(G,A) \subset M(A \otimes C_0(G))$ by

$$\alpha(a)(s) = \alpha_s(a), \; \alpha^{-1}(a)(s) = \alpha_s^{-1}(a),$$

and embeddings of A, G, $C_0(G)$ in $M(A \otimes \mathcal{K})$ by

$$k_A = (i \otimes M) \circ \alpha^{-1}, \; k_G = 1 \otimes \lambda, \; k_{C(G)} = 1 \otimes M.$$

As in [7,p.9], (k_A, k_G) is covariant and therefore gives us a homomorphism $k_A \times k_G$ of $A \times_\alpha G$ into $M(A \otimes \mathcal{K})$. We shall prove that $(A \otimes \mathcal{K}, k_A \times k_G, k_{C(G)})$ is a crossed product for $(A \times_\alpha G, G, \hat{\alpha})$. The first part is easy: a routine approximation argument shows that the elements of the form $k_A \times k_G(w) k_{C(G)}(f)$ span a dense subspace of $A \otimes \mathcal{K}$.

Next we want to prove that $(k_A \times k_G, k_{C(G)})$ is covariant - in other words, that

$$k_{C(G)} \otimes i(W_G)(k_A \times k_G(i_A(a)i_G(z)))$$
$$= [(k_A \times k_G) \otimes i(\hat{\alpha}(i_A(a)i_G(z))][k_{C(G)} \otimes i(W_G)] \qquad (+)$$

for $a \in A$, $z \in C_c(G)$. The left-hand side L of this equals

$$(1 \otimes (M \otimes i)(W_G)(k_A(a) \otimes 1)(k_G(z) \otimes 1)$$
$$= (k_A(a) \otimes 1)(1 \otimes (M \otimes i(W_G)(\lambda(z) \otimes 1)))$$
$$= (k_A(a) \otimes 1)(1 \otimes (\lambda \otimes i)(\delta_G(z)))(1 \otimes (M \otimes i)(W_G))$$

by the covariance of (λ, M) (see Example 4). Now

$$1 \otimes (\lambda \otimes i)(\delta_G(z)) = (k_G \otimes i)(\int z(s)(\delta_s \otimes \delta_s)ds)$$
$$= ((k_A \times k_G) \otimes i)(\int z(s)(i_G(s) \otimes \delta_s)ds)$$
$$= ((k_A \times k_G) \otimes i)(\hat{\alpha}(i_G(z))),$$

and L becomes

$$(k_A(a) \otimes 1)((k_A \times k_G) \otimes i)(\hat{\alpha}(i_G(z)))(1 \otimes (M \otimes i)(W_G)),$$

which is easily seen to be the right-hand side of $(+)$. Thus (a) holds.

Now suppose $(\pi \times U, \mu)$ is a covariant representation of $(A \times_\alpha G, G, \hat{\alpha})$. We claim that (μ, U) is then a covariant representation of $(C_0(G), G, \tau)$. The covariance of $(\pi \times U, \mu)$ implies that for $s \in G$

$$(\pi \times U) \otimes i(\hat{\alpha}(i_G(s))) = \mathrm{Ad}\, \mu \otimes i(W_G)(\pi \times U(i_G(s)) \otimes 1),$$

which is equivalent to

$$U_s \otimes \delta_s = \mathrm{Ad}\, \mu \otimes i(W_G)(U_s \otimes 1).$$

Thus for $f \in A(G)$ we have

$$S_f(\mu \otimes i(W_G))U_s = U_s S_{f\bullet\delta_s}(\mu \otimes i(W_G)),$$

where $\langle f\bullet\delta_s, z\rangle = \langle f, \delta_s t\rangle$ for $z \in C^*(G)$. A quick calculation shows that $f\bullet\delta_s = \tau_s^{-1}(f)$, and $S_f(\mu \otimes i(W_G)) = \mu(f)$, so this becomes

$$\mu(f)U_s = U_s\mu(\tau_s^{-1}(f)),$$

and (μ, U) is covariant as claimed. We therefore obtain a representation $\mu \times U$ of $\mathcal{K}$ such that $(\mu \times U) \circ M = \mu$ and $(\mu \times U) \circ \lambda = U$ (see [7, Example 4]).

The covariance condition for $(\pi \times U, \mu)$ also implies

$$\pi(a) \otimes 1 = ((\pi \times U) \otimes i)(\hat{\alpha}(i_A(a))) = \mathrm{Ad}\,\mu \otimes i(W_G)(\pi(a) \otimes 1),$$

and slicing this gives $\pi(a)\mu(f) = \mu(f)\pi(a)$ for $f \in A(G)$. Thus we obtain a representation $\pi \otimes \mu$ of $A \otimes C_0(G)$ on $\mathcal{H}$; let $\rho = (\pi \otimes \mu) \circ \alpha$. The argument in the fourth paragraph of the proof of [7, Theorem 6] shows that $\rho(a)$ commutes with U_s, and therefore also with the range of $\mu \times U$. This gives a representation $\rho \otimes (\mu \times U)$ of $A \otimes \mathcal{K}$. It is easily verified that

$$(\rho \otimes (\mu \times U)) \circ k_G = U, \quad (\rho \otimes (\mu \times U)) \circ k_{C(G)} = \mu,$$

and the argument in [7] shows that $(\rho \otimes (\mu \times U)) \circ k_A = \pi$. Thus $(A \otimes K, k_A \times k_G, k_{C(G)})$ also satisfies (b), and we have proved that it is a crossed product for $(A \times_\alpha G, G, \hat{\alpha})$.

The uniqueness of the crossed product now implies there is an isomorphism $\psi : A \otimes K \to (A \times_\alpha G) \times_{\hat{\alpha}} G$ such that

$$\psi \circ (k_A \times k_G) = j_{A \times G} \quad \text{and} \quad \psi \circ k_{C(G)} = j_{C(G)},$$

To check that ψ intertwines $\alpha \otimes \mathrm{Ad}\,\rho$ and $\hat{(\hat{\alpha})}$, we just need to verify that

$$\psi(\alpha_s \otimes \mathrm{Ad}\rho_s(k_A \times k_G(i_A(a)i_G(z))k_{C(G)}(f))) = \hat{(\hat{\alpha})}_s(j_{A \times G}(i_A(a)i_G(z))j_{C(G)}(f)).$$

We have $\mathrm{Ad}\rho_s \circ M = M \circ \sigma_s$, $\mathrm{Ad}\rho_s \circ \lambda = \lambda$, and, by a quick calculation, $(\alpha_s \otimes \mathrm{Ad}\rho_s) \circ k_A = k_A$; the left-hand side is therefore

$$\begin{aligned}\psi(k_A(a)k_G(z)k_{C(G)}(\sigma_s(f))) &= j_{A \times G}(i_A(a)i_G(z))j_{C(G)}(\sigma_s(f)) \\ &= \hat{(\hat{\alpha})}_s(j_{A \times G}(i_A(a)i_G(z))\hat{(\hat{\alpha})}_s(j_{C(G)}(f)),\end{aligned}$$

as required. This completes the proof of the theorem.

CONCLUDING REMARKS. We know of at least two other duality theorems for crossed products by nonabelian groups: one for twisted crossed products, due to Quigg [6], and another which starts with a coaction, due to Katayama [3]. Both of these concern spatially-defined crossed products, and it would be interesting to extend them to the full crossed products along the lines we have extended Imai and Takai's theorem here. At present, we are optimistic that this can be done for Quigg's theorem, although there are technical problems, but we do not see how to adapt these methods to handle Katayama's: certainly the spatial versions of both theorems are substantially harder than the one we have discussed here.

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