The Laplace Beltrami Operator on Unbounded Homogeneous Domains in C^n

Richard C. Penney

Let Ω be a domain in C^n. A Lie group G is said to act on Ω if G act as a group and the mapping μ of $G \times \Omega$ into Ω is real analytic in G and holomorphic on Ω . The action is said to be rational if in addition it is rational on Ω.

For the remainder of this talk, we shall assume that G acts homogeneously, rationally, and that Ω carries a G invariant volume. Then Koszul [K] showed that Ω has a canonically defined two form – the Koszul form. Assume that this form is nondegenerate . Then Ω is a pseudo-Kahlerian domain. Hence Ω carries a canonical second order differential operator – the Laplace Beltrami operator. If Ω is not Kahlerian, then then this operator is not elliptic and not positive. However it is invariant under any holomorphic mapping that preserves the volume form. In particular, the spectrum of this operator (assuming that it is essentially self adjoint) is an invariant of the domain.

It is the goal of this talk to describe a method of computing the spectrum of this operator in a broad class of domains. To describe this class of domains, we shall need some structure theory (for this see [P2]). It turns out that all such domains are describable in terms of nilpotent Lie groups.

Let N be a nilpotent Lie group with Lie algebra $\mathcal{N}$. Let $\mathcal{N}_c$ be the complexification of $\mathcal{N}$ and let N_c be the corresponding connected, simply connected, Lie group. Let $\mathcal{P}$ be a complex sub algebra of $\mathcal{N}_c$. Let A denote the group of all automorphisms of $\mathcal{N}_c$ which leave both $\mathcal{N}$ and $\mathcal{P}$ invariant. Then A acts on both N_c and $X = N_c/P$. Let T be a maximal, R-split torus in the algebraic group A. We shall say that the pair $N - P$ is a Siegel $N - P$ pair if T_o (the connected component of T) has an open orbit in X. In this case there are only a finite number of open orbits. Each such orbit is a homogeneous domain for the group $S = T_o \times_s N$. Furthermore, due to the conjugacy of maximal tori, the domains do not depend on the choice of torus. We shall refer to these domains as the domains defined by the Siegel pair $N - P$. The basic structure theorem of [P2] is the following:

THEOREM. *Every rationally homogeneous,contractable domain in C^n is bi-holomorphic with a Siegel domain of type $N - P$ for some choice of N and P.*

The open orbits are easily described. Note first that A acts on the double quotient space $Y = N \backslash N_c / P$. A has an open orbit in X if and only if A has an open orbit in Y. In [P2] we prove the following:

THEOREM. *There is a real, A-invariant vector subspace $\mathcal{C}$ in $\mathcal{N}_c$ such that A has open orbits in Y if and only if A has open orbits in $\mathcal{C}$. The open $\mathcal{C}$ orbits are in one to one correspondence with the open orbits in Y under the mapping E defined by*

$$E : c \mapsto N(\exp c)P.$$

Furthermore, if $\mathcal{V}$ is an open orbit in $\mathcal{C}$, then $\mathcal{V}$ is a cone describable as the convex hull of a basis in $\mathcal{N}_c$. The domain in N_c/P is the image of $N \exp \mathcal{V}$ under the natural projection.

The dimension of the cone $\mathcal{V}$ is referred to as the rank of the domain. It is not known whether the rank is an invariant of the domain, but this is conjectured. In this talk we shall always assume that the rank is one. It follows that the boundary of Ω in N_c/P is a smooth, codimension one, real submanifold which is homogeneous under N. We shall also assume that the Levi form of the boundary is non-degenerate (although not necessarily positive) at at least one (and hence all) point. These assumptions make Ω be what we refer to as a nil-ball.

We shall require one more assumption on the domain. We shall assume that the torus T contains an element with all of its eigenvalues greater than one. This is equivalent with the assumption that N has a dilation that preserves P. Under these assumptions, we are able to completely and explicitly describe the spectral synthesis of the Laplace–Beltrami operator on Ω. Rather than attempt to describe this synthesis in general, we shall content our selves with describing it in a specific example. The general theory will follow this example very closely.

The example is the following:

$\mathcal{N}$ will be the Lie algebra spanned by the basis :

$$\{X_1, X_2, Y_1, Y_2, Z\}$$

relative to the commutation rules:

$$[X_1, Y_1] = [X_2, Y_2] = Z$$
$$[X_1, Y_2] = Y_1$$

We define the algebra $\mathcal{P}$ to be the span over C of the elements $X_1 + iY_2$ and $X_2 + iY_1$.

The group A contains the one parameter subgroup δ_t which has the X_i as eigenvectors with eigenvalues t^i , the Y_i with eigenvalues t^{3-i}, and Z with eigenvalue t^3. Then Ω is the set in C^3 defined by

$$u > x_1 x_2 + {x_2}^3/3$$

where x_1 ,x_2, and u denote the real parts of the coordinate functions in C^3.

The Lie algebra of the group S is spanned by the elements of $\mathcal{N}$ and D, where D is the generator of δ_t. Since $P \cap N = \{e\}$, S acts freely on Ω. Hence, we may identify Ω with S. In particular, the Laplace-Beltrami operator acts on $L^2(S)$. Since G acts on the left, the measure is left Haar measure and the operator is left invariant. Explicitly, the Laplace-Beltrami operator is

$$\triangle = \mathcal{D} + \mathcal{L}$$

where

(1) $$\mathcal{D} = (D - iZ)^*(D - iZ)$$

and

$$\mathcal{L} = -2(X_1 X_2 + Y_1 Y_2).$$

Here the Lie algebra acts via left invariant operators. The symbol '$*$' denotes adjoint in $L^2(S)$. Thus, $Z^* = -Z$ while, due to non-unimodularity, $D^* = 9 - D$.

To analyze this operator, we set

$$L = -i(Z^{-1}\mathcal{L})$$

where Z^{-1} is thought of as a pseudo-differential operator. From the commutation relations above, $[D, Z] = 3Z$ and $[D, \mathcal{L}] = 3\mathcal{L}$. Hence, D commutes with L. This allows us to diagonalize these operators jointly. Actually, we diagonalize L first. The methods for doing so have been described elsewhere [P1]. We obtain that L has a purely discrete spectrum consisting of the even integers. Let m be an element of this spectrum. Then , on the corresponding eigenspace, $\triangle = \mathcal{D} + imZ$.

Note that this operator depends only on the variables in the D and Z directions. Denoting these variables by t and u respectively, the eigenspace turns out to be

$$L^2(R^2, e^{-9t}dtdu) \otimes \mathcal{H}_m$$

where $\mathcal{H}_m$ is some Hilbert space. Then $\triangle = \mathcal{D}_m \otimes I$ where

$$\mathcal{D}_m = (\frac{d}{dt} - ie^{3t}\frac{d}{du})^*(\frac{d}{dt} - ie^{3t}\frac{d}{du}) + ime^{3t}\frac{d}{du}$$

We make the change of variables $t' = 3t - \log 3$ followed by $t = t'$. This changes the measure to $e^{-3t}dtdu/27$. Next, we apply the multiplication operator defined by $e^{-t}/\sqrt{27}$, which maps into $L^2(R^2, e^{-t}dtdu)$. The effect of these changes is to change $\mathcal{D}_m$ into:

$$\mathcal{D}'_m = 3(1 + \frac{d}{dt} - ie^t\frac{d}{du})^*(1 + \frac{d}{dt} - ie^t\frac{d}{du}) + 3ime^t\frac{d}{du}.$$

Let H denote the upper-half plane in C. We denote the general element of H by $u + ie^t$. In this notation, the $Sl(2, R)$ invariant measure is $e^{-t}dtdu$. The Laplace-Beltrami operator for this space is :

$$\begin{aligned} \triangle_H &= -(\frac{d}{dt} - ie^t\frac{d}{du})^*(\frac{d}{dt} - ie^t\frac{d}{du}) \\ &= \mathcal{D}'_m/3 + i(2-m)e^t\frac{d}{du} - 1 \end{aligned} \qquad (2)$$

For $m = 2$, $\mathcal{D}'$ is (up to constants) the Laplace-Beltrami operator for the upper half plane in C. For other values of m,the operator also is known. In fact, the upper-half plane is $Sl(2,R)/SO(1)$. The Iwasawa decomposition allows us to identify the upper-half plane with AN, which is just the $ax + b$ group. The Laplace-Beltrami operator is the image under the quasi-regular representation of the Casimir element of $Sl(2,R)$. The corresponding L^2 space is the space of the representation induced by the trivial representation of $SO(1)$. If instead of the trivial representation, we use a character of $SO(1)$, we may still identify the representation space with $L^2(R^2)$. The elements of $SO(1)^\wedge$ are indexed by integers. We denote the corresponding induced representation by L_k where k indexes the character. Then the image of the Casimir element under L_k is easily seen to be $\mathcal{D}'_m/3 + 1$, where $k = 2m + 4$. The Casimir element is scalar on any irreducible subspace and the eigenspaces are spanned by the irreducible subspaces. The set of representations which occure discretely in L_k is just the elements of the discrete series which have vectors of K–type k. The elements of the discrete series are parameterized by integers n where the Casimir element has eigenvalue $(1 - n^2)/4$. For each n there corresponds two representations. Such representations have vectors of type k if and only if $k = \pm(n+1+2p)$ for some non-negative integer p. Equating the two expressions for k, we see that the eigenvalues of $\mathcal{D}_m$ are $3(1 - n^2)/4 + 1$ where n is an odd integer, $n < |2m + 2|$. Summing up, we have:

THEOREM. *The operaror $\triangle$ has both discrete and continuous spectrum. The discrete spectrum is the set of integers of the form $3(1 - n^2)/4 + 1$ where n ranges over all odd integers.*

The above Theorem finishes our discussion of the specific example. The case of a general, dilated nil-ball, however, is not very different. The Laplace-Beltrami operator still splits into a sum of operators of the form (1). Of course, the constants are different and the operator $\mathcal{L}$ is more complicated. However, the spectrum of $\mathcal{L}$ still may be explicitly determined using some (nontrivial) extensions of the ideas in [P1] and some recent work of Corwin and Greenleaf [CG], although the spectrum may now have a nontrivial continuous part. The analysis of the other part of $\triangle$, on the other hand, proceeds exactly as before. After a few simple changes of varibles we again obtain the image of the Casimir element of

$Sl(2,R)$ in the representation induced by a character of $SO(1)$.

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Department of Mathematics
Purdue University
WEST LAFAYETTE INDIANA 47907 USA