FOURIER THEORY ON LIPSCHITZ CURVES

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The aim of this talk is to indicate how the theory of Fourier multipliers in $L_p(\mathbb{R})$ can be adapted when the real line $\mathbb{R}$ is replaced by a Lipschitz curve γ. Details will appear in [6].

(I) Let us start with a resumé of the usual theory concerning $L_p(\mathbb{R})$.

(Ia) The Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx$$

defines a mapping

$$L_1(\mathbb{R}) \xrightarrow{\wedge} C_0$$

where C_0 denotes the space of continuous functions on $(-\infty,\infty)$ which tend to zero at $\pm\infty$. We consider the inverse Fourier transform

$$\check{w}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} w(\xi)\, d\xi$$

$$L_p(\mathbb{R}) \xleftarrow{\vee} S$$

where S is the Schwartz space of rapidly decreasing functions on $(-\infty,\infty)$. Then

$$(1) \qquad \int_{\mathbb{R}} f(x)\, \check{w}(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\, w(-\xi)\, d\xi$$

for all $f \in L_1(\mathbb{R})$ and $w \in S$, so it is consistent with the case $p = 1$ to define

$$L_p(\mathbb{R}) \xrightarrow{\wedge} S'$$

by

$$\langle \hat{f}, w_- \rangle = 2\pi \int_{\mathbb{R}} f(x)\, \check{w}(x)\, dx\,, \quad w \in S\,,$$

for $1 < p \le \infty$, where $w_-(\xi) = w(-\xi)$ and S' is the space of the tempered distributions. We note that

(2) $\{\check{w} \mid w \in S\}$ is dense in $L_p(\mathbb{R})$, $1 \le p < \infty$, and in $C_0(\mathbb{R})$, from which it is immediate that

(3) $L_p(\mathbb{R}) \xrightarrow{\wedge} S'$ is one-one.

Of course, the following also holds:

(4) Assuming $f \in L_1(\mathbb{R})$ and $w \in S$, then $f = w$ if and only if $f = \hat{w}$.

(Ib) Next we note some facts concerning the convolution

$$(\phi * f)(x) = \int \phi(x-y)\, f(y)\, dy.$$

(5) Let $1 \le p \le \infty$. If $\phi \in L_1(\mathbb{R})$, $f \in L_p(\mathbb{R})$, then $\phi * f \in L_p(\mathbb{R})$ and $\|\phi * f\|_p \le \|\phi\|_1 \|f\|_p$. If $f \in C_0(\mathbb{R})$, then so is $\phi * f$.

Proof: (when $1 \le p < \infty$). Let $p' = p(p-1)^{-1}$. Then

$$\|\phi * f\|_p = \{\int_{\mathbb{R}} |\int_{\mathbb{R}} \phi(x-y)\, f(y)\, dy|^p\, dx\}^{\frac{1}{p}}$$

$$\le \{\int_{\mathbb{R}} \{\int_{\mathbb{R}} |\phi(x-y)|\, dy\}^{p/p'}\ \{\int |\phi(x-y)||f(y)|^p\, dy\}\, dx\}^{\frac{1}{p}}$$

$$\le \{\sup_x \int_{\mathbb{R}} |\phi(x-y)|\, dy\}^{1/p'}\ \{\sup_y \int_{\mathbb{R}} |\phi(x-y)|\, dx\}^{\frac{1}{p}}\ \|f\|_p$$

$$= \|\phi\|_1\ \|f\|_p . \qquad //$$

It is straightforward to show

(6) if ϕ, $f \in L_1(\mathbb{R})$, then $(\phi * f)^\wedge = \hat{\phi}\, \hat{f}$.

Example: Define ϕ_λ for $Im\lambda > 0$ by

$$\phi_\lambda(x) = \begin{cases} i\, e^{i\lambda x} & , \quad x > 0 \\ 0 & , \quad x < 0 , \end{cases}$$

and, for $Im\lambda < 0$, by

$$\phi_\lambda(x) = \begin{cases} 0 & , \quad x > 0 \\ -i\, e^{i\lambda x} & , \quad x < 0 . \end{cases}$$

Then $\|\phi_\lambda\|_1 = |Im\lambda|^{-1}$, and $\hat{\phi}_\lambda(\xi) = (\xi-\lambda)^{-1}$. So, for $f \in L_1(\mathbb{R})$,

$$(\phi_\lambda * f)(x) = \begin{cases} i\int_{y<x} e^{i\lambda(x-y)}\, f(y)\, dy & , \quad Im\lambda > 0 \\ \\ -i\int_{y>x} e^{i\lambda(x-y)}\, f(y)\, dy & , \quad Im\lambda < 0 \end{cases}$$

and

$$(\phi_\lambda \hat{*} f)\ (\xi) = (\xi-\hat{\lambda})^{-1}\ f(\xi) .$$

(Ic) Let $1 \le p < \infty$. Consider the operator $D = \frac{1}{i}\frac{d}{dx}$ as a closed linear operator in $L_p(\mathbb{R})$ with dense domain

$$L_p^1(\mathbb{R}) = \{f \in L_p(\mathbb{R}) \mid f' \in L_p(\mathbb{R})\} ,$$

where f' denotes the distribution derivative of f.

It is straightforward to show

(7) when $Im\lambda \neq 0$, then $(D-\lambda I)^{-1}f = \phi_\lambda * f$ for all $f \in L_p(\mathbb{R})$, so

$$\|(D-\lambda I)^{-1}\| \leq |Im\lambda|^{-1}$$

and

$$((D-\lambda I)^{-1}f)^\wedge(\xi) = (\xi-\lambda)^{-1}\hat{f}(\xi).$$

Note in particular that the spectrum $\sigma(D)$ is contained in $\mathbb{R}$. (Actually $\sigma(D) = \mathbb{R}$.) These results also hold when $L_p(\mathbb{R})$ is replaced by $C_0(\mathbb{R})$ with the norm $\|f\| = \sup|f(x)|$.

(Id) Let $b \in L_\infty(-\infty,\infty)$. Then b is an $L_p(\mathbb{R})$- Fourier-multiplier means there exists a bounded linear operator B in $L_p(\mathbb{R})$ such that

$$(Bf)^\wedge = b\hat{f} \quad , \quad f \in L_p \cap L_1(\mathbb{R}).$$

If $p = \infty$, $L_p(\mathbb{R})$ is replaced by $C_o(\mathbb{R})$.

We denote the set of $L_p(\mathbb{R})$ - Fourier multipliers by $M_p(\mathbb{R})$, and, in analogy with (Ic), we write $b(D)$ for B when $b \in M_p(\mathbb{R})$.

Let us list some conditions which ensure that a function b belongs to $M_p(\mathbb{R})$. By S_ρ^o we mean the double sector

$$S_\rho^o = \{z \in \mathbb{C} \mid |Im\ z| < \rho|Re z|\}$$

and by $H_\infty(S_\rho^o)$ we mean the space of bounded holomorphic functions on S_ρ^o.

(i) $1 \leq p \leq \infty$. If $\phi \in L_1(\mathbb{R})$ then $\hat{\phi} \in M_p(\mathbb{R})$ and $\hat{\phi}(D)f = \phi * f$.

(ii) $1 \leq p \leq \infty$. If $b \in H_\infty(S_\rho^o)$ for some $\rho > 0$, and

$$\begin{cases} |b(\zeta)-b_0| \leq c|\zeta|^s, & |\zeta| \leq 1 \\ |b(\zeta)| \leq c|\zeta|^{-s}, & |\zeta| \geq 1, \end{cases}$$

for some b_0, c, $s > 0$, then $b \in M_p(\mathbb{R})$.

(iii) $p = 2$. If $b \in L_\infty$, then $b \in M_2(\mathbb{R})$.

(iv) $1 < p < \infty$. If $b \in L_\infty$ and, for all $a > 0$,

$$\int_a^{2a} |db(\xi)| \leq \text{const},$$

then $b \in M_p(\mathbb{R})$.

(v) $1 < p < \infty$. If $b \in H_\infty(S_\rho)$ for some $\rho > 0$, then $b \in M_p(\mathbb{R})$.

(vi) $1 < p < \infty$. If $b = \chi_J$, the characteristic function of an interval J, then $b \in M_p(\mathbb{R})$.

(vii) $1 < p < \infty$. If $b \in L_\infty$ and there exists $\phi: \mathbb{R} \sim \{0\} \to \mathbb{C}$ and $\sigma > 0$ such that

$$|\phi(x)| \leq \text{const.}\ |x|^{-1}$$

$$|\phi(x+h) - \phi(x)| \leq \text{const.}\ |h|^\sigma |x|^{-(1+\sigma)}, \quad |h| < \tfrac{1}{2}|x|,$$

and

$$p.v.\int_{\mathbb{R}} \phi(x)\,\check{w}(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(\xi)w(-\xi)d\xi \ , \ w \in S \ ,$$

then $$b(D)f(x) = p.v.\int_{\mathbb{R}} \phi(x-y)\,f(y)\,dy \ , \ f \in L_p(\mathbb{R}).$$

To prove (ii), apply the Cauchy formula for $b(z) + ib_o(z-i)^{-1}$ on the boundary of $S^o_{\rho/2}$. (c.f. [5] and [4].) Parts (v) and (vi) are both corollaries of (iv) which is due to Marcinkiewicz. See, e.g., [7]. The operators $X_J(D)$ are spectral projections for D. Part (vii) is essentialy proved in [7]. (c.f.[2].)

Example: If $b(\xi) = sgn(\xi-a)$ for $a \in \mathbb{R}$, then $\phi(x) = (\pi ix)^{-1}\exp(ixa)$, so

$$sgn(D-a)\ f(x) = \frac{1}{\pi i}\ p.v \int_{\mathbb{R}} \frac{1}{x-y}\ e^{ia(x-y)} f(y)\ dy \ ,$$

and

$$X_{[a,b]}(D)f(x) = \frac{1}{2\pi i}\ p.v \int_{\mathbb{R}} \frac{1}{x-y}\ \{e^{ia(x-y)} - e^{ib(x-y)}\}\ f(y)\ dy \ .$$

(II) Henceforth g denotes a real-valued Lipschitz function with

$$\|g'\|_\infty \le N < \infty \ ,$$

$$\gamma = \{x + ig(x) \in \mathbb{C} \mid x \in \mathbb{R}\}$$

and

$$\Gamma = \{z-\zeta \mid z \in \gamma \ , \ \zeta \in \gamma\} \ .$$

Note that

$$\Gamma \subset S_N = \{z \in \mathbb{C} \mid |Imz| \le N|Rez| \ \} \ .$$

Our aim is to see what happens when $\mathbb{R}$ is replaced by γ in the results of (I). We work in the spaces $L_p(\gamma)$ for which

$$\|f\|_p = \{\int_\gamma |f(z)|^p\,|dz|\}^{\frac{1}{p}} < \infty$$

(where the integral is with respect to arc-length) and in $C_0(\gamma)$. Let us first consider convolution on γ.

(IIb) If $f \in L_p(\gamma)$ and ϕ is defined on Γ, then $(\phi * f)(z)$ is defined by

$$(\phi * f)(z) = \int_\gamma \phi(z-\zeta)\ f(\zeta)\ d\zeta$$

whenever the right hand side makes sense. The inequality in (5) is no longer correct, but the proof goes through except for the final equality. So we have, for $1 \le p \le \infty$,

$$\|\phi * f\|_p \le \{\sup_{z\in\gamma}\int_\gamma |\phi(z-\zeta)|\ |d\zeta|\}^{\frac{1}{p'}}\ \{\sup_{\zeta\in\gamma}\int_\gamma |\phi(z-\zeta)|\ |dz|\}^{\frac{1}{p}}\ \|f\|_p$$

if $\phi(z-.)$ and $\phi(.-\zeta) \in L_1(\gamma)$ for all z and $\zeta \in \gamma$.

Example: Define $\phi_\lambda : S_N \to \mathbb{C}$ when $Im\lambda > 0$ by

$$\phi_\lambda(z) = \begin{cases} i\, e^{i\lambda z} & , \quad Re\ z > 0 \\ 0 & , \quad Re\ z \leq 0 \end{cases} ,$$

and, when $Im\ \lambda < 0$, by

$$\phi_\lambda(z) = \begin{cases} 0 & , \quad Re\ z \geq 0 \\ -i\, e^{i\lambda z} & , \quad Re\ z < 0 . \end{cases}$$

Write $z = x + iy$ and $\lambda = \mu + i\nu$. When $\nu x > 0$,

$$|\phi_\lambda(z)| = \exp(Re(i\lambda z)) = \exp(-\nu x - \mu y)$$
$$\leq \exp(-\nu x(1 - N|\mu/\nu|))$$

which tends to zero exponentially as $|x| \to \infty$ if $\lambda \in S_N$.
Computation gives, for $\lambda \in S_N$,

$$\|\phi_\lambda * f\|_p \leq \{\text{dist}(\lambda , S_N)\}^{-1} \|f\|_p \quad , \quad f \in L_p(\gamma) .$$

(IIc) Let $1 \leq p < \infty$. Define the closed linear operator D_γ with domain $L_p^1(\gamma)$ dense in $L_p(\gamma)$ as follows.

If $u\colon \gamma \to \mathbb{C}$ and $z \in \gamma$, define $u'(z)$ by

$$u'(z) = \lim_{\substack{h \to 0 \\ z+h \in \gamma}} \left\{\frac{u(z + h) - u(z)}{h}\right\}$$

provided the limit exists. Let $C_c^1(\gamma)$ be the space of continuous functions u with compact support for which u' exists and is continuous on γ . Then $C_c^1(\gamma)$ is dense in $L_p(\gamma)$. Let

$$L_p^1(\gamma) = \left\{f \in L_p(\gamma) \ \middle|\ \exists\, h \in L_p(\gamma) \ni \int_\gamma f(z)u^1(z)dz = -\int_\gamma h(z)u(z)dz \right.$$
$$\left. \text{for all } u' \in C_c^1(\gamma) \right\} .$$

and define D_γ by $D_\gamma f = i^{-1}h$ for $f \in L_p^1(\gamma)$ and h as above.

So $D_\gamma = \frac{1}{i} \frac{d}{dz}\Big|_\gamma$ in the weak sense.

It is straightforward to show that when $\lambda \in S_N$,
then $(D_\gamma - \lambda I)^{-1} f = \phi_\lambda * f$ for all $f \in L_p(\mathbb{R})$,

so

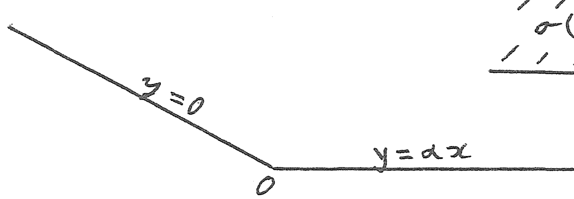
$$\| (D_\gamma - \lambda I)^{-1} \| \leq \{ \text{dist} (\lambda , S_N)\}^{-1}$$

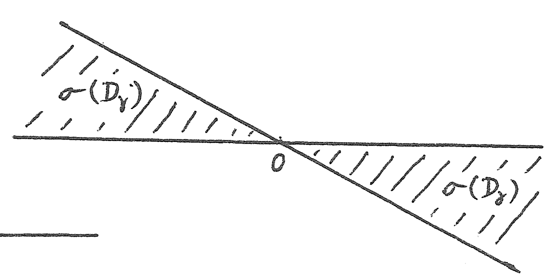
and $\sigma(D_\gamma) \subset S_N$.

These results also hold when $L_p(\mathbb{R})$ is replaced by $C_0(\mathbb{R})$.

Example:

γ:





If γ is the curve defined by the function

$$g(x) = \begin{cases} 0 & , \quad x \le 0 \\ \alpha x & , \quad x \ge 0 \end{cases} ,$$

then $\sigma(D_\gamma)$ is as shown in the right-hand sketch. In particular, if $\lambda = \mu + i\nu$ where $-\alpha\mu < \nu < 0$, then $\exp(i\lambda z) \in L_p(\gamma)$ and $C_0(\gamma)$, and

$$(D_\gamma - \lambda I)\, e^{i\lambda z} = 0 ,$$

which means that such a number λ is a eigenvalue of D_γ (acting in $L_p(\gamma)$ or in $C_0(\gamma)$).

This example shows in particular that $\sigma(D_\gamma)$ is not necessarily contained in $\mathbb{R}$, so it is not reasonable to try to define $b(D_\gamma)$ for $b \in L_\infty(-\infty,\infty)$.

Of the results listed in (Id), there are however natural ways to define $b(D_\gamma)$ in the following cases.

(i') $1 \le p \le \infty$. If $\psi \in L_1(0,\infty)$ where

$\psi(r) = \sup\{|\phi(z)| \mid z \in \Gamma, |z| = r\}$, and $\hat{\phi}(D_\gamma) f = \phi * f$,

then $\hat{\phi}(D_\gamma)$ is bounded in $L_p(\gamma)$.

(ii') $1 \le p \le \infty$. If $b \in H_\infty(S^o_\rho)$ for some $\rho > N$, and

$$|b(z) - b_0| \le c|z|^s \quad , \quad |z| \le 1$$
$$|b(z)| \le c|z|^{-s} \quad , \quad |z| \ge 1,$$

for some b_0, c and $s > o$, then $b(D_\gamma)$ can be defined as a bounded operator in $L_p(\gamma)$ (or $C_0(\gamma)$) using contour integration on the boundary of S_σ, where $N < \sigma < \rho$. (c.f. [5] and [4].)

(v') $1 < p < \infty$. If $b \in H_\infty(S^o_\rho)$ for some $\rho > N$, then $b(D_\gamma)$ can be defined as a bounded operator in $L_p(\gamma)$.
This can be achieved when p = 2 using quadratic estimates.

$$\int_0^\infty \| tD_\gamma(I + t^2 D_\gamma^2)^{-1} f \|^2 \, t^{-1} dt \le const.\|f\|^2 \quad , \quad f \in L_2(\gamma)$$

(c.f.[5]), or using singular integrals as in [3]. Both methods depend on the type of estimates first proved in [1]. The methods can be adapted to work when p ≠ 2. See [4] and [3].

(III) To proceed further we make an additional assumption, namely that the function g defining γ satisfies

$$\sup |g(x)| \le M < \infty .$$

Then

$$\Gamma \subset \{z \in S_N \mid |Im\, z| \le M\} .$$

The functions ϕ_λ defined in (IIb) now satisfy an additional estimate, namely

$$|\phi_\lambda(z)| = \exp(-\nu x - \mu y) \le \exp(|\mu|M)\, \exp(-\nu x)$$

when $\nu x > 0$. We find that, whenever $\nu = Im\lambda \ne 0$,

$$\|\phi_\lambda * f\|_p \le |\nu|^{-1} \exp(2|\mu|M) \sqrt{1+N^2} \|f\|_p .$$

So $\sigma(D_\gamma) \subset \mathbb{R}$. (Actually $\sigma(D_\gamma) = \mathbb{R}$) and we now have the possibility of defining Fourier multipliers. Let us start with Fourier transforms.

(IIIa) Define, for $-\infty < \alpha < \infty$, E_α to be the space of measurable functions w on $(-\infty,\infty)$ for which

$$\|w\|_{E_\alpha} = \left(\int_{-\infty}^{\infty} e^{2\alpha|\xi|} |w(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty ,$$

and

$$E_\alpha^2 = \{ w \in E_\alpha \mid w', w'' \in E_\alpha \} .$$

(The space $\bigcup_{\alpha > M} E_\alpha$ was used in [3].)

The Fourier transform

$$\hat{f}(\xi) = \int_\alpha e^{-iz\xi} \hat{f}(z) \, dz$$

defines a mapping

$$L_1(\gamma) \xrightarrow{\wedge} E_{-\beta}$$

provided $\beta > M$. It can be shown that the material in (Ia) goes through provided S is replaced by E_β^2 and S' by $(E_\beta^2)'$.

To be precise consider the inverse Fourier transform

$$\check{w}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iz\xi} \, w(\xi) \, d\xi$$

as a mapping

$$L_p(\gamma) \xleftarrow{\vee} E_\beta^2$$

again with $\beta > M$. Then

$$\int_\gamma f(z) \, \check{w}(z) \, dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \, w(-\xi) \, d\xi$$

for all $f \in L_1(\gamma)$ and $w \in E_\beta^2$, so it is consistent with the case $p = 1$ to define

$$L_p(\gamma) \xrightarrow{\wedge} (E_\beta^2)'$$

by

$$\langle \hat{f}, w_- \rangle = 2\pi \int_\gamma f(z) \check{w}(z) \, dz \qquad , \; w \in E_\beta^2$$

for $1 < p \le \infty$. Again we can show that the mapping $L_p(\gamma) \xrightarrow{\wedge} (E_\beta^2)'$ is one-one, and that, whenever $f \in L_1(\gamma)$ and $w \in E_\beta^2$, then $\hat{f} = w$ if and only if $f = \check{w}$, but first we need some density results, in particular that $\{\check{w} \mid w \in E_\beta^2\}$ is dense in $L_p(\gamma)$ for $1 \le p < \infty$ and in $C_0(\gamma)$. When $1 < p < \infty$, this follows from the results of [3], but we need density in $L_1(\gamma)$. (Actually we need it in $L_1 \cap L_p(\gamma)$. To see this, proceed as follows. Let

$$h_n(\zeta) = \begin{cases} \exp(-(\zeta/n)^{\alpha}) & , \quad \mathrm{Re}\,\zeta > 0 \\ \exp(-(-\zeta/n)^{\alpha}) & , \quad \mathrm{Re}\,\zeta < 0 \end{cases}$$

where α is a little larger than 1, and define $h_n(D_\gamma)$ as in (ii').
For $f \in L_p(\gamma)$ or $C_0(\gamma)$,
let $f_n = h_n(D_\gamma) f$ and $f_{n,m}(z) = (1 + z^2/m^2)^{-1} f_n(z)$.

Then $\hat{f}_{n,m} \in E^2_\beta$, and $f_n \to f$ and $f_{n,m} \to f_n$ in the appropriate topologies.

A different argument will be presented in [6].

(IIId) Let $f \in L_\infty(-\infty,\infty)$. Then b is an $L_p(\gamma)$ - <u>Fourier multiplier</u> means there exists a bounded linear operator $b(D_\gamma)$ in $L_p(\gamma)$ such that

$$(b(D_\gamma) f)^\wedge = b\hat{f} \qquad , \ f \in L_p \cap L_1(\gamma) .$$

If $p = \infty$, $L_p(\gamma)$ is replaced by $C_0(\gamma)$.

The set of $L_p(\gamma)$ - Fourier multipliers is denoted by $M_p(\gamma)$.

The following theorem is useful in studying $M_p(\gamma)$.

Theorem: Let $1 \le p \le \infty$. Suppose $w \in E^2_\beta$ and the support of w is contained in $[-L, L]$. Then

$$\|\check{w}\|_{L_p(\gamma)} \le C\sqrt{1+N^2}\, \exp(2ML)\, \|\check{w}\|_{L_p(\mathbb{R})}$$

and

$$\|\check{w}\|_{L_p(\mathbb{R})} \le C\sqrt{1+N^2}\, \exp(2ML)\, \|\check{w}\|_{L_p(\gamma)}$$

for some universal constant c.

Proof: Let θ be a C^2 function with support in $[-2,2]$ such that

$\theta(\xi) = 1$ if $|\xi| \le 1$, and let $\theta_L(\xi) = \theta(\xi/L)$. Then $\check{\theta}_L$ is an entire function which
satisfies $|\check{\theta}_L(z)| \le f_L(|z|)$ for all z such that $|Im\, z| \le M$,
where $\|f_L\|_1 \le C \exp(2LM)$.

Now $w = \theta_L w$, so

$$\check{w}(z) = \int_{\mathbb{R}} \check{\theta}_L(z-x)\check{w}(x)\, dx \quad , \quad z \in \gamma ,$$

and

$$\check{w}(x) = \int_{\gamma} \check{\theta}_L(x-z)\check{w}(z)\, dz \ , \quad x \in \mathbb{R} .$$

On proceeding as in (Ib) we obtain the required estimates.

Corollary: If $b \in M_p(\mathbb{R})$ and $sppt(b) \subset [-L,L]$, then $b \in M_p(\gamma)$. More generally we can show that if $b(\xi)\exp(2\beta|\xi|) \in M_p(\mathbb{R})$ for some $\beta > M$, then $b \in M_p(\gamma)$. Using this, we obtain the following results.

(iii') $p = 2$. If $|b(\xi)| \leq c\exp(-2\beta|\xi|)$, then $b \in M_2(\gamma)$.

(iv') $1 < p < \infty$. If $|b(\xi)| \leq c\exp(-2\beta|\xi|)$ and, for all $a > 0$,

$$\int_a^{2a} |dg(\xi)| \leq const.,$$

where $g(\xi) = b(\xi)\exp(2\beta|\xi|)$, then $b \in M_p(\gamma)$.

(vi') $1 < p < \infty$. If $b = X_J$, the characteristic function of an interval J, then $b \in M_p(\gamma)$.

(vii') $1 < p < \infty$. If $b \in M_2(\gamma)$ and there exists $\phi: \Gamma\sim\{0\} \to \mathbb{C}$ and $\sigma > 0$ such that

$$|\phi(z)| \leq const.|z|^{-1}$$

$$|\phi(z+h) - \phi(z)| \leq const.|h|^{\sigma}|z|^{-(1+\sigma)}, \quad |h| \leq \tfrac{1}{2}|z|$$

and

$$p.v.\int_{\gamma-\zeta} \phi(z)\check{w}(z)\,dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(\xi)\,w(-\xi)\,d\xi, \quad w \in E_\beta^2,$$

for all $\zeta \in \gamma$ (where $\gamma - \zeta = \{z - \zeta \mid z \in \gamma\}$), then $b \in M_p(\gamma)$ and

$$b(D_\gamma)f(z) = p.v.\int_\gamma \phi(z-\zeta)f(\zeta)\,d\zeta, \quad f \in L_p(\gamma).$$

Part (vi') is a consequence of the preceding theorem when J is a bounded interval, of (v') when $J = (-\infty, 0]$ or $[0,\infty)$, and of a combination of the two when J is an unbounded interval. The operators $X_J(D_\gamma)$ are spectral projections for D_γ.

An alternative approach to (vi') if via the L_p-boundedness of $\operatorname{sgn}(D_\gamma - a)$ for $a \in \mathbb{R}$. Use (vii') to write

$$\operatorname{sgn}(D_\gamma - a)f(z) = \frac{1}{\pi i}\ p.v.\int_\gamma \frac{1}{z-\zeta}\, e^{ia(z-\zeta)} f(\zeta)\,d\zeta.$$

i.e. $\operatorname{sgn}(D_\gamma - a) = G_a \operatorname{sgn}(D_\gamma)\, G_{-a}$

where G_a denotes multiplication by $\exp(iaz)$. Now

$$\|G_a f\|_p \leq \exp(|a|M)\|f\|_p, \quad f \in L_p(\gamma),$$

so

$$\|\operatorname{sgn}(D_\gamma - a)\| \leq \exp(2|a|M)\|\operatorname{sgn}(D_\gamma)\|,$$

and $\operatorname{sgn}(D_\gamma)$ is bounded by (v'). Indeed $\operatorname{sgn}(D_\gamma)$ is the Cauchy operator, the boundedness of which was first shown in [1].

REFERENCES

1. Coifman R.R., McIntosh A. and Meyer Y., "L'intégral de Cauchy définit un operateur borné sur L^2 pour les courbes lipschitziennes", Annals of Mathematics, 116 (1982), 361-387.

2. Coifman R.R. and Meyer Y., "Au-delà des opérateurs pseudo-différentiels", astérisque 57, société mathématique de France, 1978.

3. Coifman R.R. and Meyer Y., "Fourier analysis of multilinear convolutions, Calderón's theorem, and analysis on Lipschitz curves", Lecture Notes in Mathematics, 779 Springer-Verlag, Berlin, 1980.

4. Cowling M, McIntosh A, and Yagi A. "Banach space operators with an H_∞-functional calculus", in preparation.

5. McIntosh, Alan, "Operators which have an H_∞-functional calculus", Miniconference on Operator Theory and Partial Differential Equations, 1986, Proceedings of the Centre for Mathematical Analysis, ANU, Canberra, 14 (1986).

6. McIntosh, Alan and Qian Tao, "Fourier multipliers on Lipschitz curves", in preparation.

7. Stein, Elias M., "Singular Integrals and Differentiability Properties of Functions", P.U.P., Princeton, 1970.

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